COSMOLOGICAL BACKGROUNDS of GRAVITATIONAL WAVES

DANIEL G. FIGUEROA
IFIC, Valencia, Spain

II Joint ICTP-Trieste/ICTP-SAIFR School on Particle Physics (June 22 - July 3 2020)

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June 22 – July 3, 2020

São Paulo, Brazil

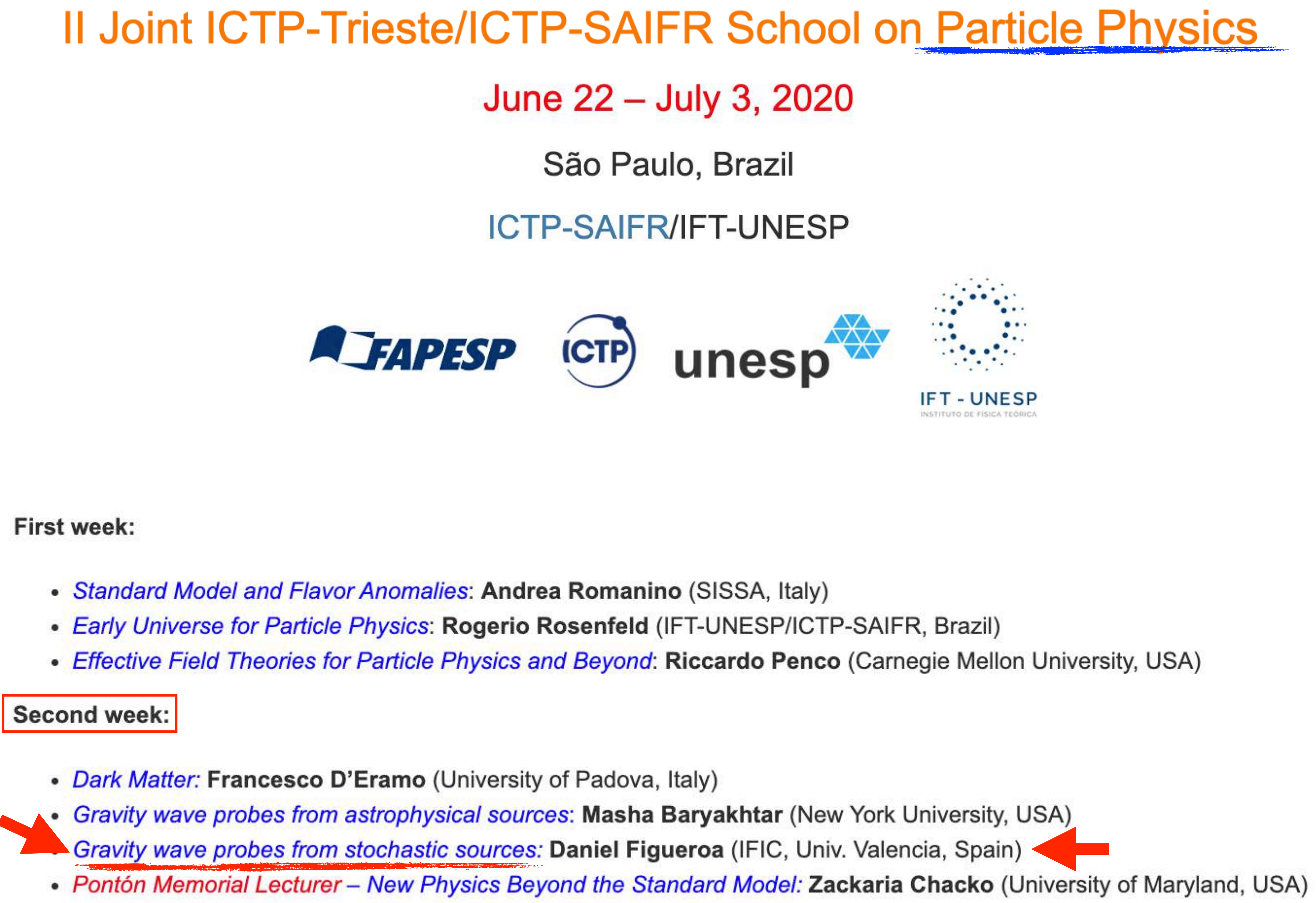
ICTP-SAIFR/IFT-UNESP

First week:

- *Standard Model and Flavor Anomalies*: **Andrea Romanino** (SISSA, Italy)
- *Early Universe for Particle Physics*: **Rogerio Rosenfeld** (IFT-UNESP/ICTP-SAIFR, Brazil)
- *Effective Field Theories for Particle Physics and Beyond*: **Riccardo Penco** (Carnegie Mellon University, USA)

Second week:

- *Dark Matter:* **Francesco D’Eramo** (University of Padova, Italy)
- *Gravity wave probes from astrophysical sources*: **Masha Baryakhtar** (New York University, USA)
- *Gravity wave probes from stochastic sources:* **Daniel Figueroa** (IFIC, Univ. Valencia, Spain)
- *Pontón Memorial Lecturer – New Physics Beyond the Standard Model:* **Zackaria Chacko** (University of Maryland, USA)



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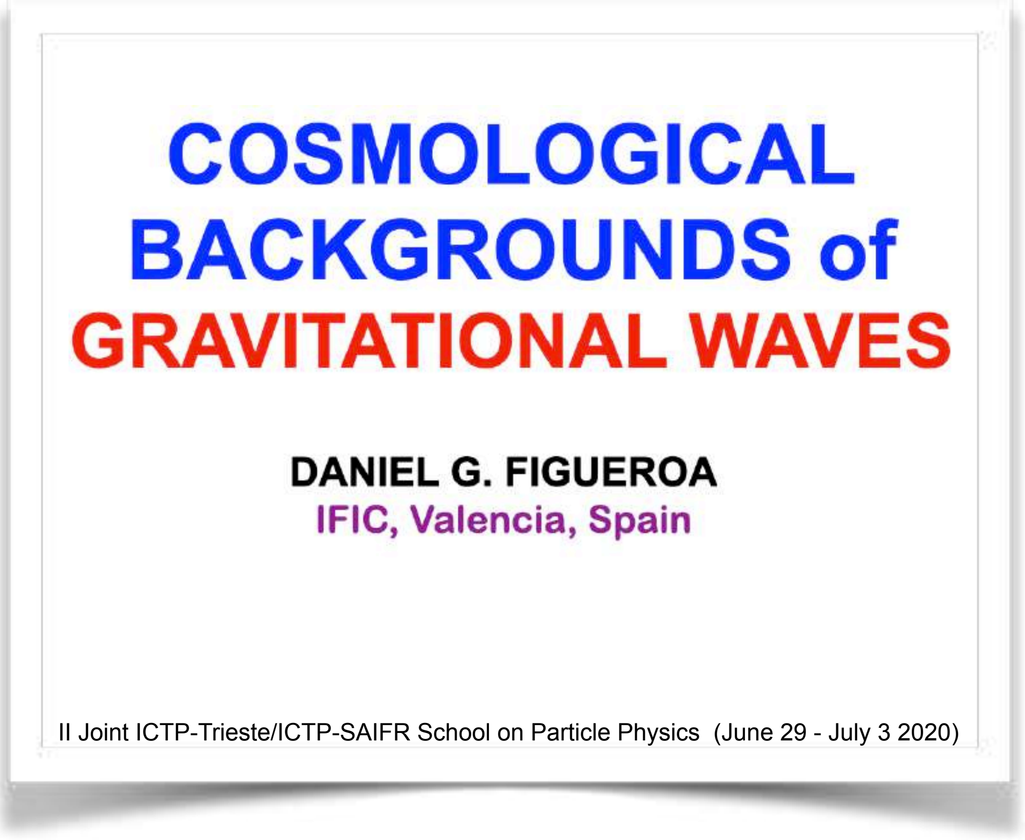

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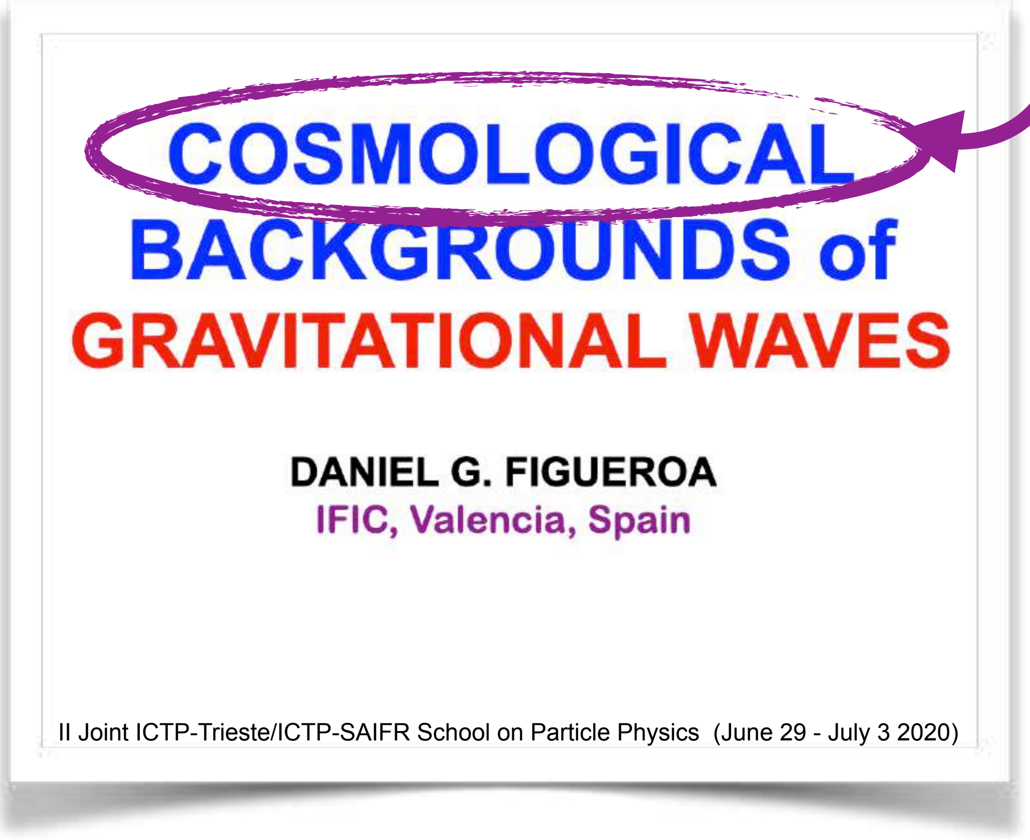

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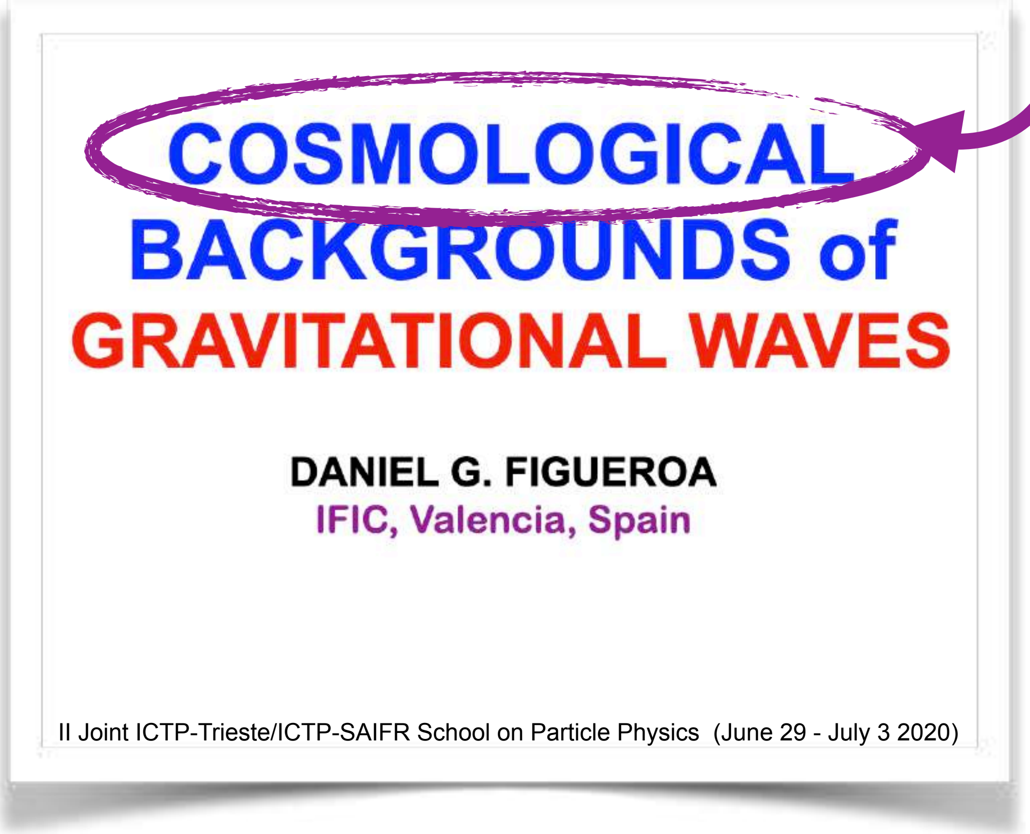

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?

COSMOLOGICAL BACKGROUNDS

$$\equiv$$

STOCHASTIC BACKGROUNDS

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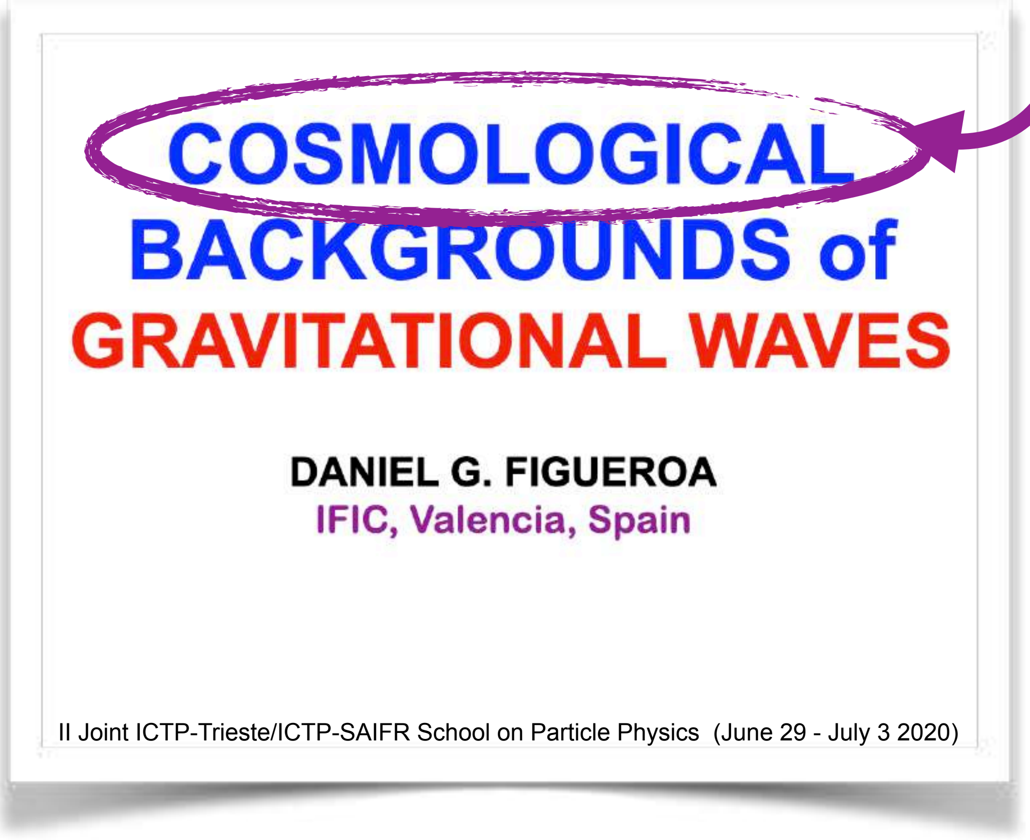

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?

COSMOLOGICAL BACKGROUNDS

≡

STOCHASTIC BACKGROUNDS

GW = Gravitational Waves

II Joint ICTP-Trieste/ICTP-SAIFR School on Particle Physics

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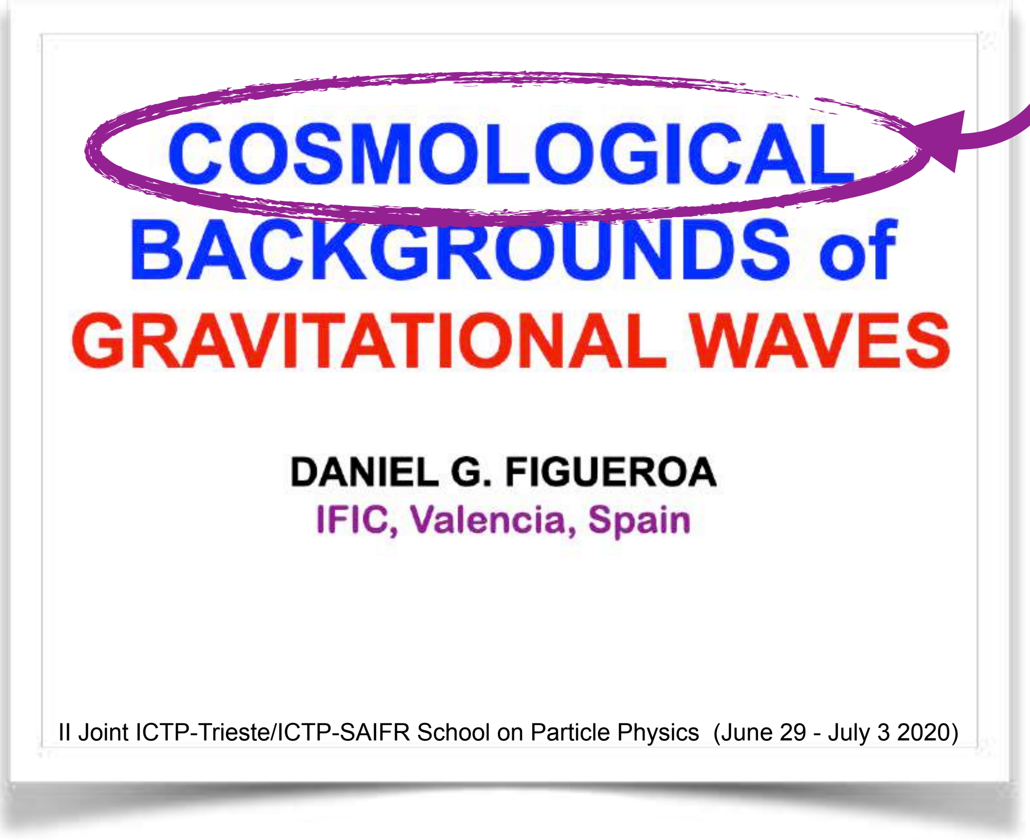

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?

