Wondering about an open-closed correspondence

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Plan:

- Recall A_∞-algebras and their typical properties
- Relation to field theory
- Other homotopy algebras
- Open-closed correspondence

- On what I study recently (Homological mirror symmetry)

Def. [A_∞-algebra (Stasheff'63)]

$(A, \mathfrak{m} := \{m_k\}_{k\geq 1})$ *is an* A_∞-**algebra** $\Leftrightarrow$

$A = \oplus_{r\in\mathbb{Z}} A^r$ *: $\mathbb{Z}$-graded vector space,*

$\mathfrak{m} := \{m_n : A^{\otimes n} \to A\}_{n\geq 1}$ *: linear maps of degree* $|m_n| = (2-n)$ *satisfying the* A_∞*-relations:*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm\, m_k(a_1, \cdots, a_j, m_l(a_{j+1}, \cdots, a_{j+l}), a_{j+l+1}, \cdots, a_n)\ ,$$

for $n = 1, 2, ...,$ *where* $a_i \in A^{|a_i|}$, $i = 1, ..., n$.

The A_∞-relations for $n = 1, 2, 3$:

for $m_1 = d$, $m_2 = \cdot$, $\quad x, y, z \in V$:

$$i) \quad d^2 = 0 \ ,$$

$$ii) \quad d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) \ ,$$

$$iii) \quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z).$$

$i) \Leftrightarrow (A, d)$ forms a complex.

$ii) \Leftrightarrow$ Leibniz rule of d w.r.t. the product $\cdot$.

$iii)$ $\cdot$ is associative **up to homotopy**.

In particular, if $m_3 = 0$, the product $\cdot$ is strictly associative.

An A_∞-algebra $(A, \mathfrak{m})$ with $m_3 = m_4 = \cdots = 0$ is a **DG algebra**.

An A_∞-algebra $(A, \mathfrak{m})$ with $m_1 = 0$ is called **minimal**.

An Example of minimal A_∞-algebra

A generated by $e^0 = id, e^2, e^5$

nontrivial A_∞-product:

$$m_2(e^0, e^2) = m_2(e^2, e^0) = e^2, \qquad m_3(e^2, e^2, e^2) = e^5$$

Def. *Given two* A_∞*-algebras* $(A, \mathfrak{m})$ *and* $(A', \mathfrak{m}')$*, an* A_∞**-morphism** $\mathfrak{f} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *is a collection of degree* $(1-k)$ *multilinear maps* $\mathfrak{f} := \{f_k : A^{\otimes k} \to A'\}_{k\geq 1}$ *s.t.*

$$\sum_{i\geq 1} \sum_{k_1+\cdots+k_n=n} \pm m_i'(f_{k_1} \otimes \cdots \otimes f_{k_i})(a_1, ..., a_n)$$
$$= \sum_{\substack{i+1+j=k \\ i+l+j=n}} \pm f_k(\mathbf{1}^{\otimes i} \otimes m_l \otimes \mathbf{1}^{\otimes j})(a_1, ..., a_n)$$

for $n = 1, 2, ...$

Note: For $n = 1$: $m_1' f_1 = f_1 m_1 \Leftrightarrow$

$f_1 : (A, m_1) \to (A', m_1')$ forms a chain map.

Def. $\mathfrak{f} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *is called an* A_∞-**(quasi)-isomorphism** *iff* $f_1 : (A, m_1) \to (A', m_1')$ *is a (quasi)-isomorphism.*

Note that A_∞-quasi-isomorphisms define an equivalence relation.

Important theorems:

Minimal model theorem (Kadeishvili'83)

For any A_∞-algebra $(A, \mathfrak{m})$, there exists an A_∞-algebra $(H(A), \mathfrak{m}')$ and an A_∞-quasi-isomorphism $(H(A), \mathfrak{m}') \to (A, \mathfrak{m})$.

Note that $\mathfrak{m}' = \{m'_1 = 0, m'_2, m'_3, \ldots\}$. Such an A_∞-algebra $(H(A), \mathfrak{m}')$ is called a **minimal model** of $(A, \mathfrak{m})$.

$\star$ Minimal models of $(A, \mathfrak{m})$ are unique up to A_∞-isomorphisms on $H(A)$.

More generally ...

Homological perturbation theory (HPT) (1982∼)

(Kadeishvili, Gugenheim, Lambe, Stasheff, Huebschmann,...)

For an A_∞-algebra $(A, \mathfrak{m})$,

strongly deformation retract (SDR) data is

$$(V, d) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (A, m_1)\ , \qquad h : A^r \to A^{r-1}$$

s.t. $m_1 h + h m_1 = id_A - \iota \circ \pi, \qquad \pi \circ \iota = id_V$.

