Gauge field theory

Deformations, obstructions and higher structure

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Fundamental Aspects of String Theory
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Introduction

In physics, a *field theory* refers to a set-up in which there are ‘fields’, often sections of some bundle.

A *gauge field theory* is a field theory presented as having a group $\mathcal{G}$ of local symmetries.
In physics, ‘local’ means $\mathfrak{g}$ acts as fibre preserving morphisms of the bundle. Think of vertical vector fields as the analogous Lie algebra $\mathfrak{g}$.

The designation *gauge transformation* refers to the infinitesimal action of $\mathfrak{g}$ on the fields.
History of $L_\infty$ in physics

- 1982 D’Auria and Fré
- 1984 Berends, Burgers and van Dam
- 1983 Batalin, Fradkin, Vilkovisky and Batalin, Vilkovisky
- 1989 Zwiebach CSFT
Giovanni Felder remarked: “the $\infty$-virus had a long incubation time and the outbreak came after I left the infection zone.”

Since then, it has expanded to epidemic proportions in the field theoretic physics community, being useful and popular in two major ways:

- Description of a physical problem leads to a structure, later recognized as an $L_{\infty}$-algebra - BBvD’s higher spin algebras, Zwiebach’s CSFT, BFV and BV complexes
- Use of $L_{\infty}$-algebras to better organize a generalized gauge field theory
It has been suggested that every classical perturbative gauge theory, including its dynamics and more, can be organized using an $L_\infty$-structure that encodes the fields, the gauge structure, field equations, Noether identities and deformations of the theory; in short, everything one would like to know about a classical theory.

A significant part of the story is allowing ‘field dependent’ gauge transformations, which goes back to at least BBvD.
$L_\infty$-algebras are a generalization of dg Lie algebras

- a dg vector space $(\mathcal{X} = \bigoplus_m \mathcal{X}_m, d_{\mathcal{X}})$
- a coherent set of generalized brackets

\[
l_n : \mathcal{X} \otimes \mathcal{X} \otimes \cdots \otimes \mathcal{X} \to \mathcal{X}
\]

where $l_1 = d_{\mathcal{X}}$.

Coherent means satisfying generalized Jacobi identities.
For an $L_\infty$ algebra $\mathcal{L}$, the relevant cochain algebra is the same as for a dg Lie algebra $\mathfrak{g}$, the $CE_\infty$-algebra

$$\text{Hom}(\Lambda^\bullet \mathcal{L}, \mathcal{L})$$

but with the differential using all the brackets $\ell_n : \Lambda^n \mathcal{L} \to \mathcal{L}$.

$\Lambda^\bullet \mathcal{L}$ denotes the free graded commutative COalgebra on $\mathcal{L}$. 
Before they were identified in the Chevalley-Eilenberg-algebra, generators were dubbed ‘ghosts’ in physics and the differential was called BRST.

All three, the higher bracket, the dg algebra and the coalgebra descriptions have their individual advantages; it is occasionally very useful to switch between them.

The $CE_\infty$-coalgebra is particularly useful for defining $L_\infty$-morphisms even of strict dg Lie algebras. Example: Kontsevich’s proof of formality.
The Batalin and Vilkovisky complex $\mathcal{BV}$ deals with variational problems such as Euler-Lagrange equations with symmetries/gauge transformations which form a group $\mathfrak{g}$ with infinitesimal Lie algebra $\mathfrak{g}$. 
Start with a Lagrangian $\mathbf{L}$ in terms of fields regarded as sections of some bundle $\rho : E \to M$, its jet extension $JE$ and use $\text{Loc}(E)$, the algebra of Local functions on $E$.

A Lagrangian $\mathbf{L}$ determines a solution surface or shell $\Sigma \subset JE$ such that $\phi$ is a solution of the variational problem iff $j_\infty \phi$ has its image in $\Sigma$.

The corresponding algebra for $\Sigma$ is the quotient $\text{Loc}(E)/\mathcal{I}$ where the stationary ideal $\mathcal{I}$ consists of local functions which vanish ‘on shell’, i.e. when restricted to the solution surface $\Sigma$. 
Extend \( \text{Loc } E \) by adjoining generators of various degrees to form a dgca over \( \text{Loc}(E) \):

the Koszul-Tate resolution \((KT, d_{KT})\) for \( \text{Loc}(E)/\mathcal{I} \).