COSMOLOGICAL BACKGROUNDS

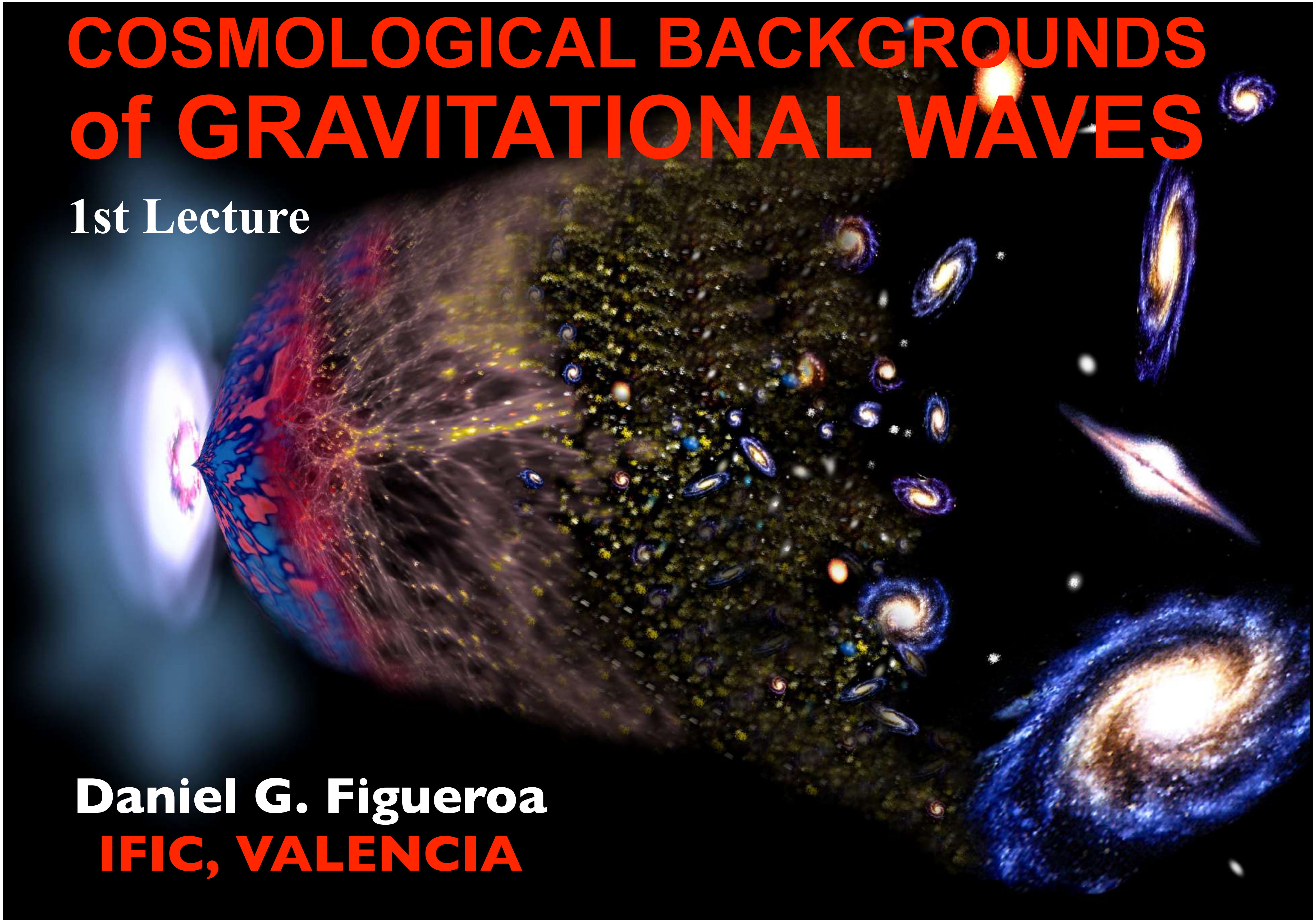
$$\equiv$$

STOCHASTIC BACKGROUNDS

GW: COSMOLOGICAL, ergo STOCHASTIC

COSMOLOGICAL BACKGROUNDS of GRAVITATIONAL WAVES

1st Lecture



Daniel G. Figueroa

IFIC, VALENCIA

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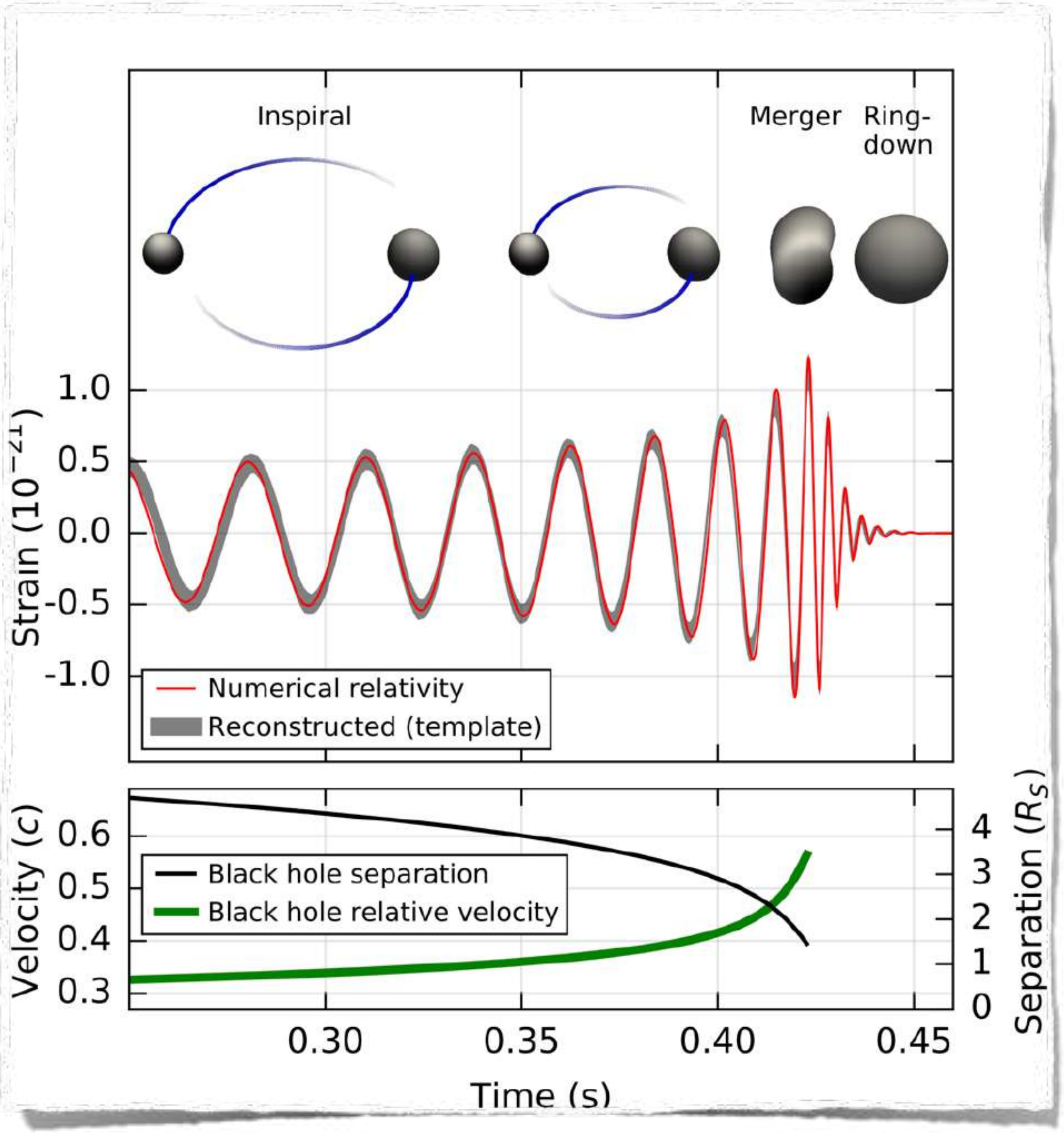
MOTIVATION

(cosmologist biased)

Let us celebrate …

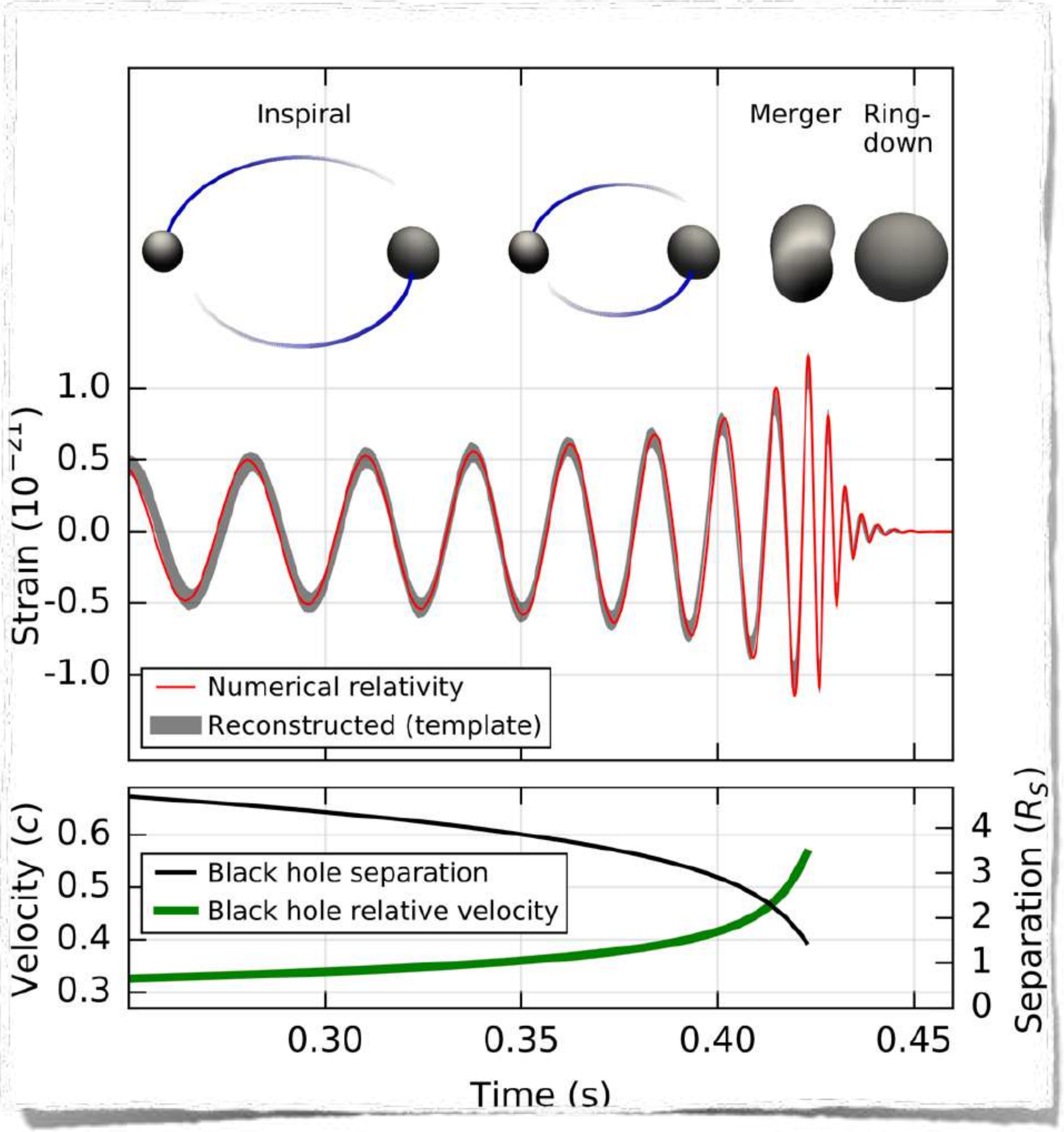


[LIGO & Virgo Scientific Collaborations (arXiv:1602.03841)]

Gravitational Waves (GWs) detected !

[by LIGO/VIRGO]

Let us celebrate …



[LIGO & Virgo Scientific Collaborations (arXiv:1602.03841)]

Gravit al

)

[b VIRGO]

milestone in physics

Einstein 1916 … LIGO/VIRGO 2015/16/17

Let us celebrate …

* O(10) Solar mass
Black Holes (BH) exist

* We can test the
BH's paradigm and
Neutron Star physics

* We can further test
General Relativity (GR)
[so far no deviation]

* We can observe the
Universe through GWs

* …

Grav al
)

[b O/VIRGO]

milestone
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* We can observe the Universe through GWs

* …

→ (binaries)

Extremely interesting !

BUT …

… We will focus on something else !

Let us celebrate …

* O(10) Solar m s
Black H st

N (binaries)

* W test
Gen lativity (GR)
[so far no deviation]

* We can observe the
Universe through GWs

* …

Stay tuned !

more fun
guaranteed
to come …

Extremely
interesting !

BUT …

… We will focus
on something else !

Let us celebrate …

* O(10) Sc s
Blac st

N (binaries)

* W
Gen ity (GR)
[sc no deviation]

* We can observe the
Universe through GWs

* …

See
Lectures
by Masha
Baryakhtar,
Francesco
D'Eramo

Extremely
interesting !

BUT …

… We will focus
on something else !

Let us celebrate …

* O(10) Solar mass Black Holes (BH) exist

* We can test the BH's paradigm and Neutron Star physics

* We can further test General Relativity (GR)
[so far no deviation]

* We can observe the Universe through GWs

* …

(binaries)

Extremely interesting !

BUT …

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* We can observe the Universe through GWs

*** We can observe the Universe through GWs**

* We can observe the Universe through GWs

* Cosmology with GWs

* We can observe the Universe through GWs

* **Cosmology** with **GWs**

* Late Universe:

* Early Universe:

*** We can observe the Universe through GWs**

* Cosmology with GWs

*** Late Universe:**

Standard sirens: distances in cosmology;
Measuring H0 and EoS dark energy;
cosmological parameters;
modify gravity, lensing, …

*** We can observe the Universe through GWs**

* Cosmology with GWs

- **Late Universe:**

- **Early Universe:** High Energy Particle Physics

*** We can observe the Universe through GWs**

* Cosmology with GWs

*** Late Universe:**

*** Early Universe:** High Energy Particle Physics

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*** We can observe the Universe through GWs**

* Cosmology with GWs

*** Late Universe:**

*** Early Universe:** High Energy Particle Physics

*** We can observe the Universe through GWs**

* Cosmology with GWs

- **Late Universe:**

- **Early Universe:** High Energy Particle Physics

Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?

GWs: probe of the early Universe

Why ?

One and ONLY One reason …

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

2. **ADVANTAGE**: GW → Probe for Early Universe

$$\rightarrow \left\{ \begin{array}{l} \textbf{Decouple} \rightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\Leftrightarrow\; \text{Specific GW} \end{array} \right.$$

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

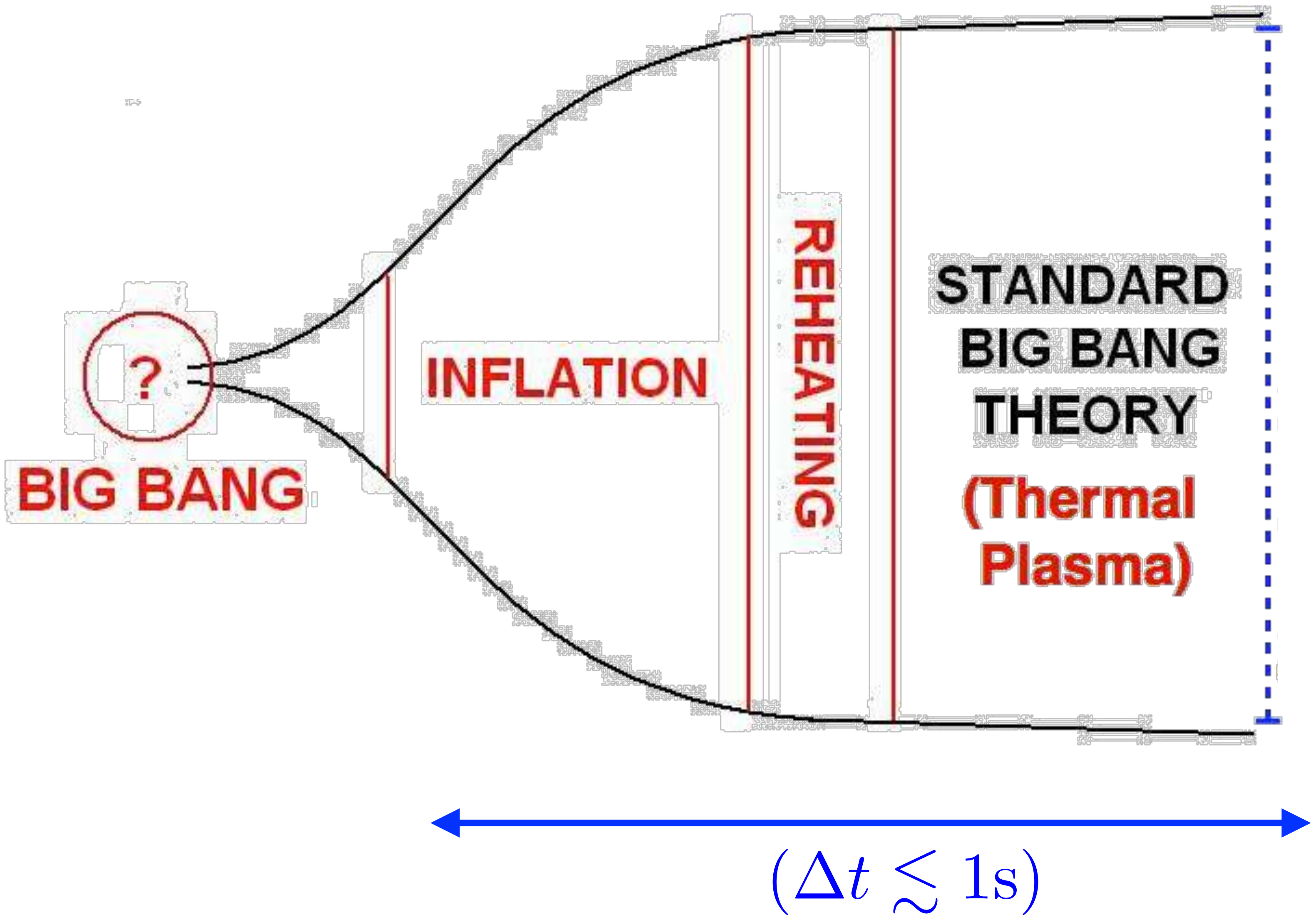
   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

2. **ADVANTAGE**: GW → Probe for Early Universe

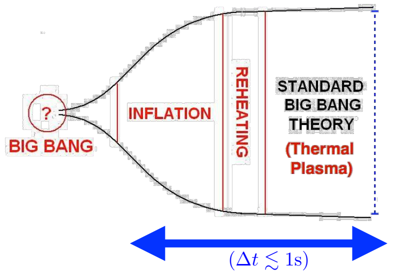
$$\rightarrow \begin{cases} \textbf{Decouple} \rightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\Leftrightarrow\; \text{Specific GW} \end{cases}$$

What processes of the early Universe ?

