IX Southern-Summer School on Mathematical Biology

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Lecture VI

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Outline

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Vegetation in semi-arid regions

Eremology: science of arid regions.

Figura: Arid and semi-arid regions of the world
Vegetation in semi-arid regions

Eremology: science of arid regions.

Figura: Arid and semi-arid regions of the world
Consider the vegetation cover in water-poor regions.

In this case, water is a *limiting factor*.

Quite different from tropical regions, where competition for water is irrelevant. One of the main limiting factors is light.

We want to build a mathematical model — *(simple, please)* — to describe the mutual relation between water in soil and biomass in semi-arid regions.

Let us do it.
Water and vegetation, in a first approximation, entertain a relation similar to *predator-prey dynamics*.

The presence of water is incremental for vegetation; vegetation consumes water.

But note that water does not originate from water. It is an abiotic variable.

The usual predator-prey dynamics does not apply.

Consider two variables:

- $w$, the amount of water in soil.
- $u$, the vegetation biomass (proportional to the area with vegetation cover).
Klausmeier Model

Equation for the amount of water in soil

\[ \frac{dw}{dt} = a \quad \text{precipitation} - bw \quad \text{evaporation} - cu^2 w \quad \text{absorption by vegetation} \]

Water in soil increases due to precipitation \((a)\), evaporates at a constant *per volume* rate \((b)\), and is absorbed by vegetation in a per volume rate that depends on \(u^2\) \((c)\). This is phenomenological law coming from lab fittings.
Klausmeier Model

Equation for the amount of water in soil

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Equation for biomass

\[ \frac{du}{dt} = -du + eu^2w \]

Vegetation has a natural death rate, \((d)\) and absorbs water at a per volume rate \((e)\) proportional to \(uw\).
Let us begin by defining two new variables, rescaled ones:

\[ W = w \left[ \frac{e}{\sqrt{b^3 c}} \right] \]

\[ U = u \sqrt{bc} \]

\[ T = tb \]

They are dimensionless.

Plug them into the equations and you will get....
Analysis of the model

\[ \frac{dW}{dT} = A - W - WU^2 \]
\[ \frac{dU}{dT} = WU^2 - BU \]

where

\[ A = \frac{ae}{\sqrt{b^3c}} \]

and

\[ B = d/b \]

⇒ the equations depend only on two parameters, instead of five.

What do these equations tell us?
Analysis of the model

\[ \frac{dW}{dT} = A - W - WU^2 \]
\[ \frac{dU}{dT} = WU^2 - BU \]

Let us look for fixed points:

- The points \( U^* \) e \( W^* \) such that

\[ \frac{dW^*}{dT} = 0 \]
\[ \frac{dU^*}{dT} = 0 \]

or

\[ A - W^* - W^*(U^*)^2 = 0 \]
\[ W^*(U^*)^2 - BU^* = 0 \]
Analysis of the model

\[ \frac{dW}{dT} = A - W - WU^2 \]
\[ \frac{dU}{dT} = WU^2 - BU \]

- The algebraic equations has three roots:

\[ U^* = 0 \]
\[ W^* = A \]

If \( A > 2B \)

\[ U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}} \]
\[ W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2} \]

If \( A > 2B \)

\[ U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}} \]
\[ W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2} \]
Analysis of the Model

Interpretation

Our first conclusion:

- If $A < 2B$ the only solution is $U^* = 0$ and $W^* = A$.
- This represents a bare state. A desert.
- The condition $A > 2B \Rightarrow a > \frac{2d\sqrt{bc}}{e}$ shows that there must be a minimum amount of precipitation to sustain vegetation.
- Moreover, the higher $e$ the easier to have a state with vegetation. Recall that $e$ represents the absorption rate. The higher, the better.
- On the other, the higher ($b$) and the death rate of the population, ($d$) easier it is to have a vegetationless solution.
- Seems reasonable!
Analysis of the model

So, let $A > 2B$

- If $A > 2B$, we can have two more fixed points.
- What about their stability?
- The linear stability analysis results in:
  - The fixed point $U^* = 0$ and $W^* = A$ is always stable, even if $A \leq 2B$.
  - The fixed point
    
    $$U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}}, \quad W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

    is always unstable.
  - The fixed point
    
    $$U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}}, \quad W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

    is stable provided $B < 2$. 
Analysis of the Model

- So, if $A > 2B$, and $B < 2$, we have **two** stable fixed points, each of them with its basin of attraction.
- A pictorial view is as follows:

![Graph showing stable fixed points](image)

**Figura**: $B$ is fixed, and we plot $U^*$ (the biomass) in terms of $A$. The solution representing a desert ($U^* = 0$) and the solution corresponding to vegetation cover are both stable.
Hysteresis

- The existence of a region of bi-stability \((A > 2B, B < 2)\), can take us to the following situation.

- Take a fixed \(B\). Consider that \(A\) can change slowly.

- Let us begin in the bi-stability region. And let \(A\) decrease. At a certain moment, \(A\) will cross the critical value \(A = 2B\).

- At this time a **sudden transition** occurs, a jump, in which \(U^* \rightarrow 0\). **Desertification!!!**

- Suppose now that \(A\) begins again to increase - slowly. As \(U^* = 0\) is stable, even with \(A > 2B\) we will continue in the "desertic" region, as at the moment of crossing back the critical point we were in its the basin of attraction.

- **In summary**: if we begin with a certain \(A\), decrease it \(A < 2B\) and then come back to our initial value of \(A\), the state of the system can transit from \(U^* \neq 0\) to \(U^* = 0\).

- This is called **Hysteresis**.
Hysteresis

Figura: $B$ is fixed. $A$ begins at a value $A'$ with $U^* = U'$, decreases, crosses a critical point at $A = 2B$. It goes to, $U^* \to 0$. When we increase again $A$, even with $A > 2B$, we have $U^* = 0$.

Once the "desertic" state is attained it is not sufficient to change the external conditions back (in our model, this is the rainfall) in order to get back a vegetation-cover state. Terrible!
Glory and Misery of the Model

Glory

Figura: Estimated vegetation cover in the region of Sahara, over a long time span. We see a sudden change around 5500 BP.

- Existence of sudden transitions can be understood rather simply. The same kind of phenomenon appears in other systems as well.
- The model is Simple.
Glory and Misery of the Model

Misery

- The model is very simple
- The transition is towards a completely vegetationless state. Actual desertification processes allow for remnants of vegetation.
- The model predicts an infinite bi-stability region... We could think that enough rain could reverse desertification.
- There are indeed better models.

**Figura**: Desertification Region in China

**Figura**: Senegal, at the Sahel region, south to Sahara.
More realistic models give bifurcation diagrams like the one below.

Biomass in terms of the rainfall, in a static case. The blue curve represents two transition regions. The one to the left implies a vegetation $\rightarrow$ desert transition. To the right, a reversed transition.
More realist models

Still another curve.

This diagram is similar to the preceding one, but $U^*$ does not tend to zero.
Bibliography

Online Resources

- http://ecologia.ib.usp.br/ssmb/

Thank you for your attention