For the symmetries, similarly adjoin generators’ (called ‘ghosts’) to form the Chevalley-Eilenberg-complex for \( \text{Loc}(E) \) as a module over \( g \) with differential \( d_{CE} \).
Batalin and Vilkovisky ‘walk on 2 feet’, introducing at each stage a new variable $u$ and an ‘anti’ $u^*$, giving rise to the odd Poisson bracket $\{u, u^*\} = 1$.

For example, for each field $A$, an ‘anti-field’ $A^*$ of degree 1.
Unfortunately (or fortuitously?), the total $d_{CE} + d_{KT}$ does not square to 0, because the ‘structure constants’ of $\mathfrak{g}$ are now structure functions.

The brilliance of Batalin and Vilkovisky and of Batalin, Fradkin and Vilkovisky was to add a sequence of terms of higher order $d_1, d_2, \ldots$ to achieve $D^2 = 0$ for $D = d_{CE} + d_{KT} + d_1 + d_2 + \cdots$ so as to kill the discrepancy step by step.
Organizing Principle

Perturbative gauge theory, including its dynamics, can be organized using an $L_\infty$-structure.

A significant part of the story is allowing field dependent gauge transformations, going back to at least BBvD.
Hohm and Zwiebach began investigating the $L_\infty$-structure of general gauge invariant perturbative field theories. They focused on the part of the theory dealing with gauge parameters, gauge fields and “equations of motion” organized as a dg graded vector space $\mathfrak{X}$:

$$\cdots \to \mathfrak{X}_0 \to \mathfrak{X}_1 \to \mathfrak{X}_2 \to \cdots$$

$\lambda \quad A \quad E$

The differential $\ell_1 = d_\mathfrak{X}$ and is theory dependent.
They emphasized the terms of higher order in $A$

the gauge transformations and gauge algebra:

$$\delta_\lambda(A) = \ell_1(\lambda) - \frac{1}{2!} \ell_2(\lambda, A) - \frac{1}{3!} \ell_2(\lambda, A, A) + \cdots$$

$$[\delta_\lambda, \delta_\mu](A) = \ell_2(\lambda, \mu) - \frac{1}{3!} \ell_3(\lambda, \mu, A) + \frac{1}{4!} \ell_4(\lambda, \mu, A, A) + \cdots$$

*Exactly the BBvD pattern*

and adding the field equations:

$$\ell_1(A) + \frac{1}{2!} \ell_2(A, A) - \frac{1}{3!} \ell_2(A, A, A) + \cdots = 0$$
Jurco, Raspollini, Sämann and Wolf and their team extend $\mathcal{X}$ to

\[ \begin{align*}
\text{gauge symmetries} & \quad \rightarrow \\
\mathcal{X}_0 & \quad \rightarrow \\
\text{classical fields} & \quad \rightarrow \\
\mathcal{X}_1 & \quad \rightarrow \\
\text{equations of motion} & \quad \rightarrow \\
\mathcal{X}_2 & \quad \rightarrow \\
\text{Noether identities} & \quad \rightarrow \\
\mathcal{X}_3 & \quad \rightarrow \\
\end{align*} \]

The classical BV complex $\mathcal{B}\mathcal{V}$ provides precisely such a differential graded algebra.
Additional structures

Add a suitable inner product $\langle \ , \ \rangle$ to define an action principle.

An $L_\infty$-algebra is called \textit{cyclic}

if $\langle a_0, \ l_n(a_1, \cdots, a_n) \rangle = \langle a_n, \ l_n(a_0, \cdots, a_{n-1}) \rangle$

Čirić, Giotopoulos, Radonović and Szabó have added another ‘twist’, a Drinfel’d-type \textit{braiding}.
Bootstrapping is a process which starts with a known theory and then from initial data \( \ell_1, \ell_2 \) creates a deformation as an \( L_\infty \)-algebra by inductively solving the \( L_\infty \)-relations to determine the rest of the \( L_\infty \)-structure.

For example, initiate a deformation using the star commutator = Poisson bracket. 

A key observation is the use of field dependent gauge transformations as well as an assumption that guarantees “closed implies exact”.

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*Hierarchy* refers to a structure much like a Postnikov tower or a Sullivan model.

*Tensor hierarchies* form a class of objects found in supergravity theories, arising from Leibniz algebras as gauge algebras.