The Early Universe



The Early Universe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BIG BANG

INFLATION

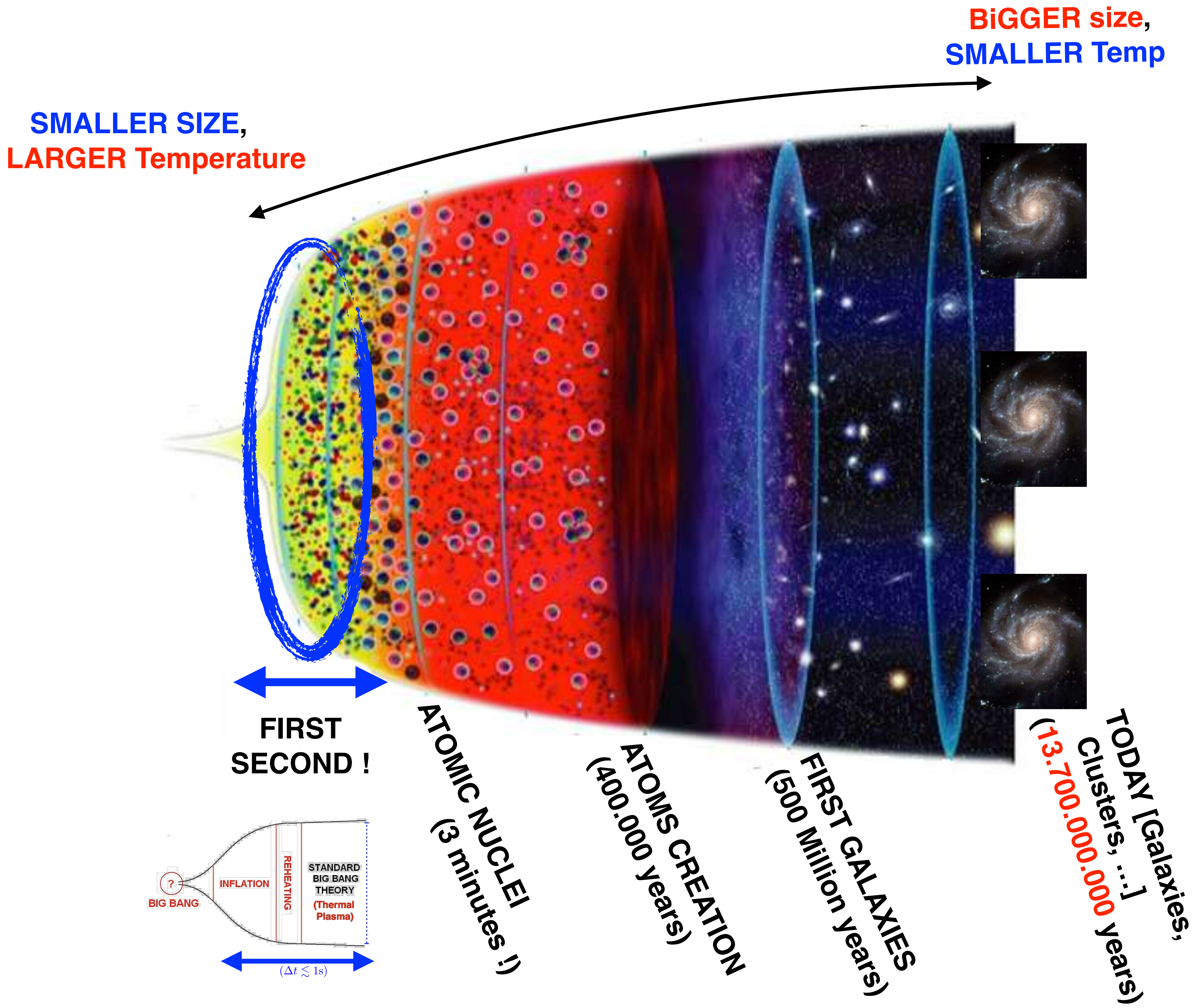
REHEATING

STANDARD BIG BANG THEORY

(Thermal Plasma)

$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



BiGGER size,
SMALLER Temp

SMALLER SIZE,
LARGER Temperature

FIRST SECOND !

ATOMIC NUCLEI (3 minutes !)

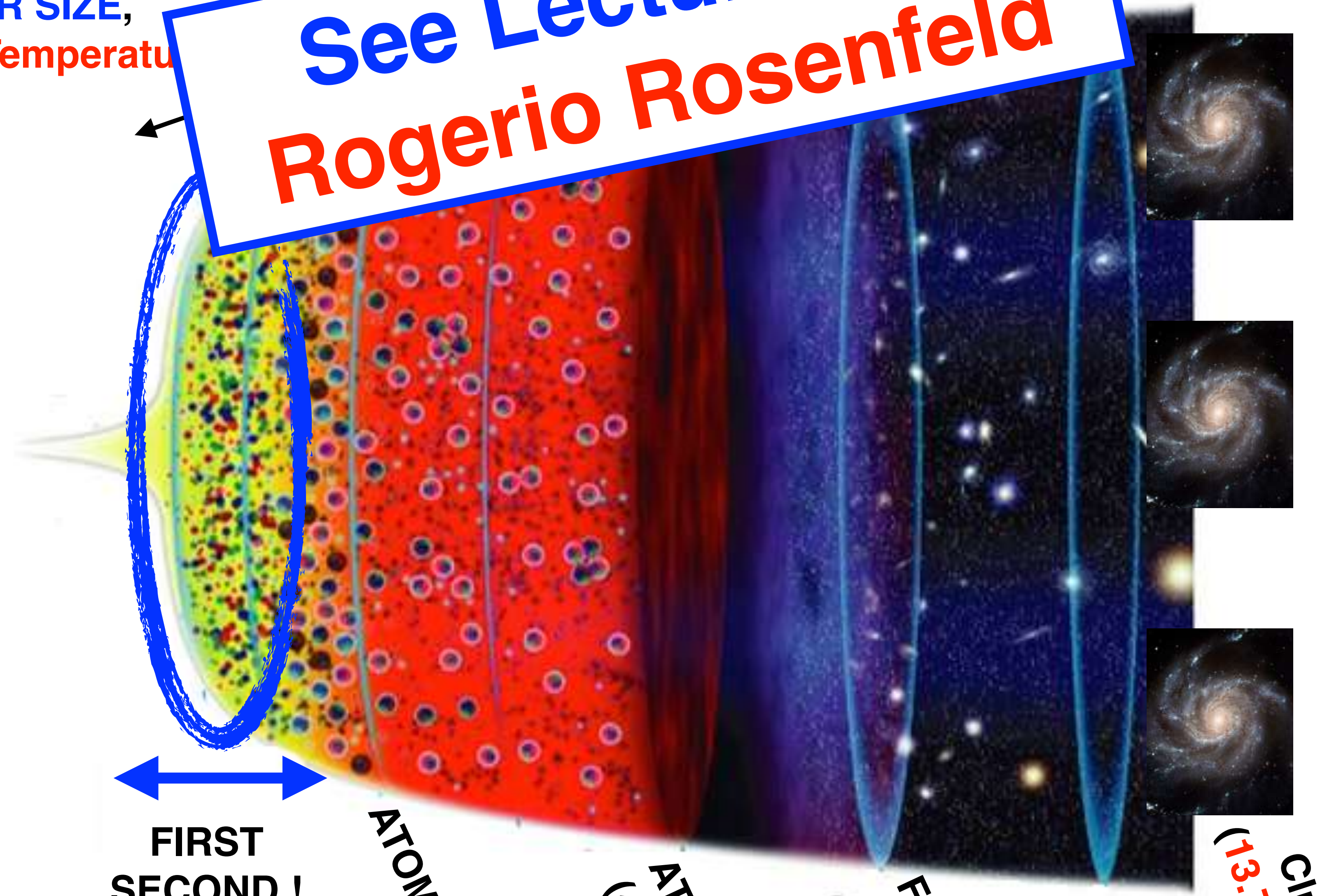
ATOMS CREATION (400.000 years)

FIRST GALAXIES (500 Million years)

TODAY [Galaxies, Clusters, ...] (13.700.000.000 years)

The Early Universe

See Lectures by
Rogerio Rosenfeld



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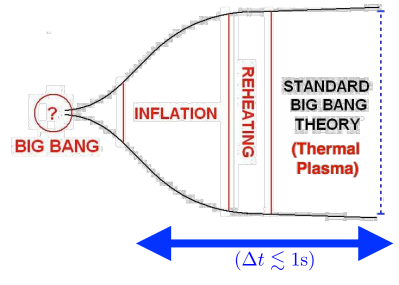
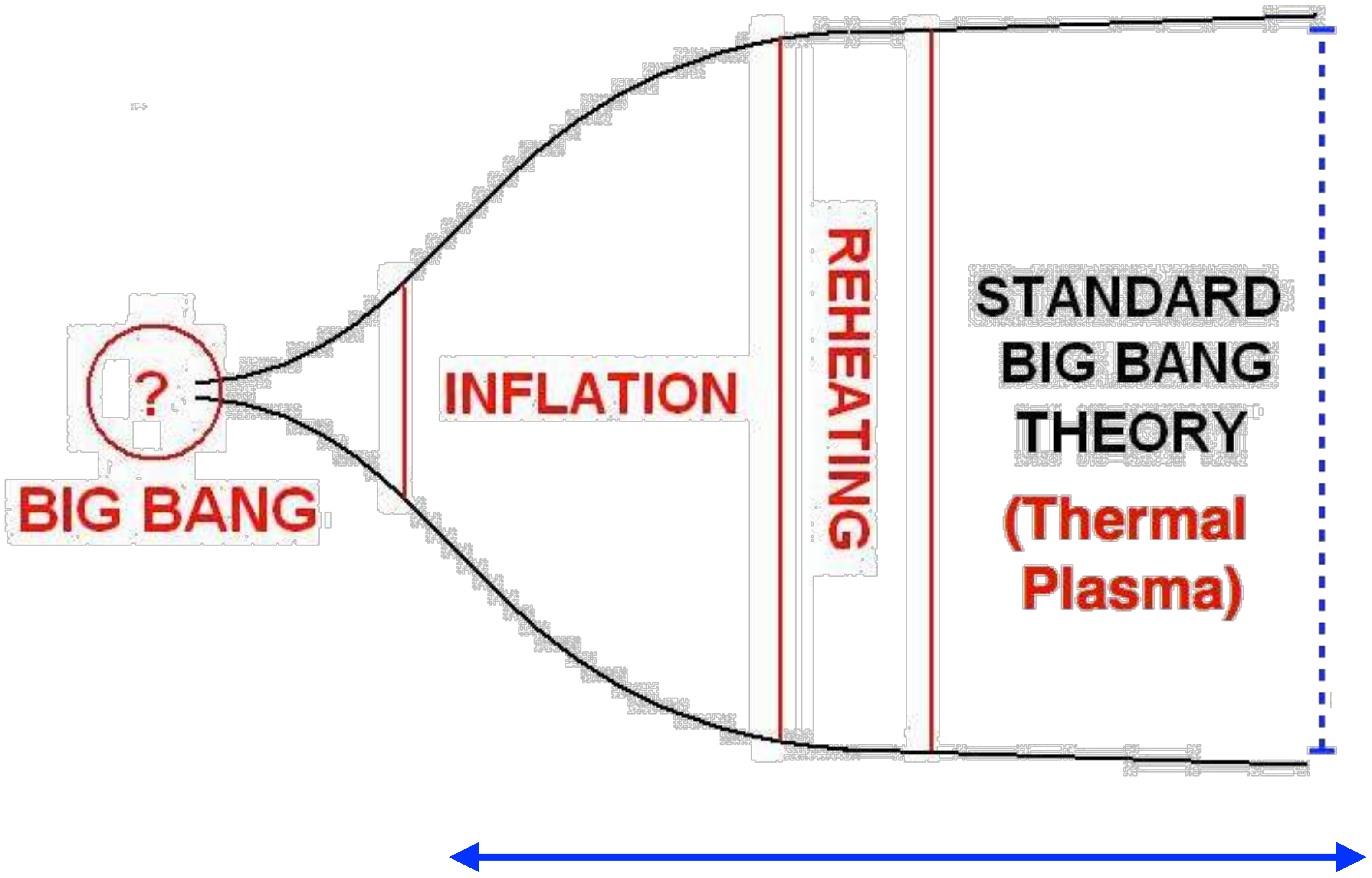
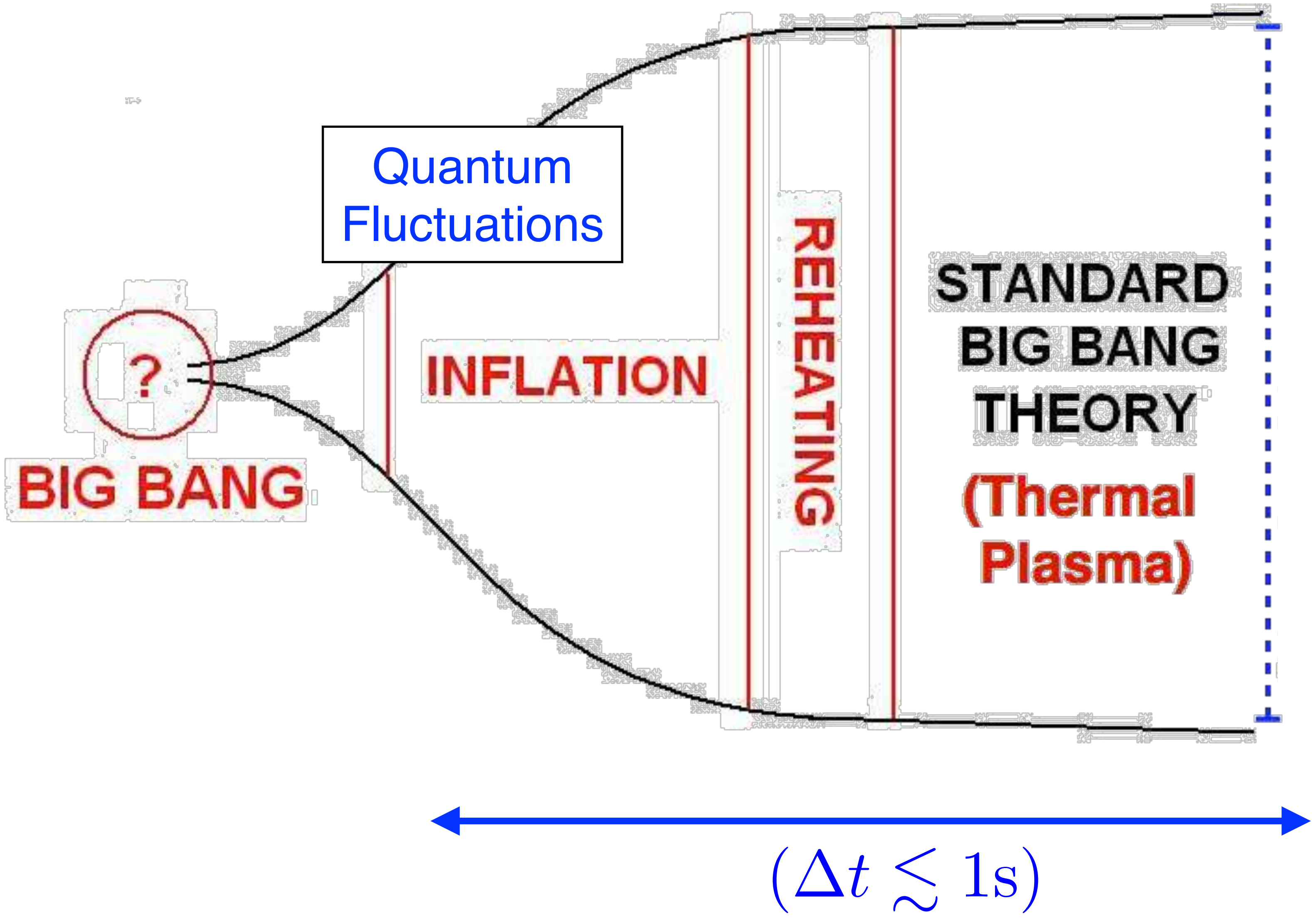


The Early Universe



$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



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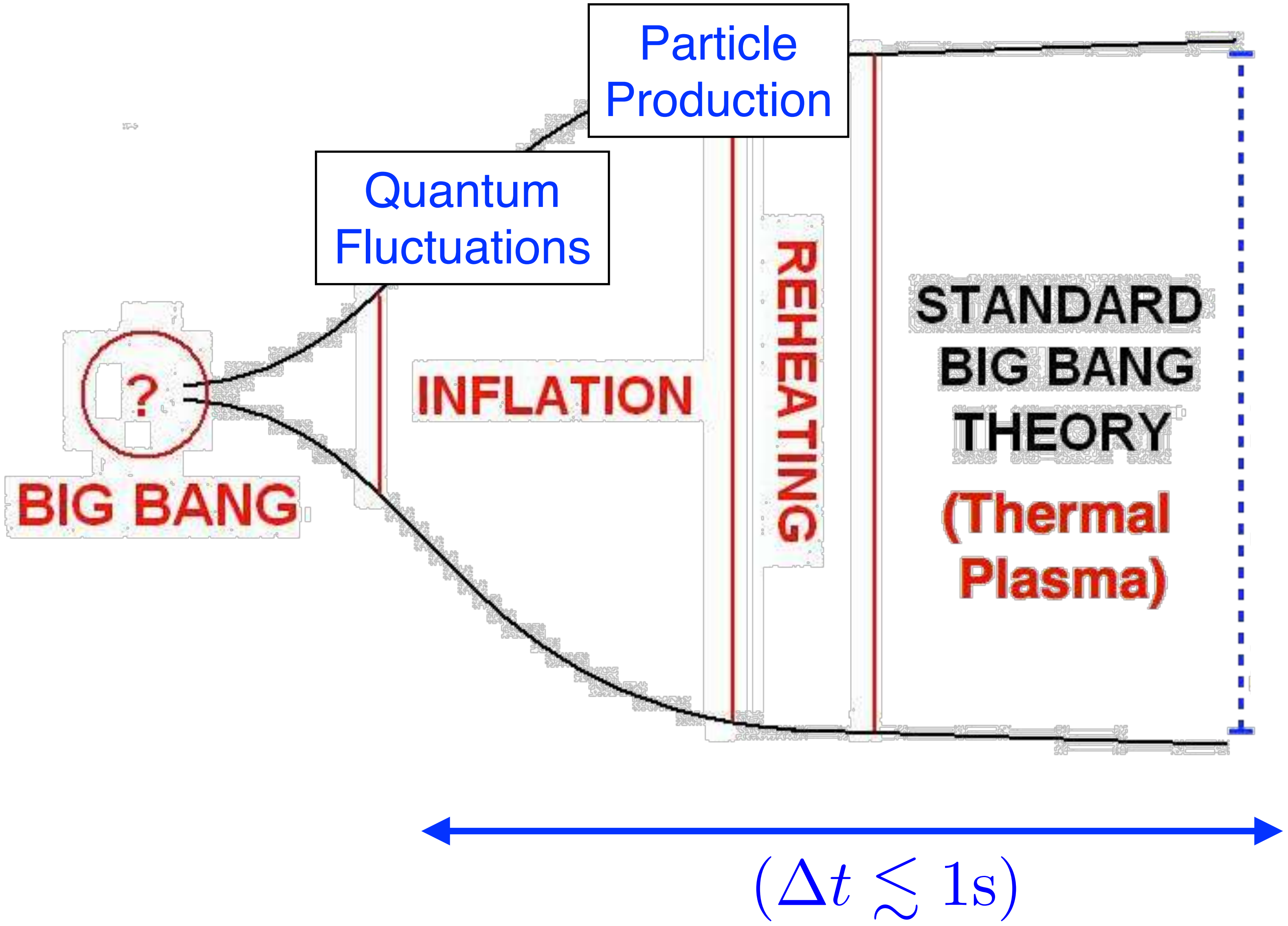
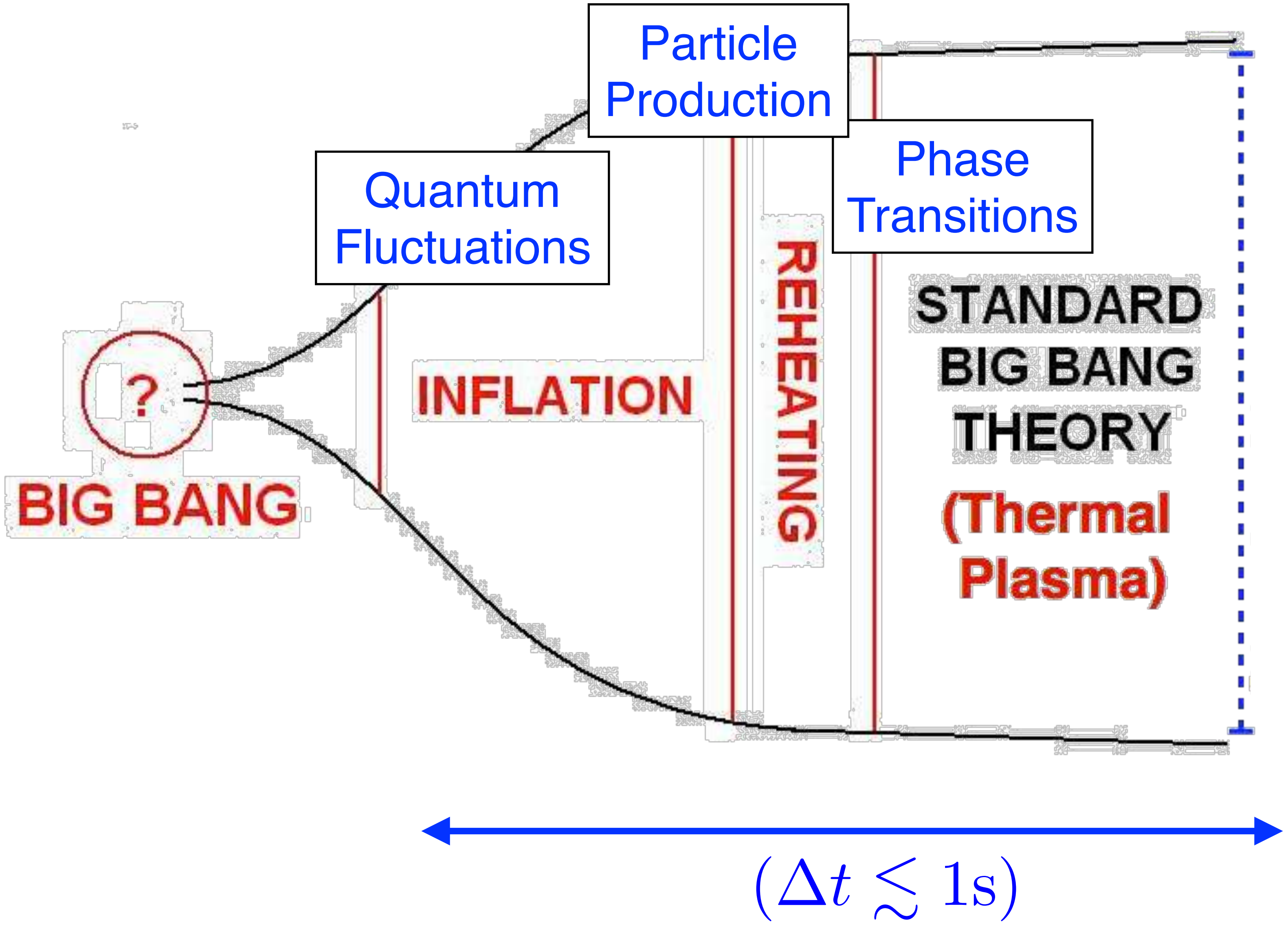


