Gravitational Waves

Small perturbation, propagation in vacuum

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = (\underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} + \underbrace{h_{\mu\nu}}_{|h| \ll 1})\,dx^\mu dx^\nu$$

"GW strain"

soln's to $\quad \Box h^{\mu\nu} = 0$.

As in E&M, solution to wave eqn can be found by integrating over the source,

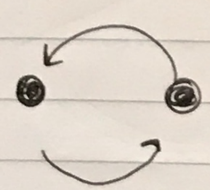
$$h_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^4} \int d^3x \, \frac{T_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|}$$

In the non-relativistic limit, weak field, reduce to quadrupole formula.

$$h_{jk} = \frac{2G}{c^4} \frac{1}{r} \ddot{\underset{\sim}{I}}_{jk}(t - r/c)$$

$$\underset{\sim}{I}^{jk} = I^{jk} - \frac{1}{3}\delta^{jk}\delta_{\ell m} I^{\ell m}, \quad I^{jk} = \int d^3x \, \rho(t, \vec{x})\, x^j x^k$$

Let's estimate the strain from a circular binary of BH's.



orbital angular frequency ω

radius R.

Kepler: $\quad \omega^2 = \dfrac{G(M_1 + M_2)}{\underbrace{a^3}_{\text{semimajor axis}}} = \dfrac{2GM}{R^3}$

Bayaktar
06/29
p2/4

Strain from binary BHs contd.

$$\rho = \delta(z)\, M\, [\delta(x-x_1)\,\delta(y-y_1) + \delta(x-x_2)(y-y_2)]$$

$\vec{x}_1 = \frac{1}{2}R\,(\cos\theta, \sin\theta, 0)$ $\vec{x}_2 = \frac{1}{2}R(-\cos\theta, \sin\theta, 0)$
and $\theta = \omega t + \phi$, set t s.t. $\phi = 0$.

$$I^{xx} = \int d^3x\, \rho\, x^2 = M(x_1^2 + x_2^2) = \frac{1}{4} M R^2 (1 + \cos 2\omega t)$$

$$\ddot{\underset{\sim}{I}} \sim O(\ddot{I}^{xx}) \sim M R^2 \omega^2 \cos(2\omega t)$$

$$h \sim \frac{2G}{r}\ddot{\underset{\sim}{I}} \sim \frac{2GMR^2\omega^2}{r}\cos(2\omega t).$$

close to merger, $\omega^2 \sim r_g / R^3$
where $GM \equiv r_g$.

$$h_{max} \lesssim \frac{r_g \cdot R^2 \cdot r_g / R^3}{r} \lesssim \frac{r_g}{r} \sim \frac{10\,\mathrm{Km}}{100\,\mathrm{Mpc}} \sim$$

$\sim 3 \times 10^{-21}$

Measuring the GW strain

Consider a plane wave propagating in x-direction.

$$h_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_{yy} & h_{yz} \\ 0 & h_{zy} & h_{zz} \end{pmatrix}$$

transverse traceless: 2 dof

$$h_+ = h_{yy} = -h_{zz}$$
$$h_\times = h_{yz} = h_{zy}$$

Assume wave is emitted in single polarization state s.t.

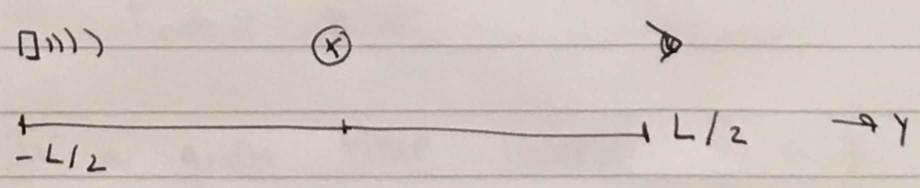
$$h_\times = 0$$
$$h_+ = h_0 \sin(kx - \omega t).$$

The invariant interval between two "events", i.e. free-falling masses, is given by

$$ds^2 = -dt^2 + \left(dx^2 + (1 + h_0 \sin(kx - \omega t))dy^2 + (1 - h_0 \sin(kx - \omega t))dz^2\right)$$

$$= 0 \quad \text{for light.}$$

along y-axis: $dt^2 = 1 + h_0 \sin(kx - \omega t)\, dy^2$



since $h_0 \ll 1$

$$\int dt = \int 1 + \frac{h_0}{2} \sin(kx - \omega t)\, dy$$

$$\Delta t \simeq L - \frac{hL}{2} \sin(\omega t) \qquad \text{for } \Delta t\omega \ll 1$$

$$\Delta t_{no\,GW} \simeq L \quad \Rightarrow \quad \boxed{\Delta t_{GW} = \frac{hL}{2} \sin \omega t.}$$

Sources of noise

→ seismic, Newtonian, suspension thermal at low frequencies.

→ over most of the frequency range, main factors constitute "quantum" noise, i.e. the fact that the light beams used to make measurements are made up of individual photons.

Shot noise.

The laser is a set of discrete photons with an average flux but independent arrival time.

This causes power fluctuations at the output from the mean

$$\bar{P}_{out} = \bar{n}\,\omega$$

In a given time interval, $\bar{N} = \bar{n}\,\tau$ photons arrive on average, with poisson distribution:

$$\frac{\sigma_N}{\bar{N}} = \frac{\sqrt{\bar{N}}}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}} = \sqrt{\frac{\omega}{\bar{P}\,\tau}}$$