Recall a Leibniz algebra \((V, \circ)\) is a vector space \(V\) with a binary operation

\[
\circ : V \otimes V \to V
\]

satisfying the (left) derivation rule:

\[
x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z).
\]
A Leibniz algebra gives rise to a tensor hierarchy and an $L_\infty$-algebra by lifting the skew-symmetric part of the Leibniz product, successively adjoining additional variables to compensate for the failure of $\circ$ to satisfy Jacobi.
$L_\infty$-connections and twisted tensor products

Maurer-Cartan conditions:
- The integrability condition for deformation theory
- The master equation for field theories
- Flatness of a connection
- Definition of a *twisting cochain*

The $L_\infty$ version (up to signs and rational fudge factors):

$$\sum \ell_n(a, \cdots, a) = 0.$$
The algebraic model of a fibration $F \to E \overset{p}{\to} B$ is a twisted tensor product $(B \otimes F, D)$ where the differential $D$ is twisted:

$$d = d_B \otimes 1 + 1 \otimes d_F + \tau;$$

the “twisting term” $\tau$ is a differential and a perturbation with respect to filtration by the $F$ degree i.e.

$$\tau : B^p \otimes F^q \subset \sum_{r < q} B \otimes F^r$$
Bonezzi and Holm have an example of such a twisted tensor product (though not so named):

\[ \Omega^\bullet(M) \otimes \mathcal{L} \] with an \( L_\infty \)-algebra \( \mathcal{L} \) as fibre.

A connection \( A \) or covariant derivative \( D_A \) with values in \( \mathcal{L} \).

If it is not flat, they then ‘flatten the curvature’ by successively using the higher brackets.
Following the work of Gerstenhaber introducing algebraic deformation theory, it was applied to quantization by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer.

A paradigm shift: Out with Hilbert space physics; in with cohomological physics.

Most recently, deformation theory appeared in the context of cohomological field theory, but there have been several other appearances in the meantime.
Gerstenhaber’s Algebraic Deformation Theory:

An associative $k$-algebra is a pair $(A, m)$ where $A$ is a $k$-vector space and $m : A \otimes A \to A$ is a bilinear map satisfying the associativity condition.

A first-order (or \textit{infinitesimal}) deformation of $m$ is given by $m_1 : A \otimes A \to A$ such that $m + t m_1 : A[t] \otimes A[t] \to A[t]$ is associative modulo $t^2$.

The associativity implies that $m_1$ determines a class $[m_1] = m_1$ in what is known as the Hochschild cohomology $HH^2(A)$. 
A formal deformation of $m$ (over $k[[t]]$) is given by a formal power series

$$m_t = m + \sum t^n m_n$$

satisfying associativity.

The full deformation theory of associative algebras can be expressed conveniently in terms of ‘higher structure’ on the Hochschild cochain complex.
Obstructions

Given a choice of $m_1$, there may or may not exist a suitable $m_2$.

Gerstenhaber’s eponymous bracket:

$$\{ , \} : HH^2 \otimes HH^2 \rightarrow HH^3$$

behaves like the Schouten-Nijenhuis bracket of multi-vector fields.
The primary obstruction to extending $m + t m_1$ to order $t^2$ is given by the Gerstenhaber bracket.

There exists $m_2$ such that $m + tm_1 + t^2m_2$ is associative mod $t^3$ if and only if $\{m_1, m_1\} = 0$. 
If the primary obstruction vanishes, there is a secondary obstruction which lies in a quotient of $HH^3(A)$.

In the case of complex structures, this was first worked out by Douady, who noticed the relation to a Massey product.
Work in progress with Vladislav Kupriyanov

For a Poisson manifold, \{ \ , \ \} can be constructed from a bi-vector \( \Theta \) by the usual local coordinate formula or as a derived bracket:

\[
\{ f, g \} = [[\Theta, f]_{SN}, g]_{SN}.
\]

The key requirement is that the Schouten-Nijenhuis bracket

\[
[\Theta, \Theta]_{SN} = 0.
\]
Kupriyanov introduced the notion of quasi-Poisson bracket, given by an arbitrary bi-vector $\Theta$.

In general, $[\Theta, \Theta]_{SN}$ will not vanish.

We seek to deform $\Theta$ to $\Theta_\infty$ so that $[\Theta_\infty, \Theta_\infty]_{SN} = 0$
Our approach is to follow the BFV procedure:

adjoin auxiliary variables and construct a symplectic embedding of the given quasi-Poisson structure in the extended space.

introduce Hamiltonian constraints and perform homological reduction to achieve the “gauge closure” condition:

the commutator of two field dependent gauge transformations is again such a field dependent gauge transformation.
George (Yuri) Rainich’s advice

In 1923, Rainich wrote:

As to the method of the study it seemed to me better
to avoid, as far as possible, the introduction of things
which have no intrinsic meaning, such as coordinates,
the g’s, the three-indices symbols,...

Rainich tried again in 1950

to introduce the idea of the tensor itself and to consider
the components as something secondary.