Particle Production

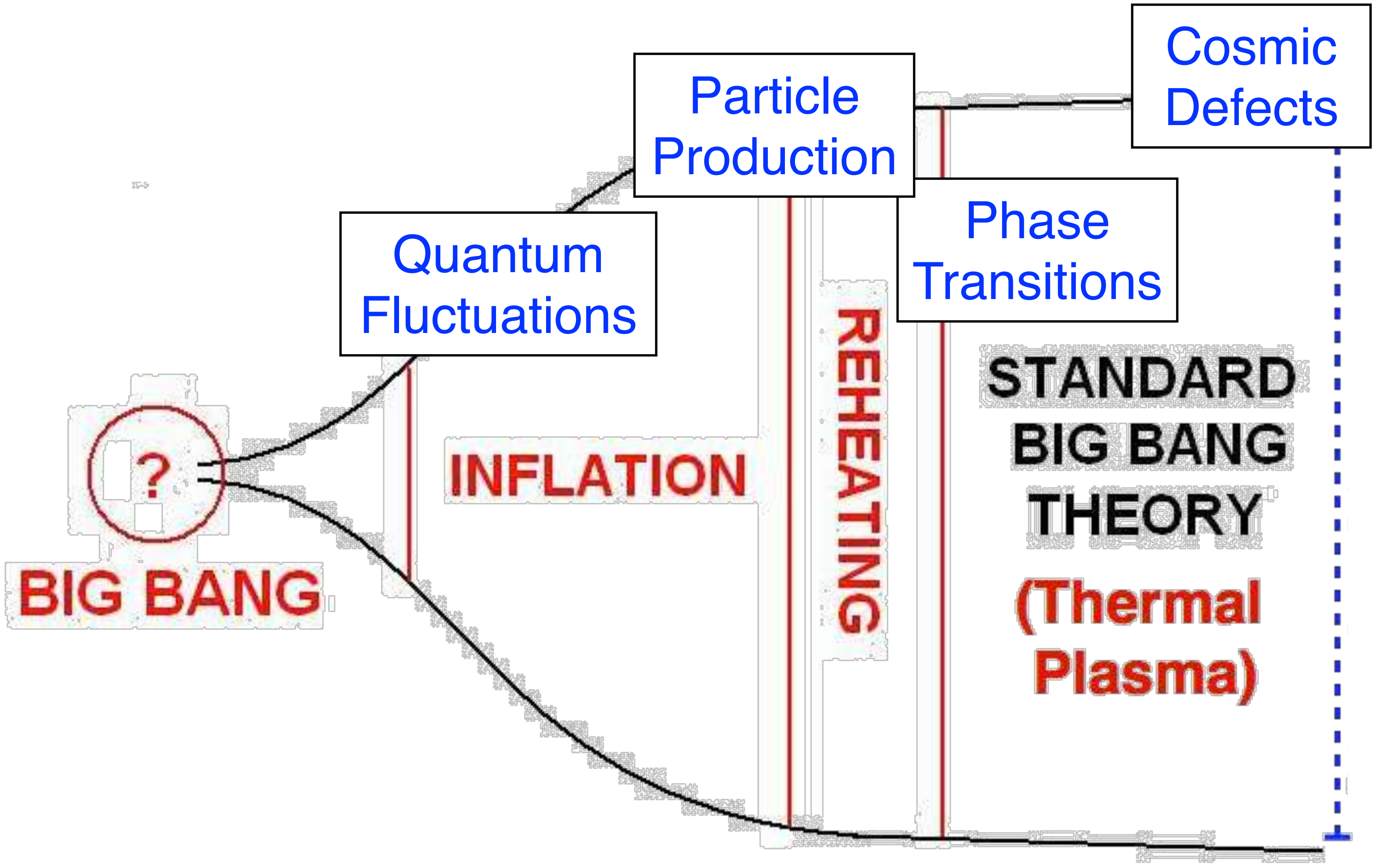
Quantum Fluctuations

$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe

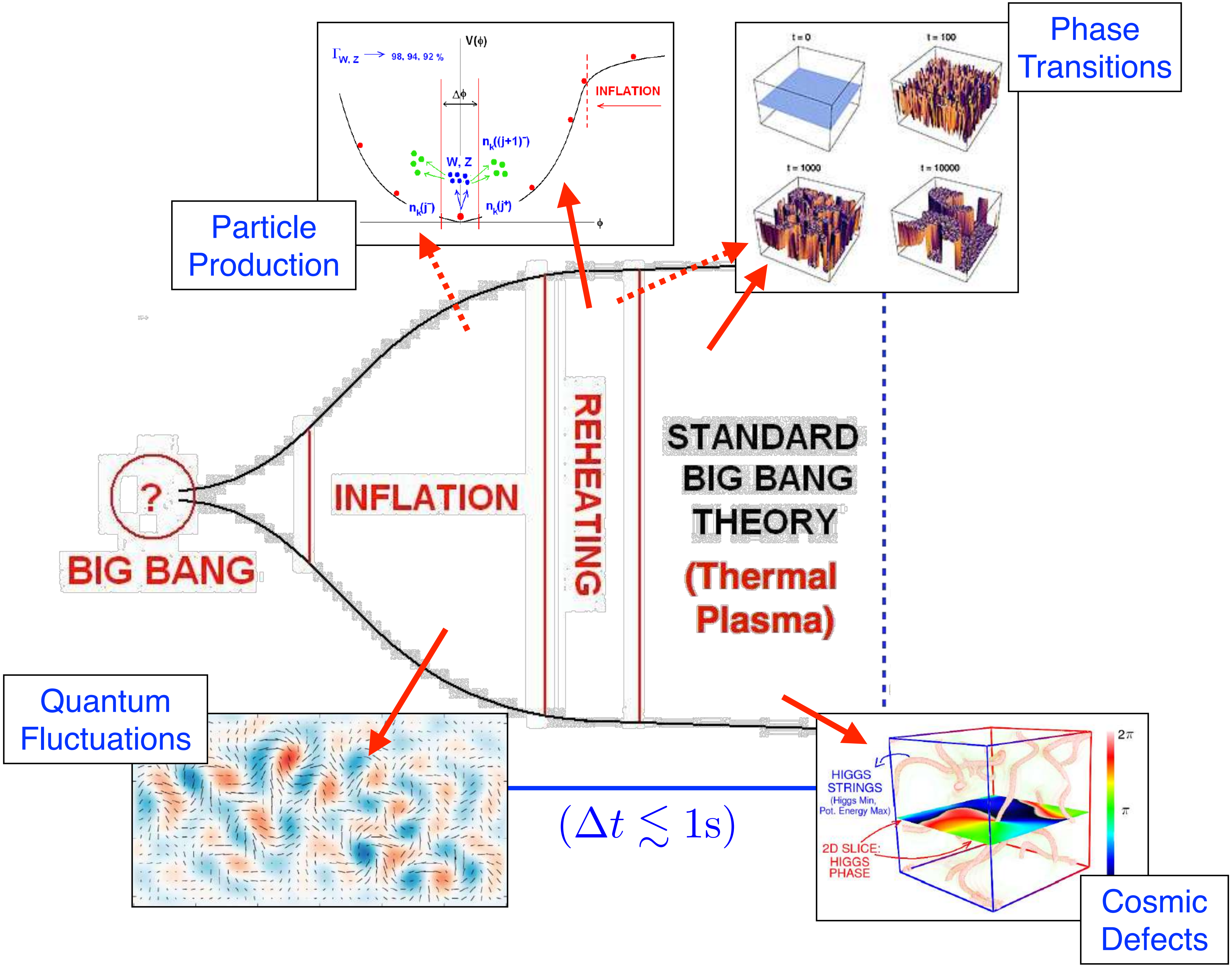


The Early Universe



$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



Particle Production

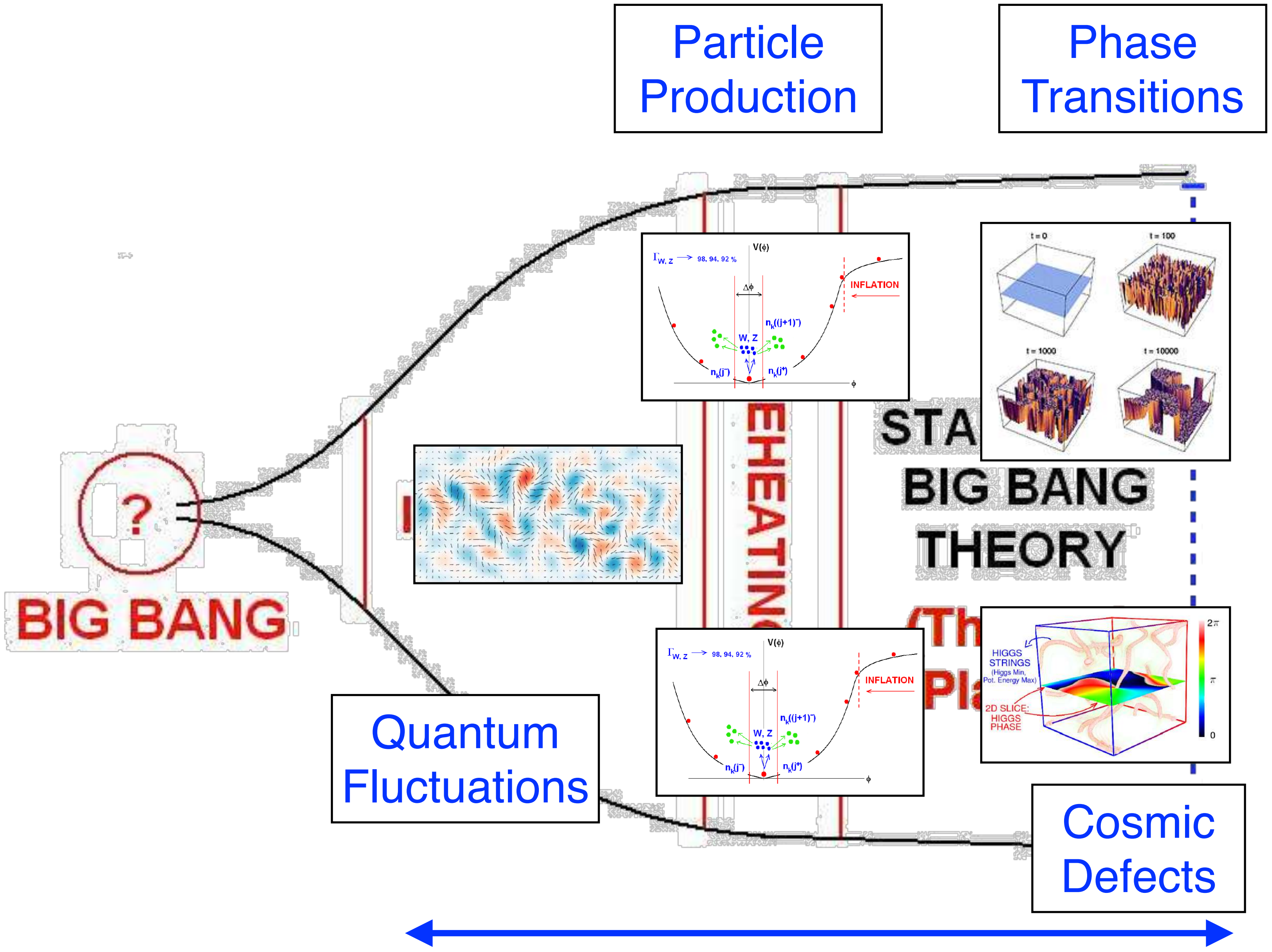
Phase Transitions

Quantum Fluctuations

Cosmic Defects

$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



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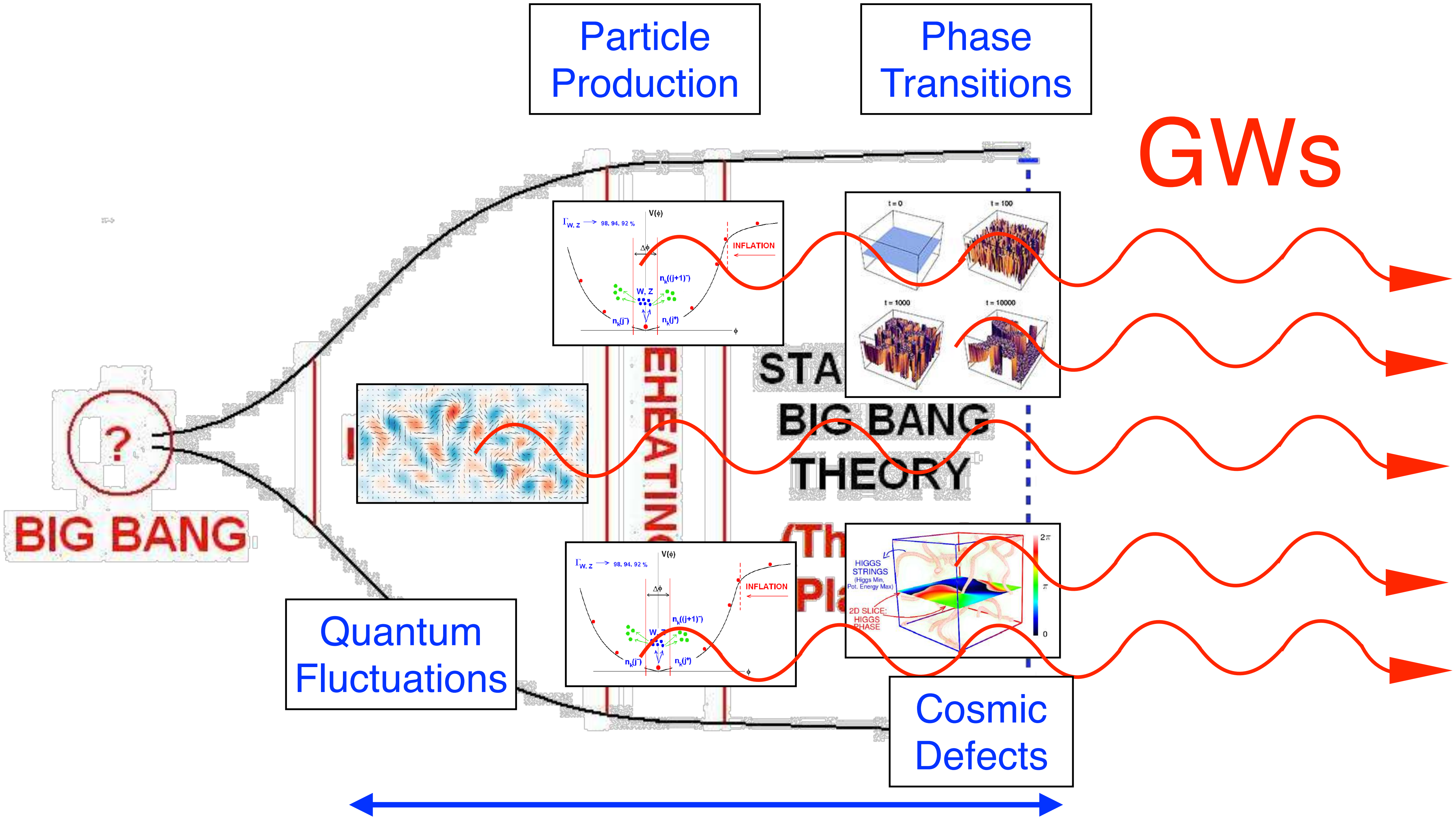

Particle Production

Phase Transitions

GWs

Quantum Fluctuations

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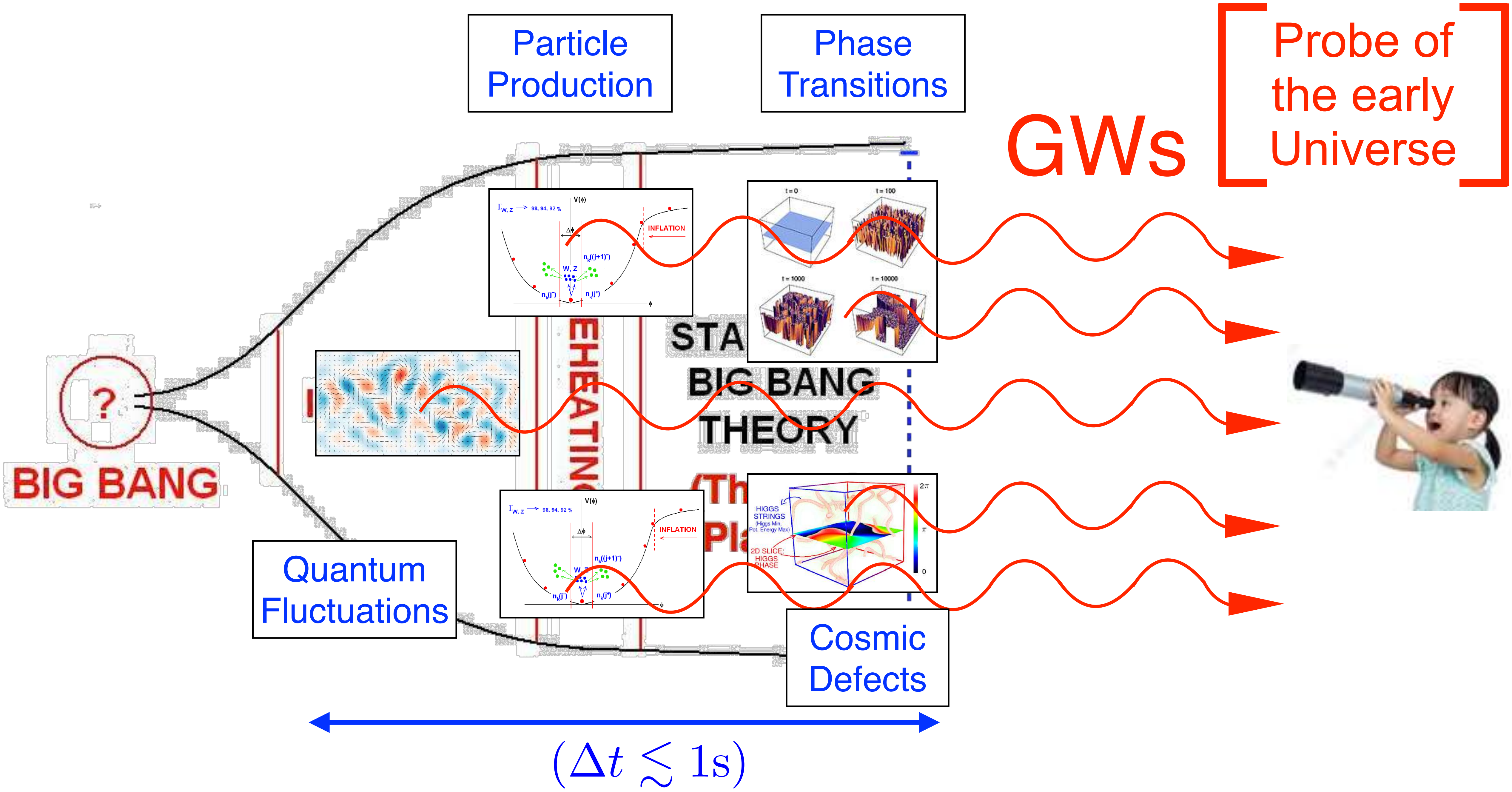
The Early Universe