Given SDR data, there exists an A_∞-algebra $(V, \mathfrak{m}')$ with $m_1' = d$ and ι, π lift to A_∞-quasi-isomorphisms.

(∃ an explicit construction using **Feynman graphs**.)

Relation to field theory

- cyclic A_∞-algebra $(A, \mathfrak{m}, \omega)$

$$\Longleftrightarrow \text{ action } \quad S = \sum_{n \geq 1} \frac{1}{n+1} \omega(\Phi, m_n(\Phi, \ldots, \Phi))$$

satisfying classical BV master eq. $(S, S) = 0$

- $(V, \mathfrak{m}', \omega')$ obtained by HPT $\Longleftrightarrow$ effective field theory of S on V

- the (cyclic) minimal model $\Longleftrightarrow$ on-shell scattering amplitudes

(cf. H.K' 02, 07 on classical open SFTs)

(Inspired by K. Fukaya's lectures in Japan)

Other homotopy algebras:

- L_∞-algebra (cf. Lada-Stasheff'92)

m_2 corresponds to a Lie bracket. This satisfies the Jacobi identity up to homotopy m_3.

- C_∞-algebra ($=$ homotopy commutative A_∞-algebra)

(Kadeishvili, Markl, ...)

- OC_∞-algebra (=OCHA) (K-Stasheff'06, cf. E.Hoefel'12):

O=open, C=closed

mixture of A_∞-algebra and L_∞-algebra

Relation to string (field) theory

A_∞-algebra $\Leftrightarrow$ open string theory (Gaberdiel-Zwiebach'97, etc)

L_∞-algebra $\Leftrightarrow$ closed string theory (Zwiebach'92)

OC_∞-algebra $\Leftrightarrow$ open-closed string theory (Zwiebach'98)

$\Uparrow$ $\Uparrow$

cyclic — extract the classical part of the action satisfying the quantum **BV master eq.**

Open-closed correspondence:

An OC_∞-algebra (which includes an A_∞-algebra A to an L_∞-algebra L) gives us an L_∞-morphism (K-Stasheff'06)

$$\mathfrak{f} : L \to (C(A^{\otimes \bullet}, A), d_{Hoch}, [\ ,\ ]_G)$$

($Hoch$ = Hochschild, G = Gerstenhaber)

For each given field theory on the world sheet, we obtain $\mathfrak{f}$.

(cf. Poisson-sigma model $\to$ Kontsevich's L_∞-quasi-isomorphism which solves the deformation quantization problem

(Cattaneo-Felder'98))

Wondering...

• For the bosonic open-closed SFT, is $\mathfrak{f}$ an L_∞-quasi-isormophism ?

(Zwiebach'92: Interpolating SFTs may be useful.)

• For the bosonic classical open SFT $(A, \mathfrak{m}, \omega)$, what is the L_∞ minimal model of the DGLA $(C(A^{\otimes\bullet}, A), d_{Hoch}, [\ ,\ ]_G)$?

If $\mathfrak{f}$ is an L_∞ quasi-isomorphism, then this should be

the on-shell scattering amplitudes of tree closed strings!

$\Rightarrow$ $(C(A^{\otimes\bullet}, A), d_{Hoch}, [\ ,\ ]_G)$ (with an appropriate cyclicity)

is a closed SFT !!

Homological mirror symmetry

$$\{\text{symplectic mfds. } M\} \overset{\text{Mirror Symmetry}}{\Longleftrightarrow} \{\text{complex mfds. } \check{M}\}$$

Homological mirror symmetry is a homological (or categorical) formulation of mirror symmetry. This claims an equivalence

$$Tr(Fuk(M)) \simeq D^b(coh(\check{M}))$$

(of triangulated categories) where

- $Fuk(M)$ is the Fukaya A_∞ category of Lagrangians in M,
- $D^b(coh(\check{M}))$ is the derived category of coherent sheaves on $\check{M}$,

⋆ Kontsevich-Soibelman'00 's proposal to obtain the equivalence

$$Tr(Fuk(M)) \to D^b(coh(\check{M})) \quad :$$

Apply HPT to a DG category $\mathcal{C}'$,

$$\mathfrak{f} : \mathcal{C} \ \to \ \mathcal{C}' \ ,$$

so that

- $\mathcal{C}'$ generates $D^b(coh(\check{M}))$
- $\mathcal{C}$ is a full subcategory $\mathcal{C} \subset Fuk(M)$.

Reformulated so that we can proceed this idea explicitly (H.K'09,14)

It actually works well for

- $M = \mathbb{R}^2$ (H.K'09)
- $M = T^2$ (H.K'11, 19 preprint)
- $\check{M}$ for some toric Fano : work in progress