Particle Production

Phase Transitions

GWs

Probe of the early Universe

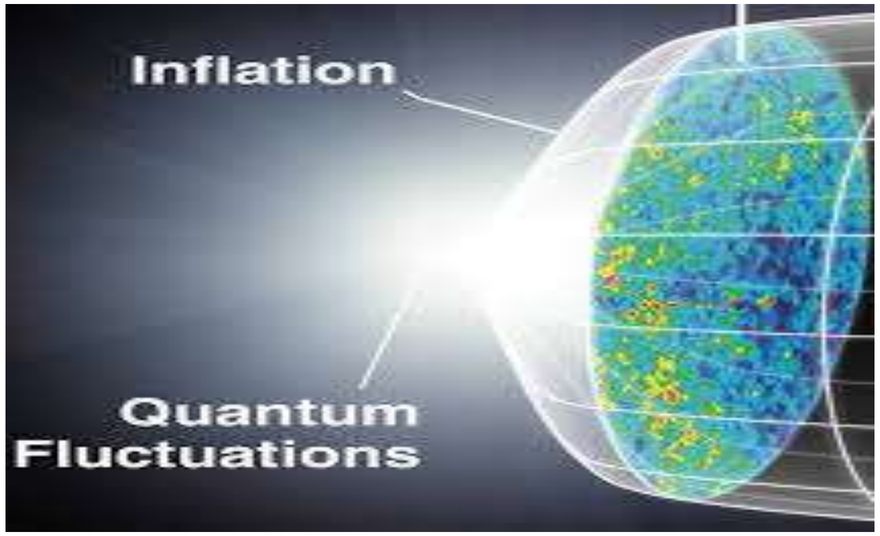
Quantum Fluctuations

Cosmic Defects

$(\Delta t \lesssim 1\mathrm{s})$

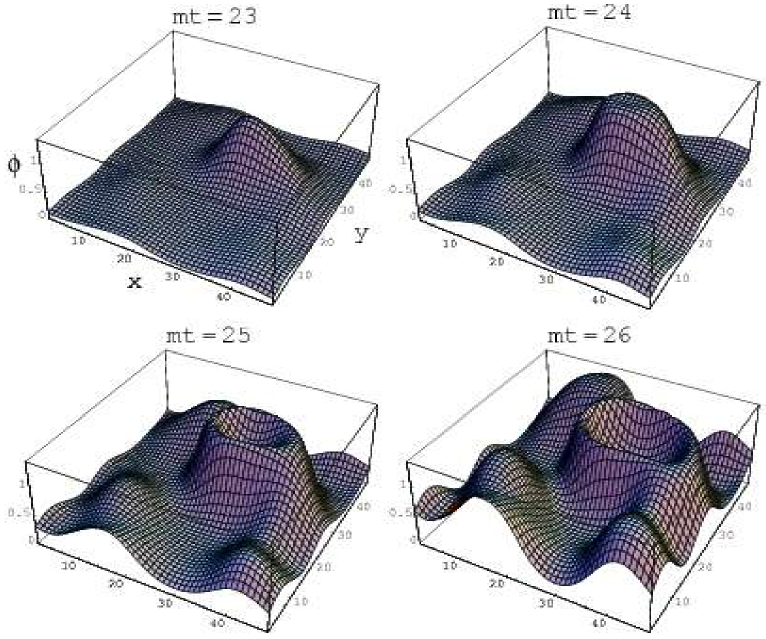
COSMOLOGICAL GRAVITATIONAL WAVES

Inflationary Period



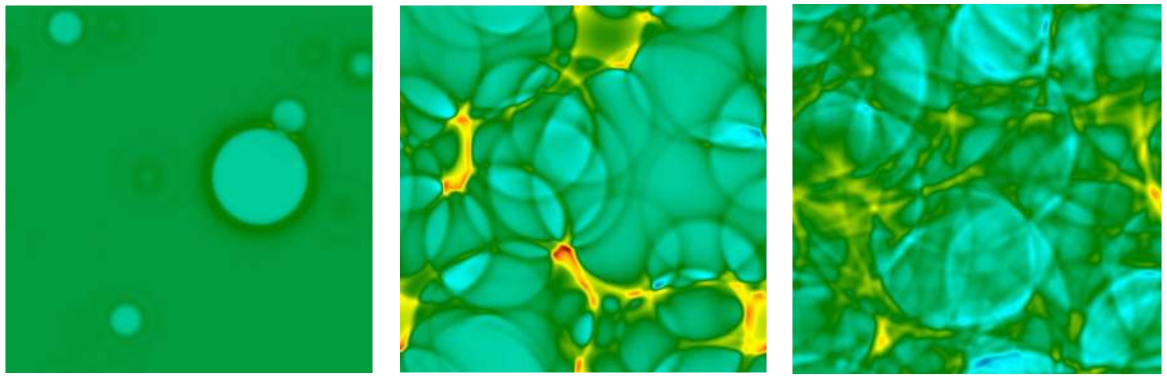
(Image: Google Search)

(p)Reheating



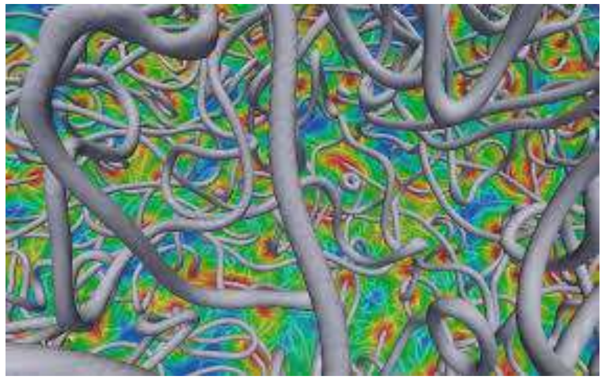
(Fig. credit: Phys.Rev. D67 103501)

Phase Transitions



(Image: PRL 112 (2014) 041301)

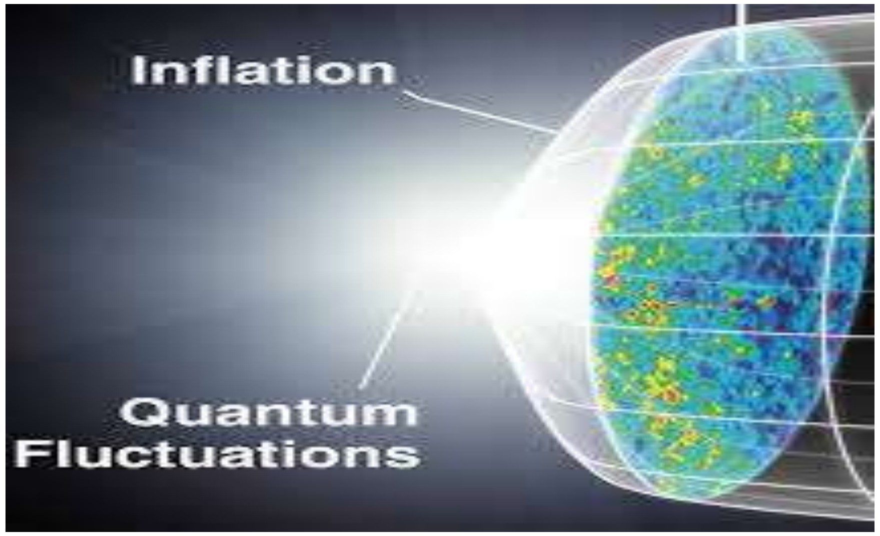
Cosmic Defects



(Image: Daverio et al, 2013)

COSMOLOGICAL GRAVITATIONAL WAVES

Inflationary Period



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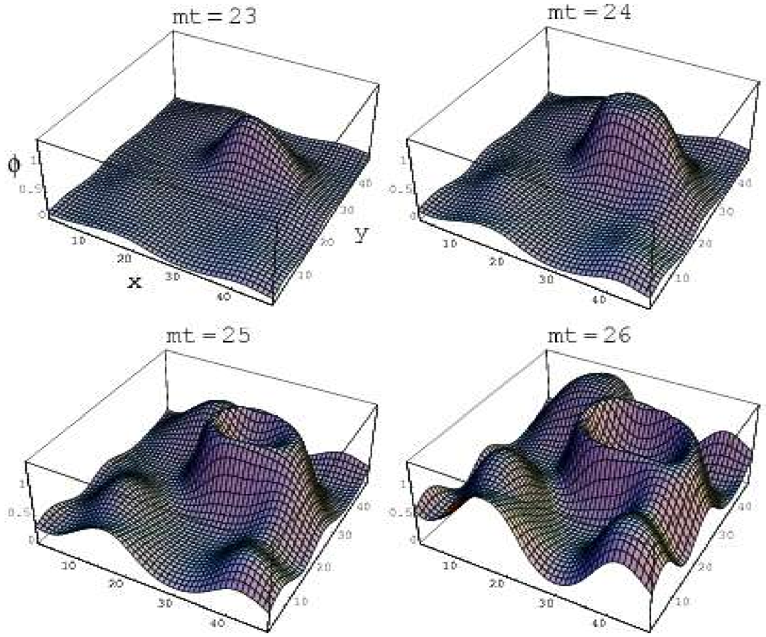
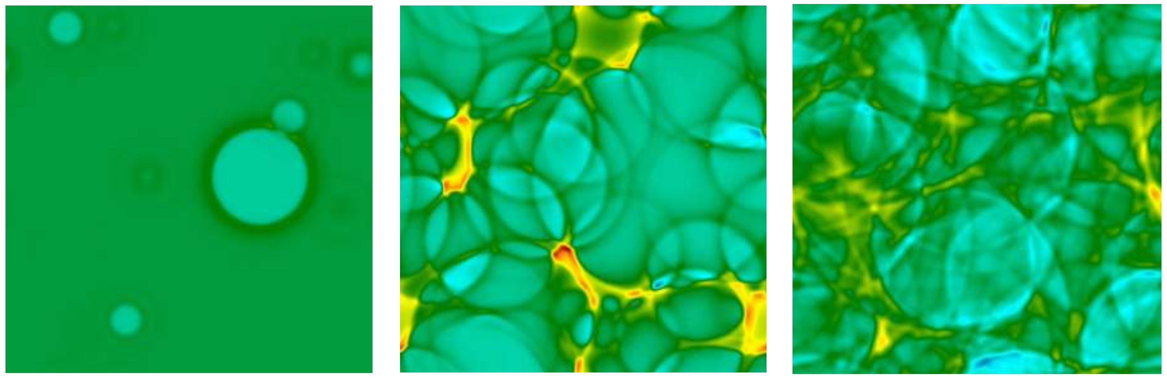


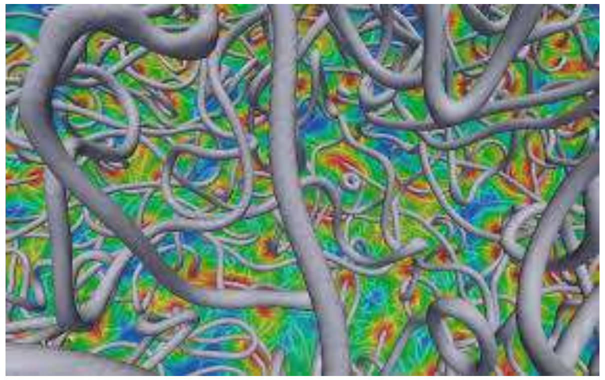
(Fig. credit: Phys.Rev. D67 103501)

Phase Transitions



(Image: PRL 112 (2014) 041301)

Cosmic Defects



(Image: Daverio et al, 2013)

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE

Early Universe

1) GWs from Inflation

2) GWs from Preheating

3) GWs from Phase Transitions

4) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

Early Universe

1) GWs from Inflation → ~ 1 h

2) GWs from Preheating → ~ 1 h

3) GWs from Phase Transitions → ~ 1 h

4) GWs from Cosmic Defects → ~ 1 h

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

0) Gravitational Waves definition

Early Universe

1) GWs from Inflation

2) GWs from Preheating

3) GWs from Phase Transitions

4) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

Early Universe

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

2nd lecture (~ 1.5 h)

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

2nd lecture (~ 1.5 h)

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

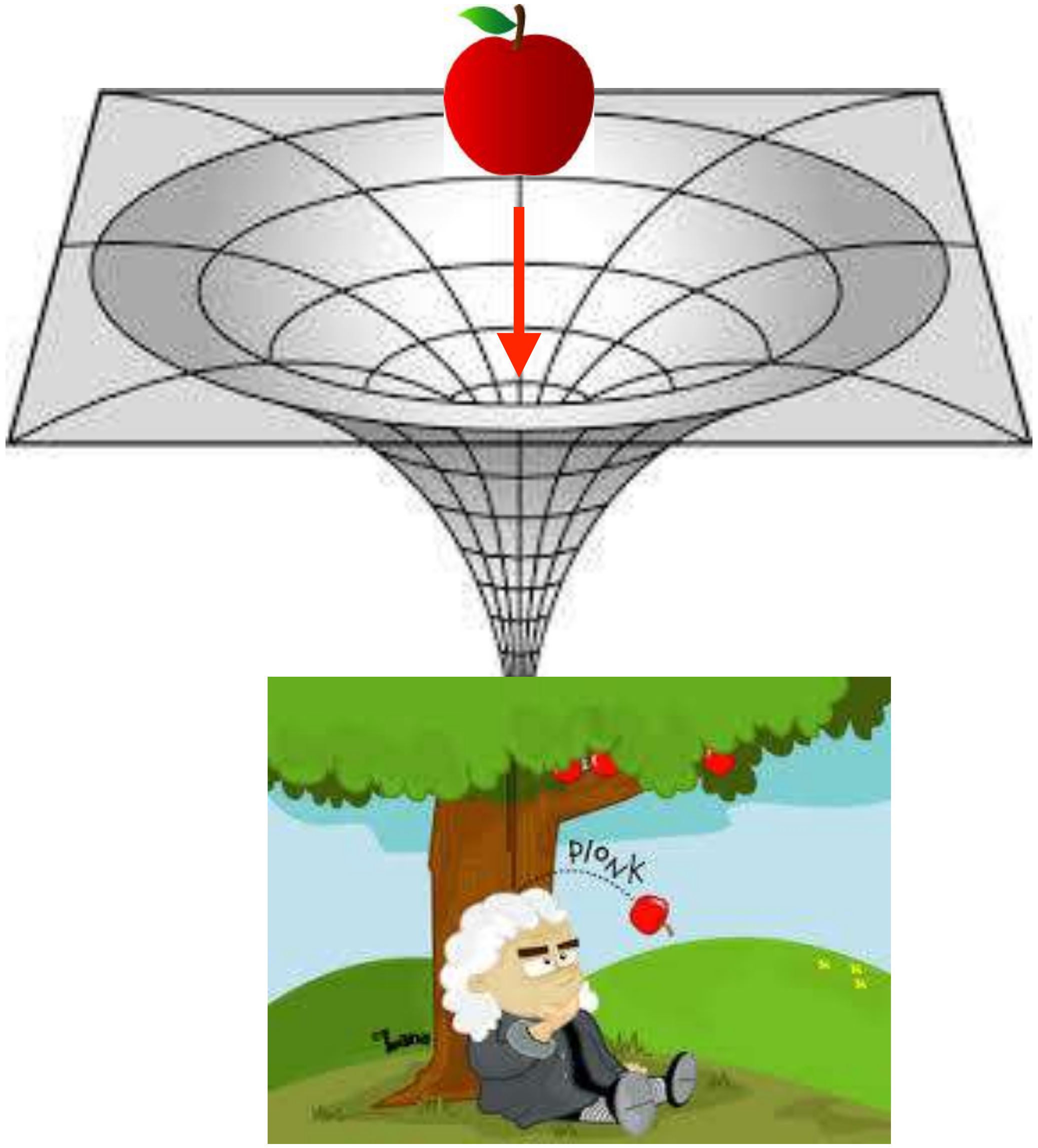
3rd lecture (~ 1.5 h)

Let’s Start !

A PRIMER ON GRAVITATIONAL WAVES

Gravitational Framework

General Relativity (GR)



Gravitational Framework

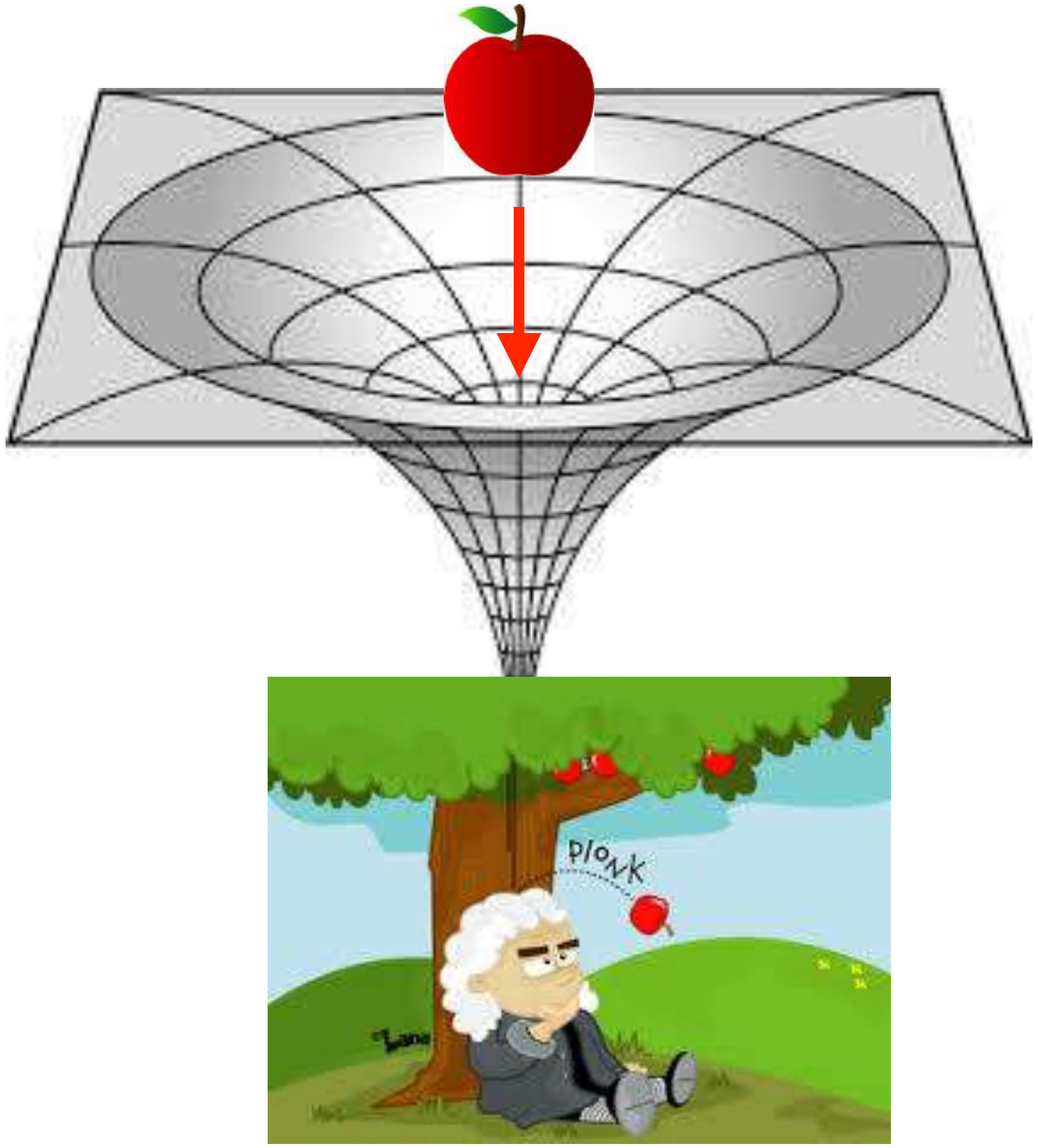
General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass



Gravitational Framework

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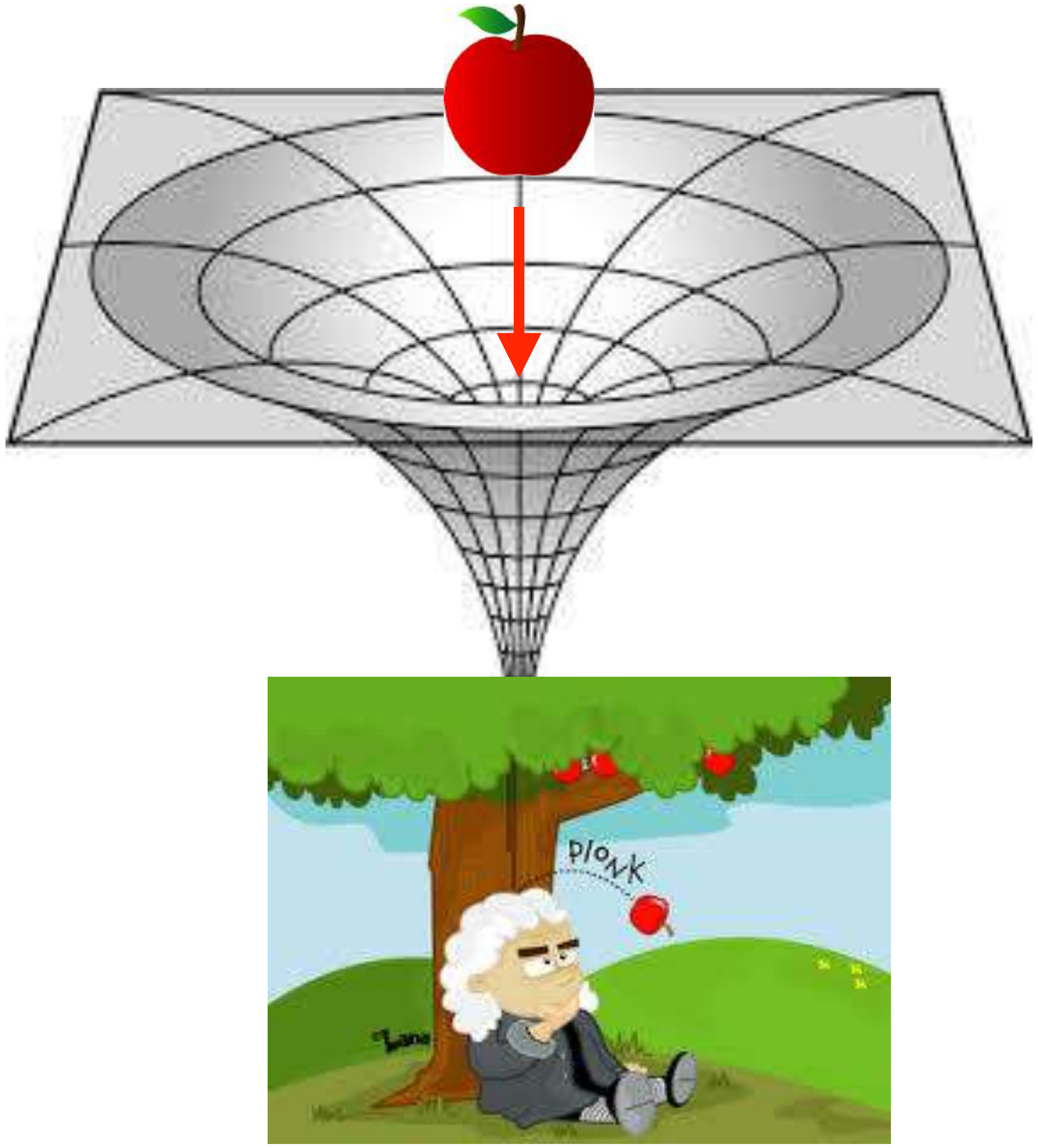


$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

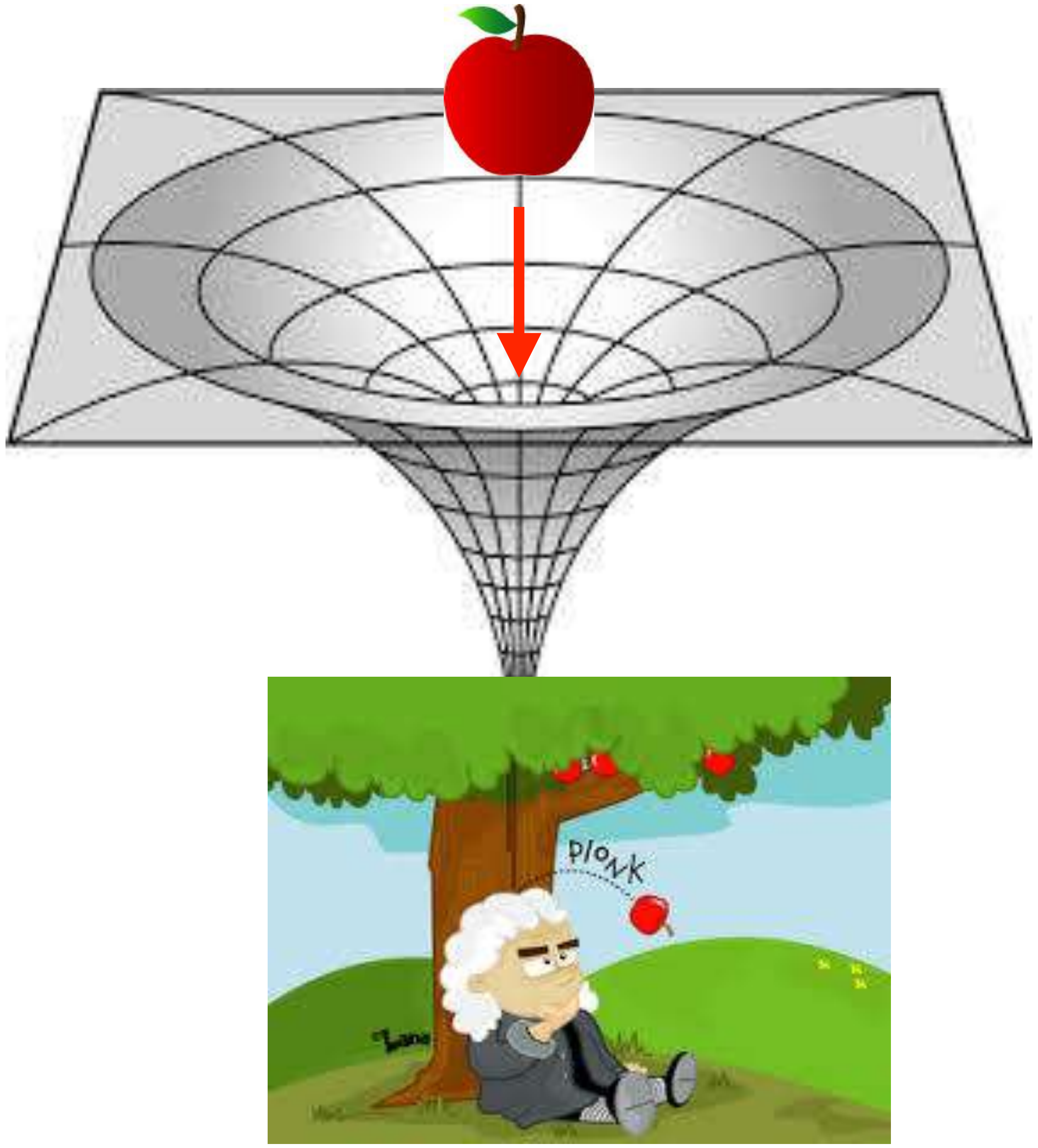
$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Gravitational Framework

General Relativity (GR)



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geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\, \mathrm{GeV} \right]$$

Reduced Planck mass

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

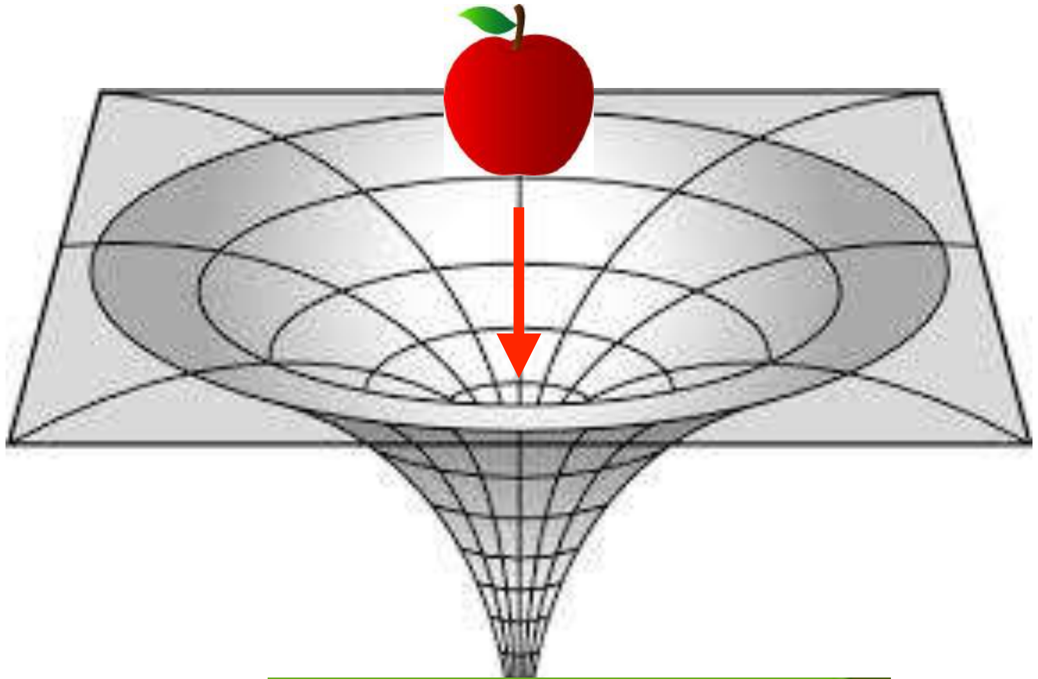
$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry

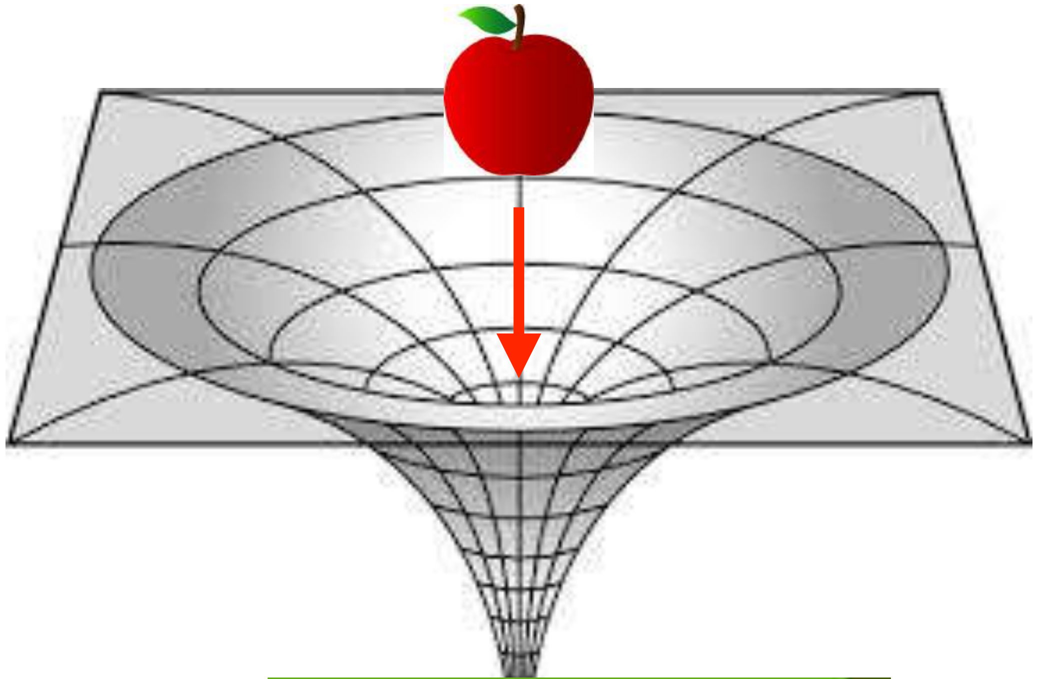
(

A PRIMER ON GENERAL RELATIVITY

The Equivalence Principle



The Equivalence Principle

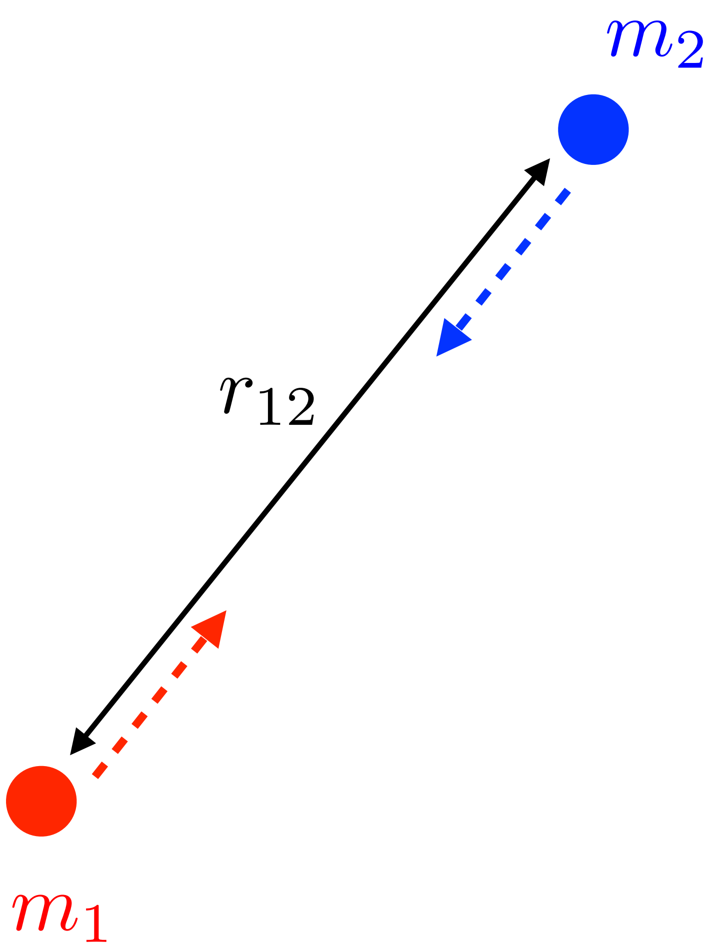


Newton's Gravitation Law

$$F = G\frac{m_1 m_2}{r_{12}^2}$$

$$G = 6.67 \cdot 10^{-11}\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{Kg}^2}$$

The Equivalence Principle



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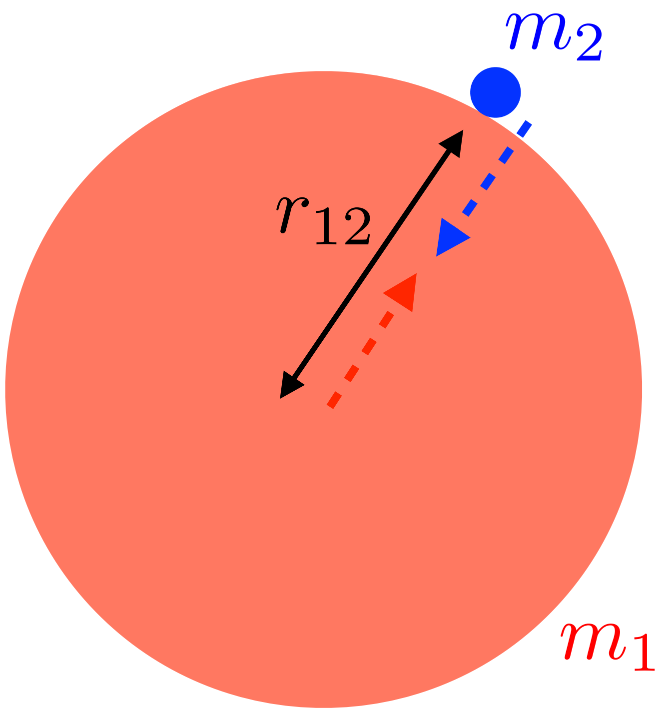

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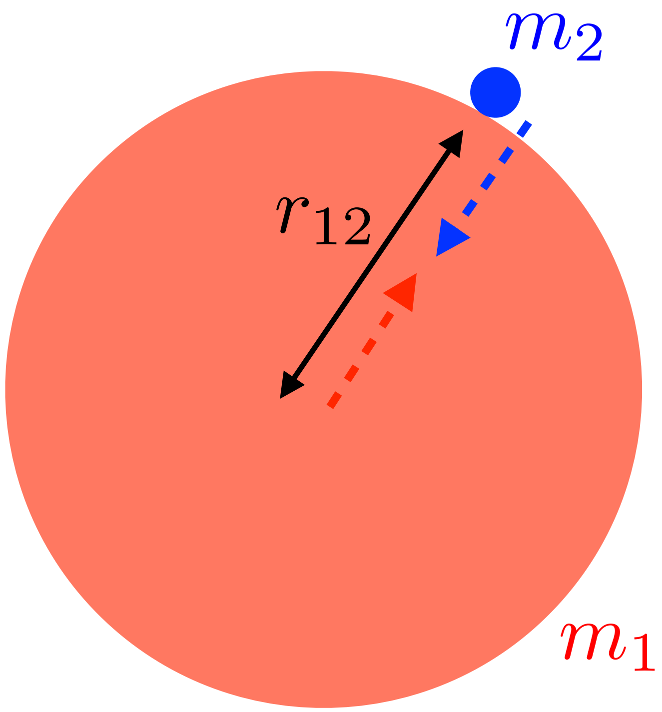
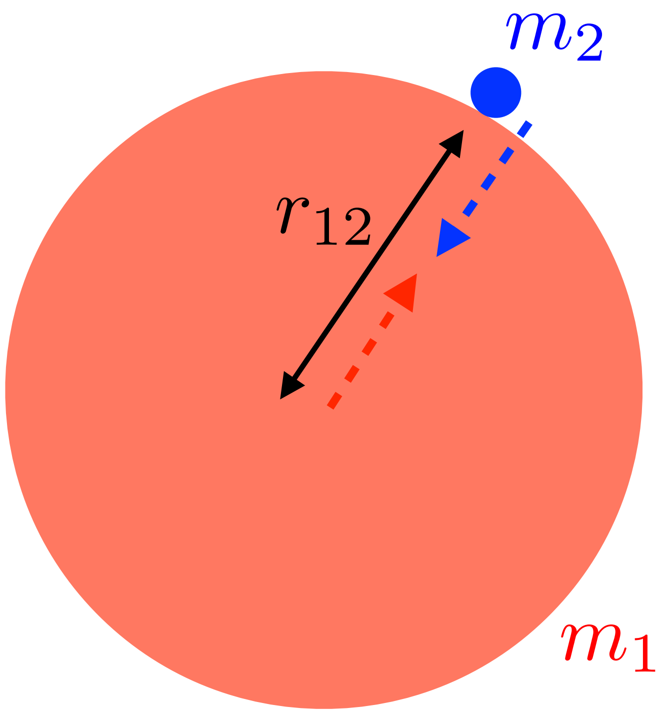


The Equivalence Principle

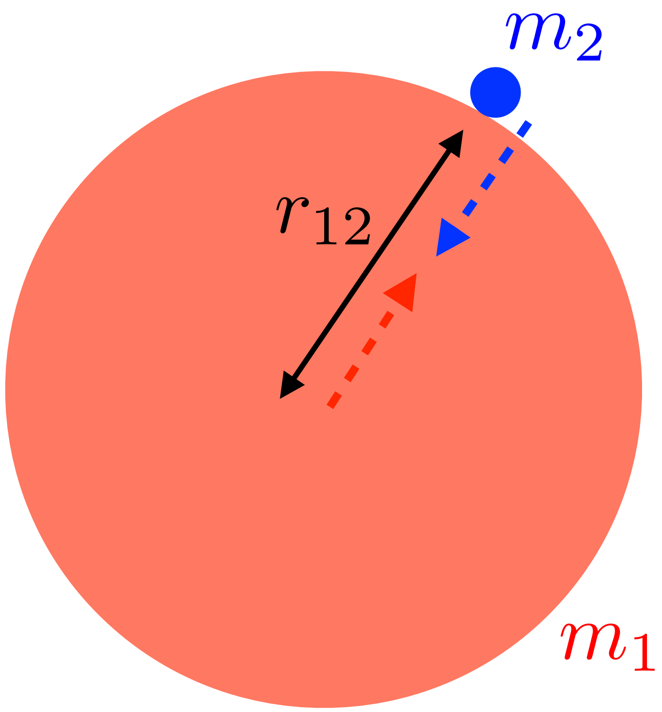
Newton's Gravitation Law

$$F = \boxed{G\frac{m_1 m_2}{r_{12}^2}} = m_2 \cdot g_\oplus$$

$$g_\oplus = 9.81\ m/s^2$$



The Equivalence Principle



Newton's Gravitation Law

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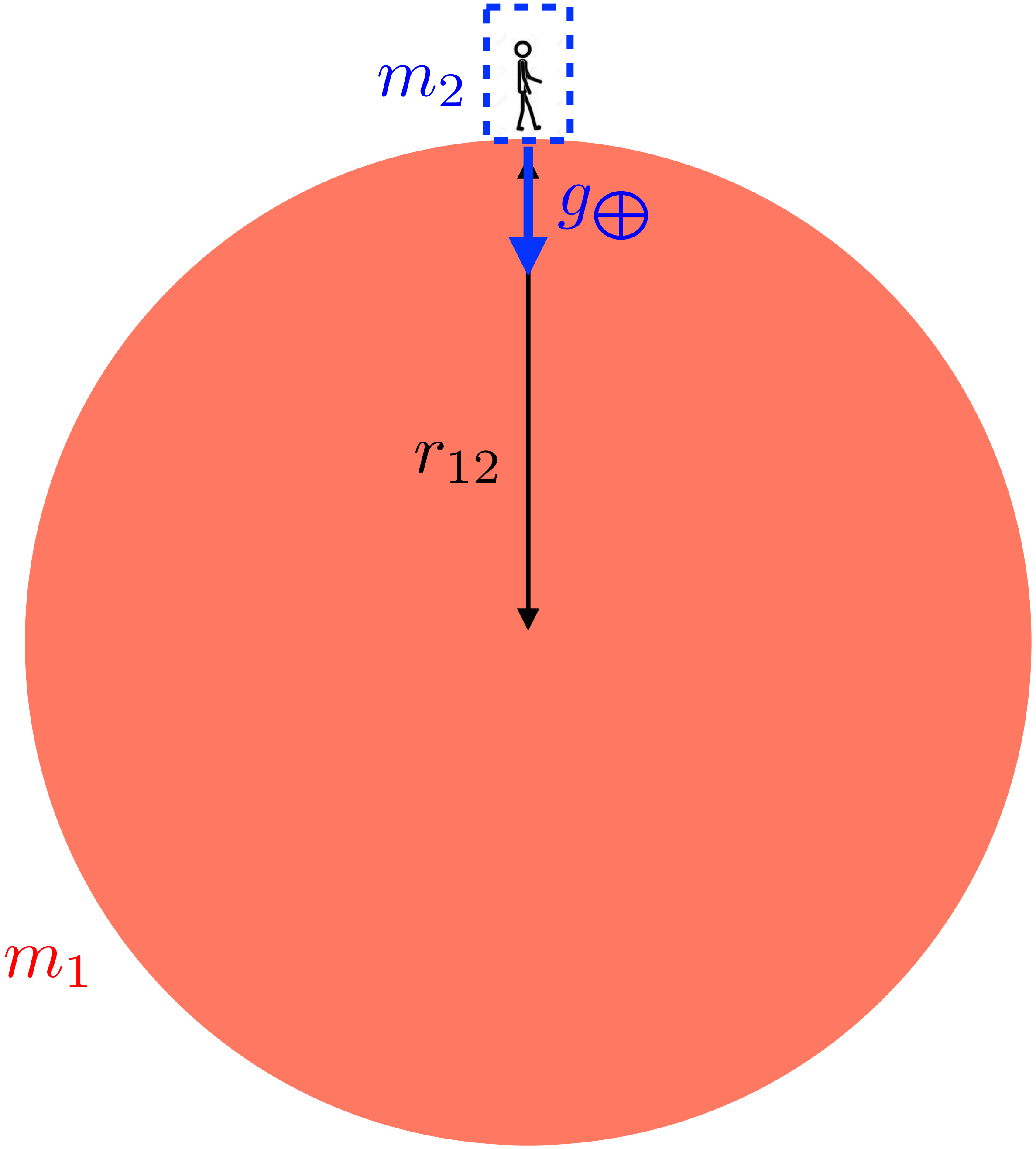
$$F = m_2 \cdot g_\oplus = m_2 \cdot a$$

$$\boxed{a = g_\oplus}$$

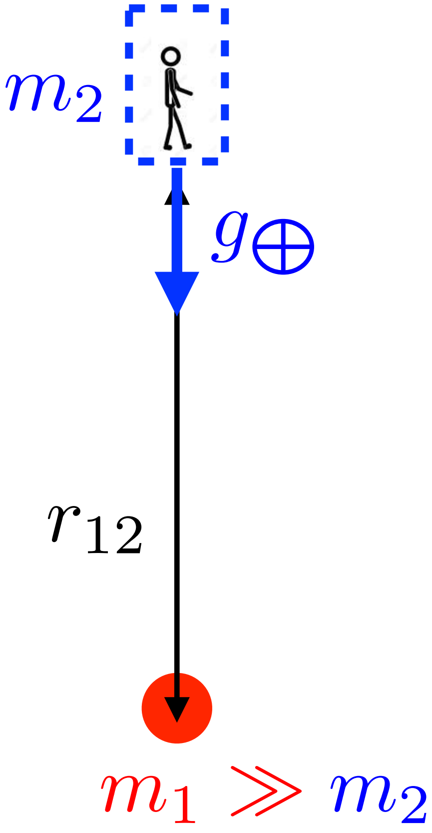
Universal

$$(\text{IF}\quad m_{\text{grav}} = m_{\text{inertial}})$$

The Equivalence Principle

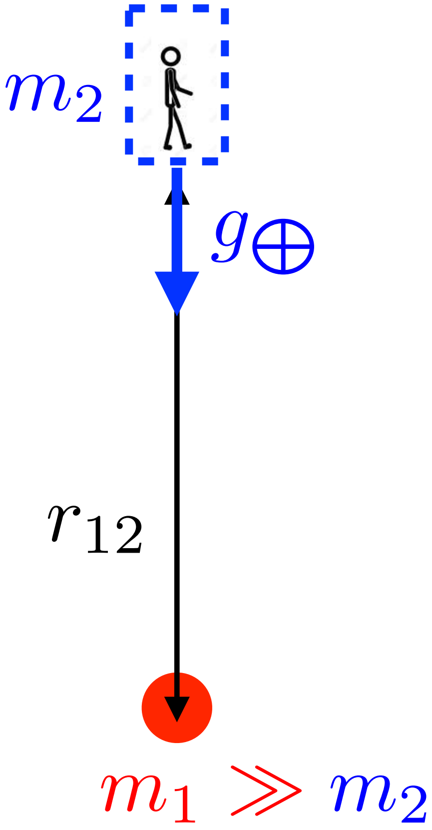


The Equivalence Principle



The Equivalence Principle

Free falling (you feel no force)



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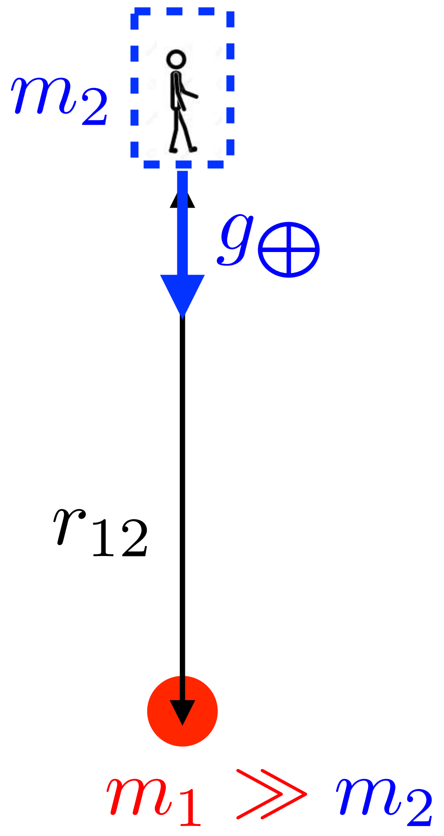
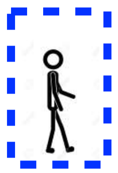


Outer space

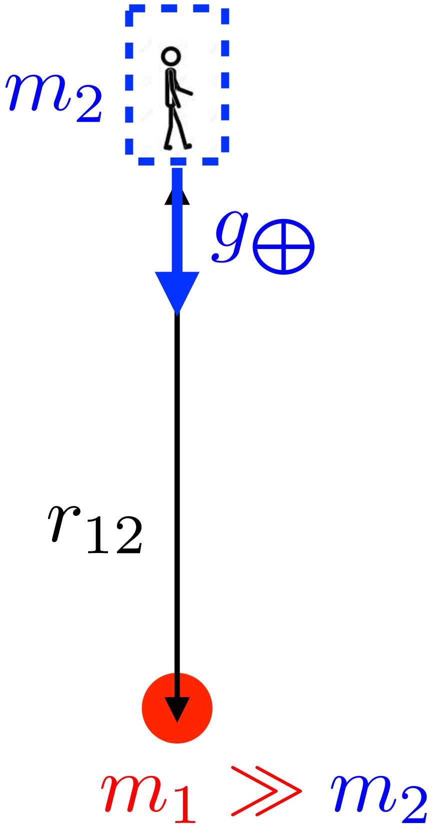


(no gravity, no forces)

The Equivalence Principle

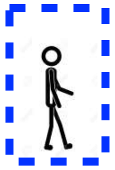
Free falling

(you feel no force)



$$\equiv$$

Outer space



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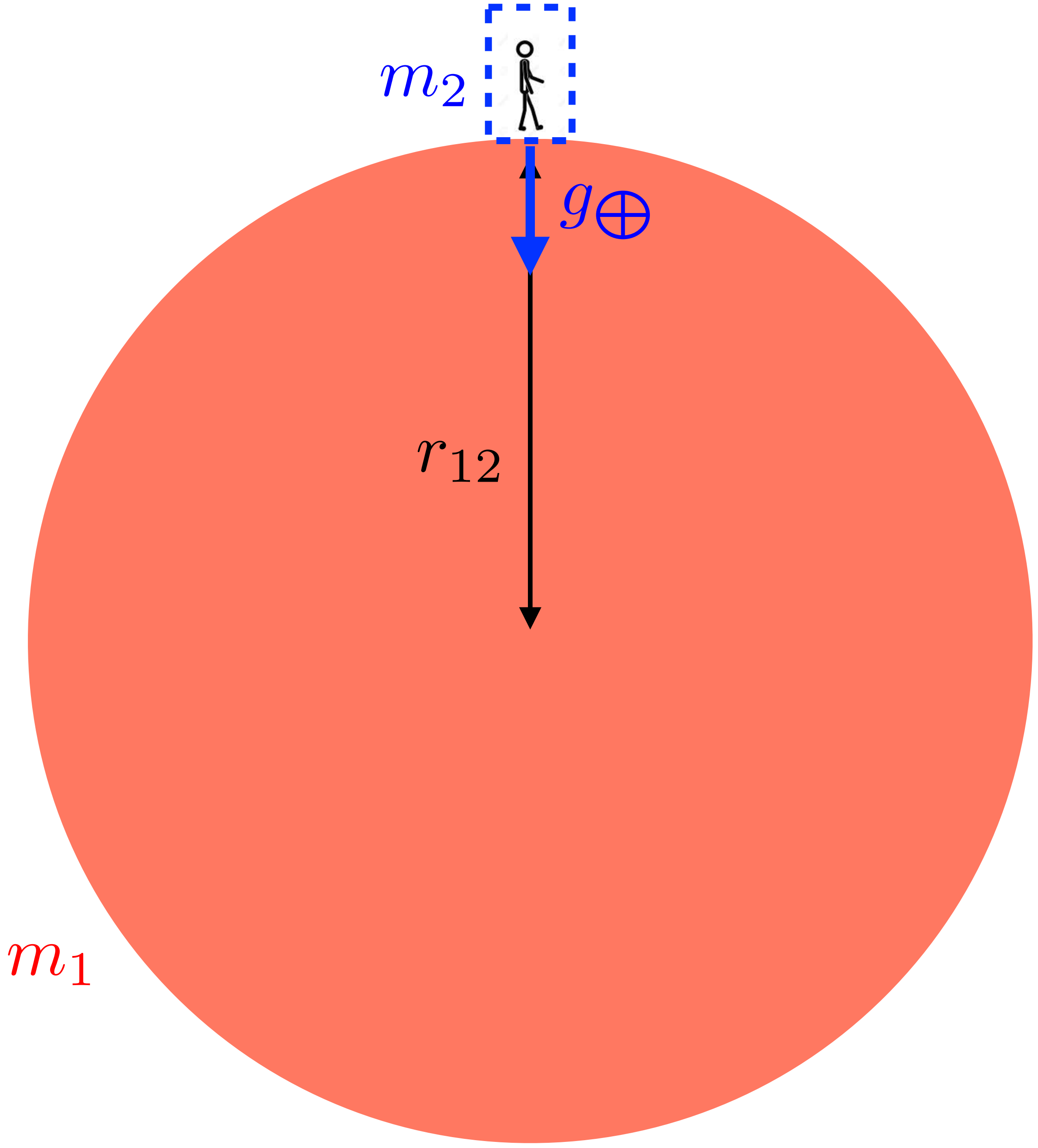
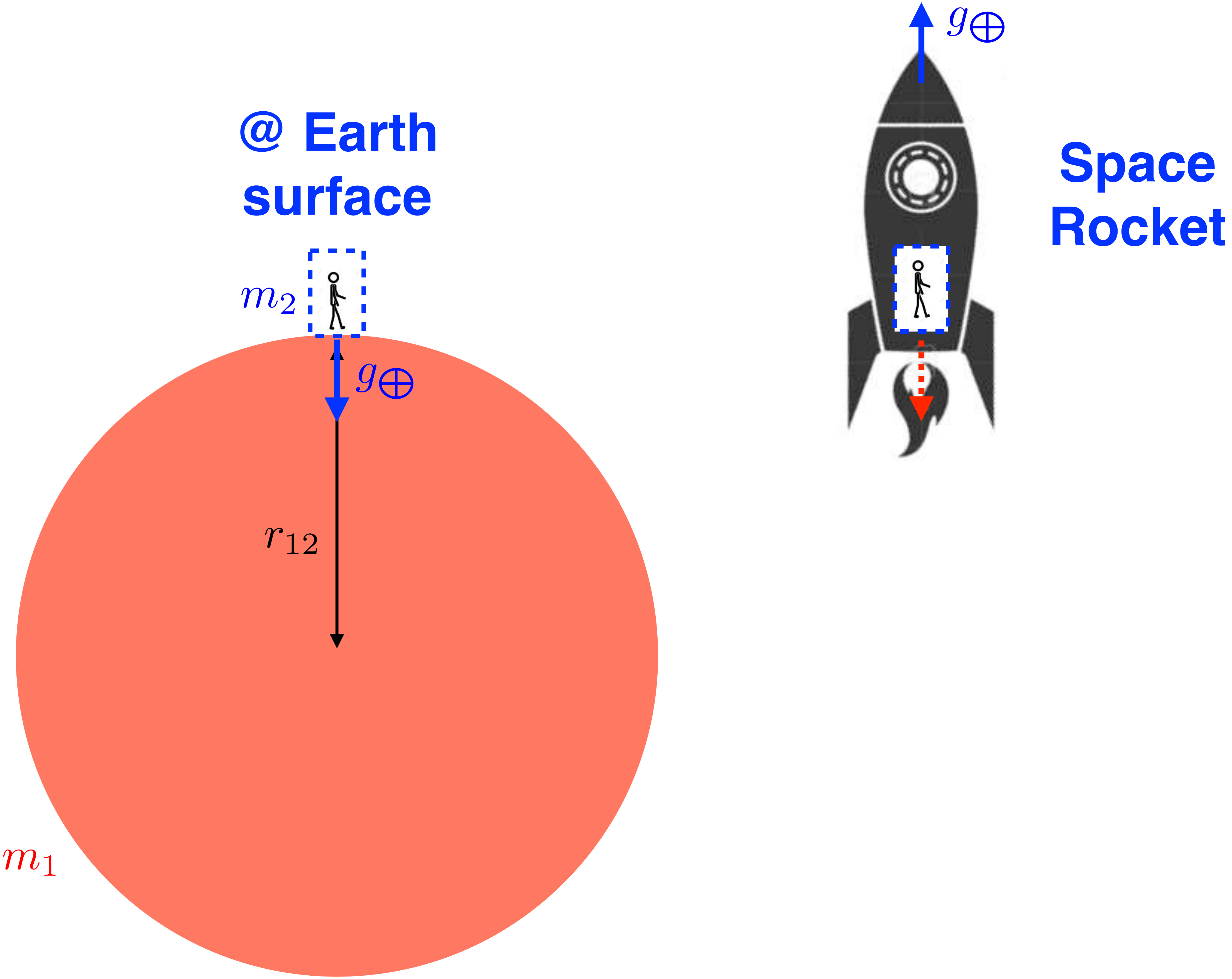


@ Earth surface

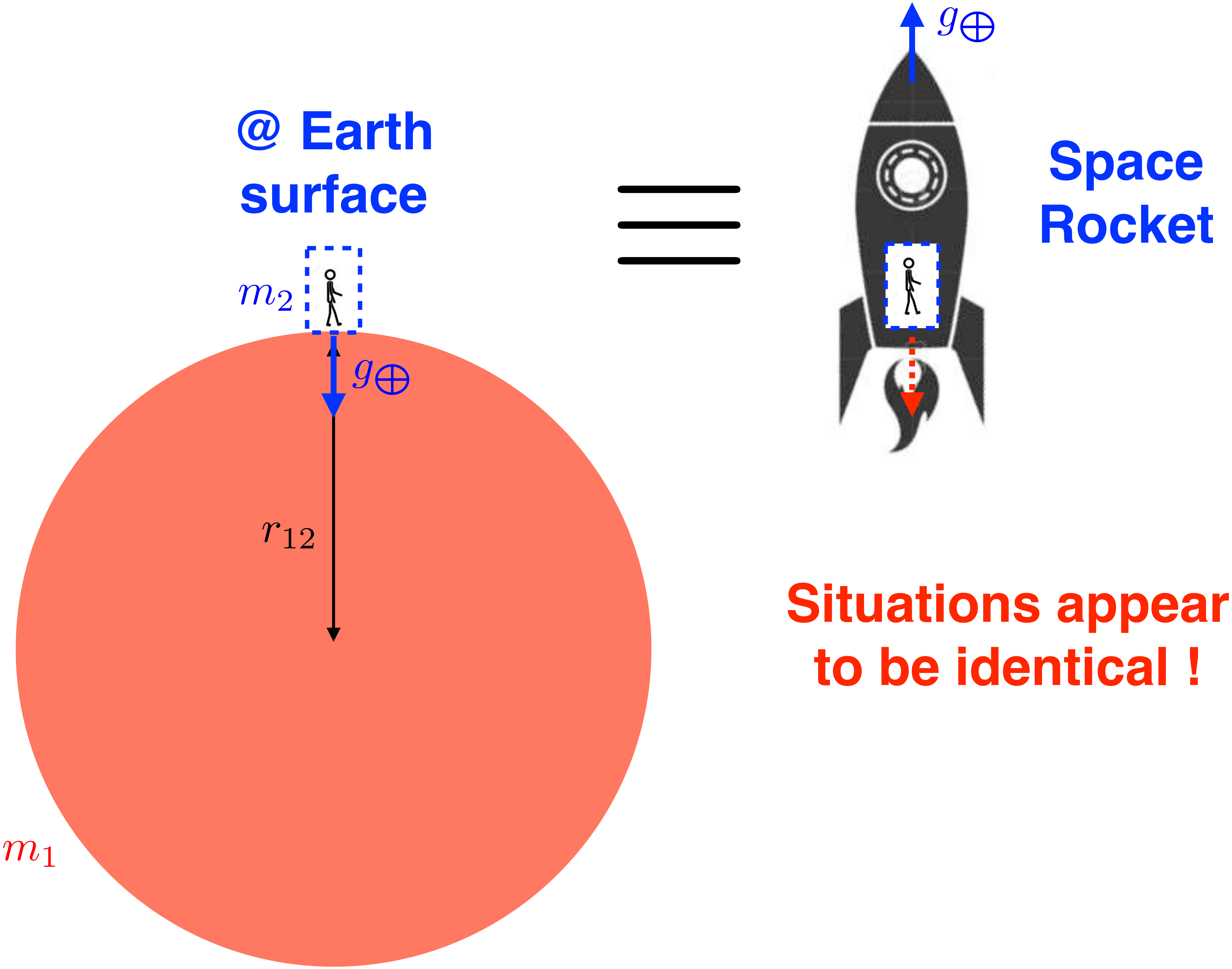
The Equivalence Principle



@ Earth surface

Space Rocket

The Equivalence Principle

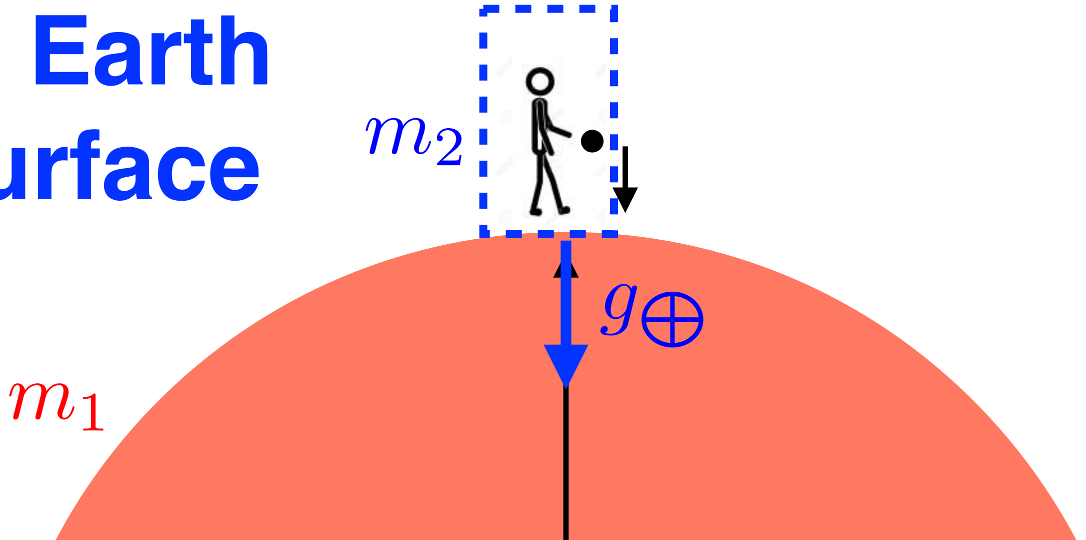


@ Earth surface

Space Rocket

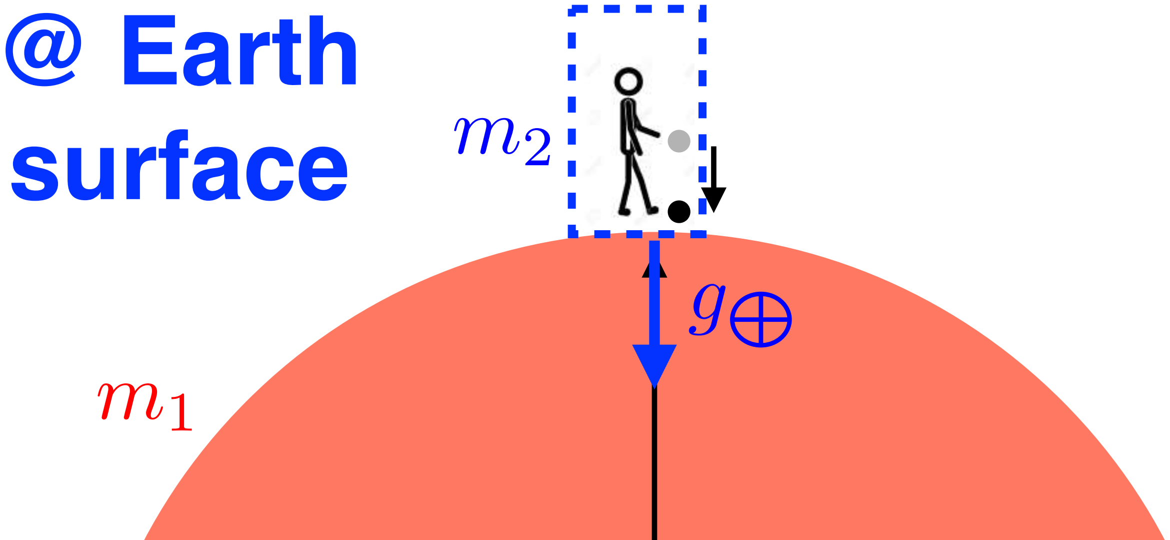
Situations appear to be identical !

The Equivalence Principle

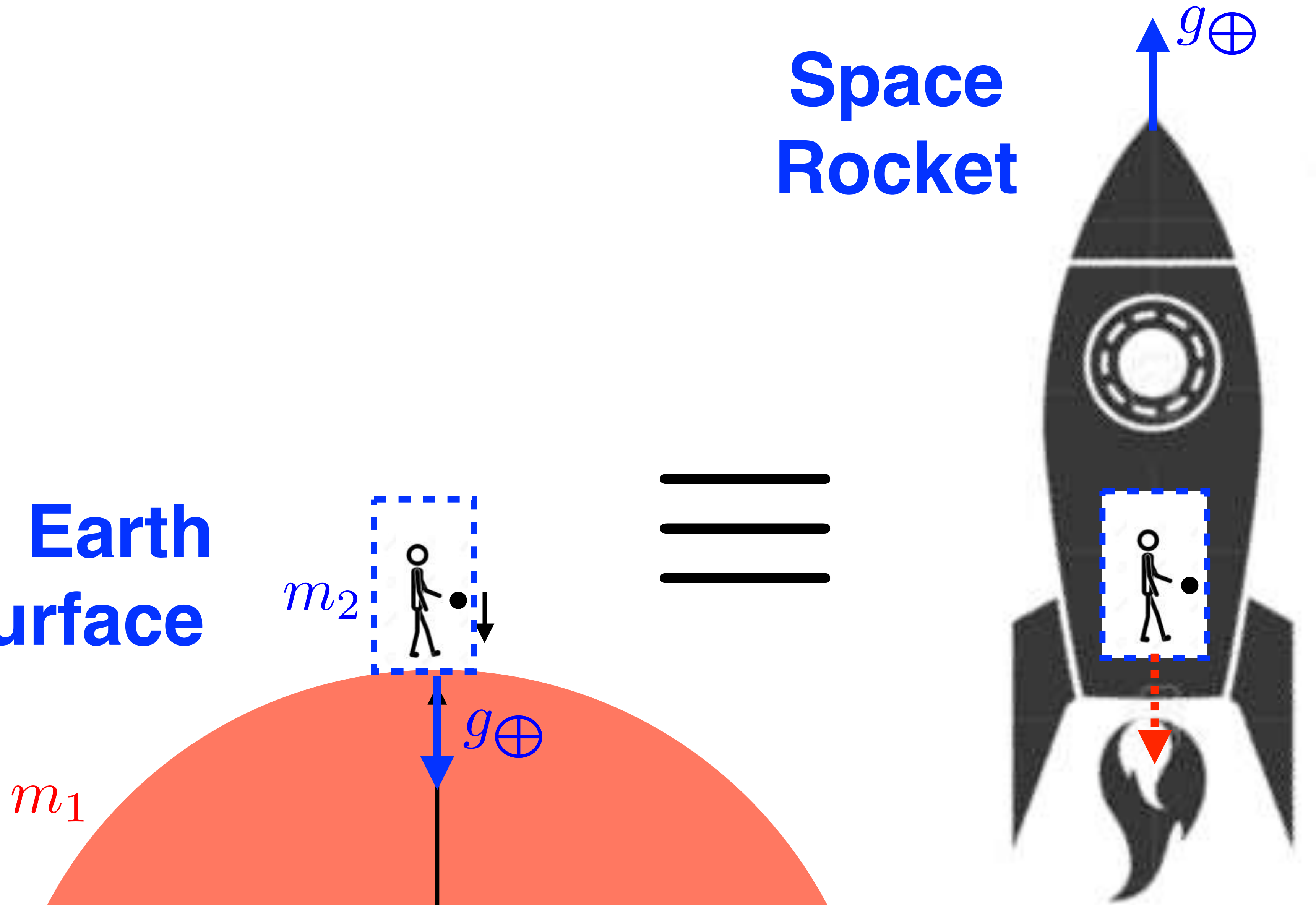


@ Earth surface

The Equivalence Principle



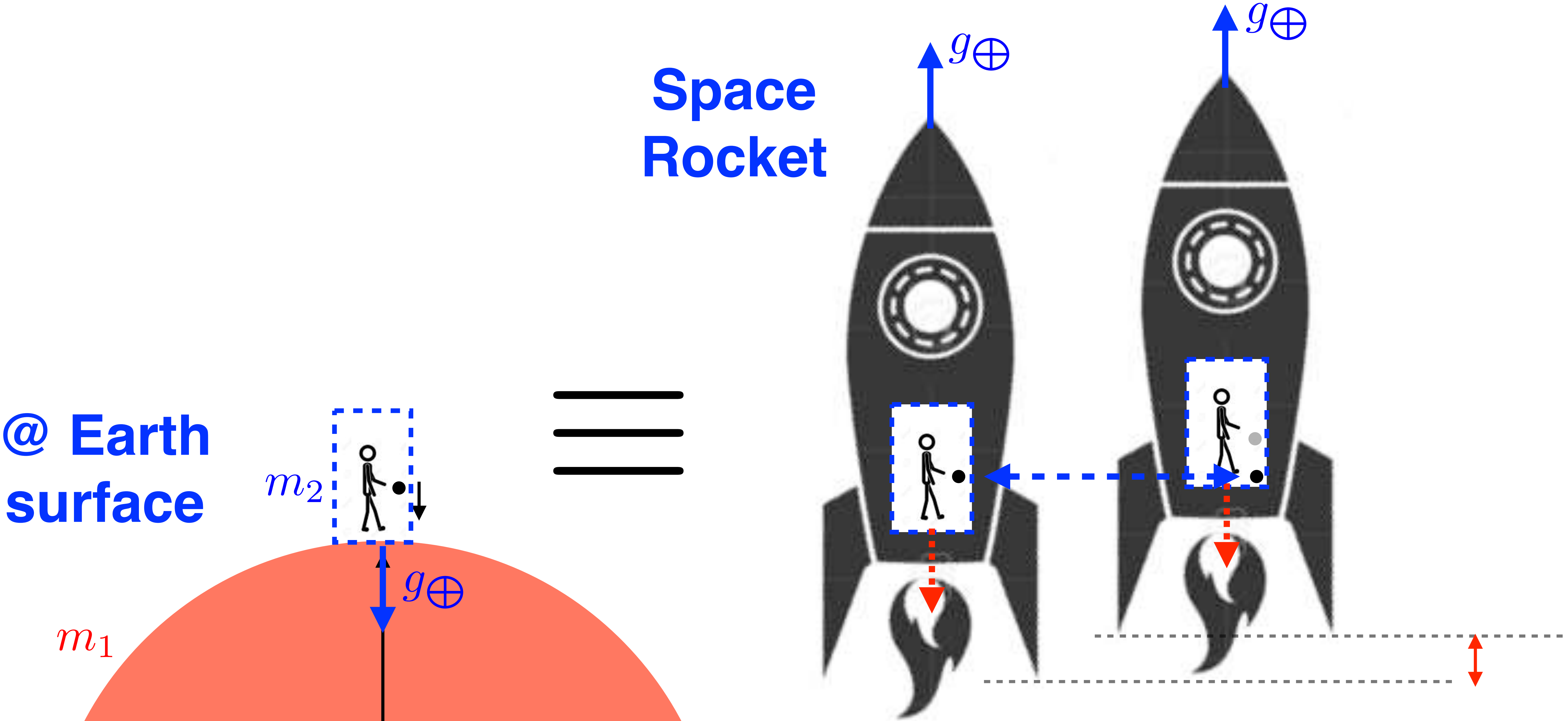
The Equivalence Principle



@ Earth surface

Space Rocket

The Equivalence Principle

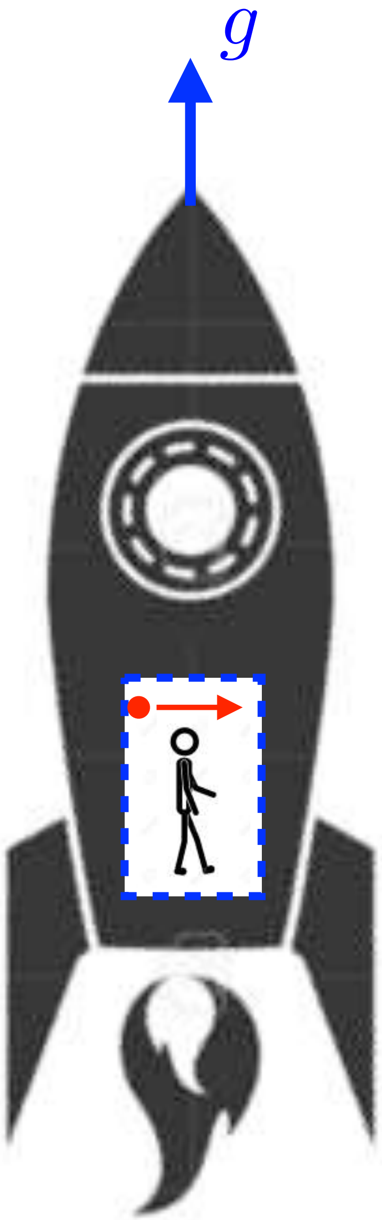


@ Earth surface

$\equiv$

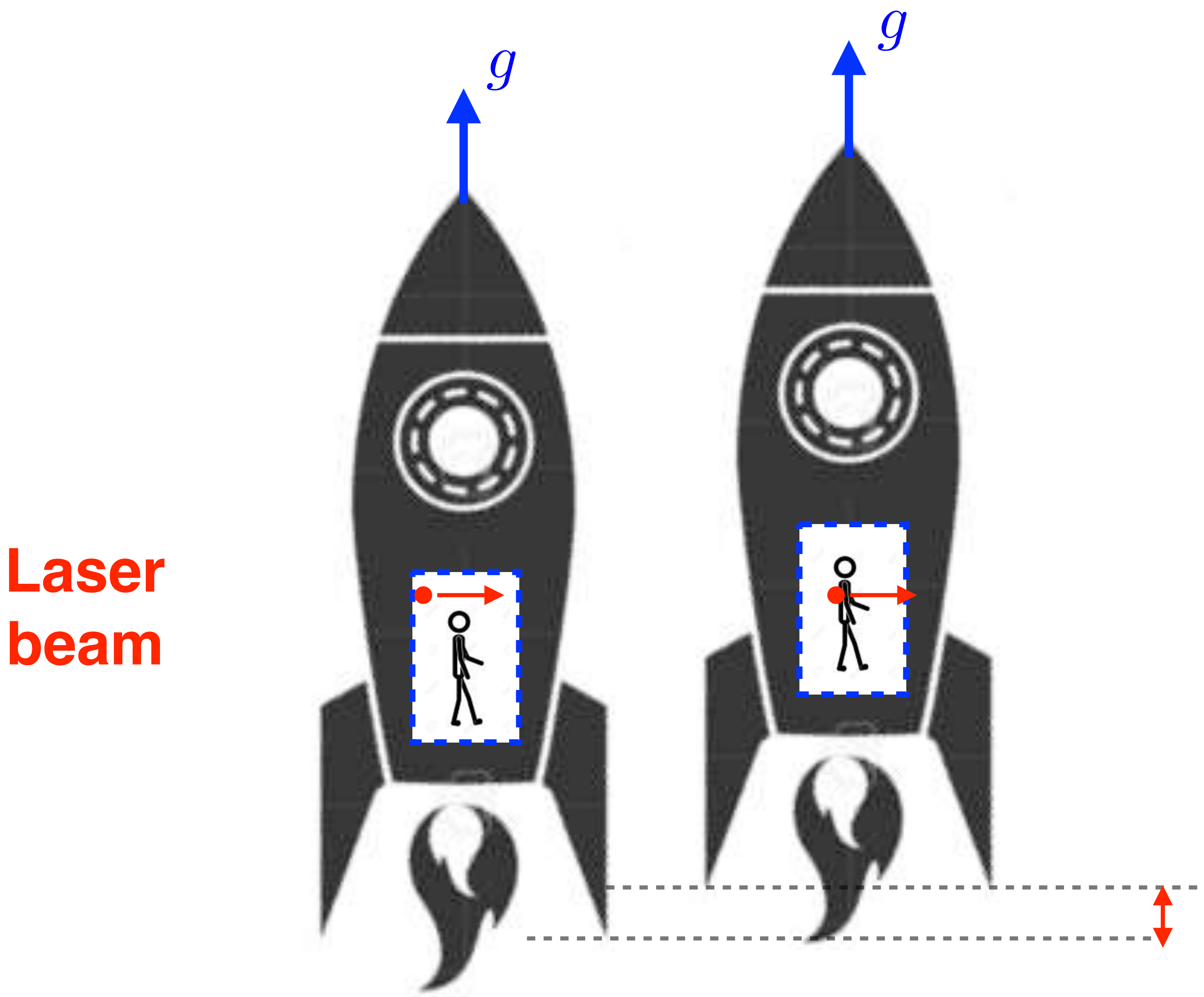
Space Rocket

The Equivalence Principle



Laser
beam

The Equivalence Principle



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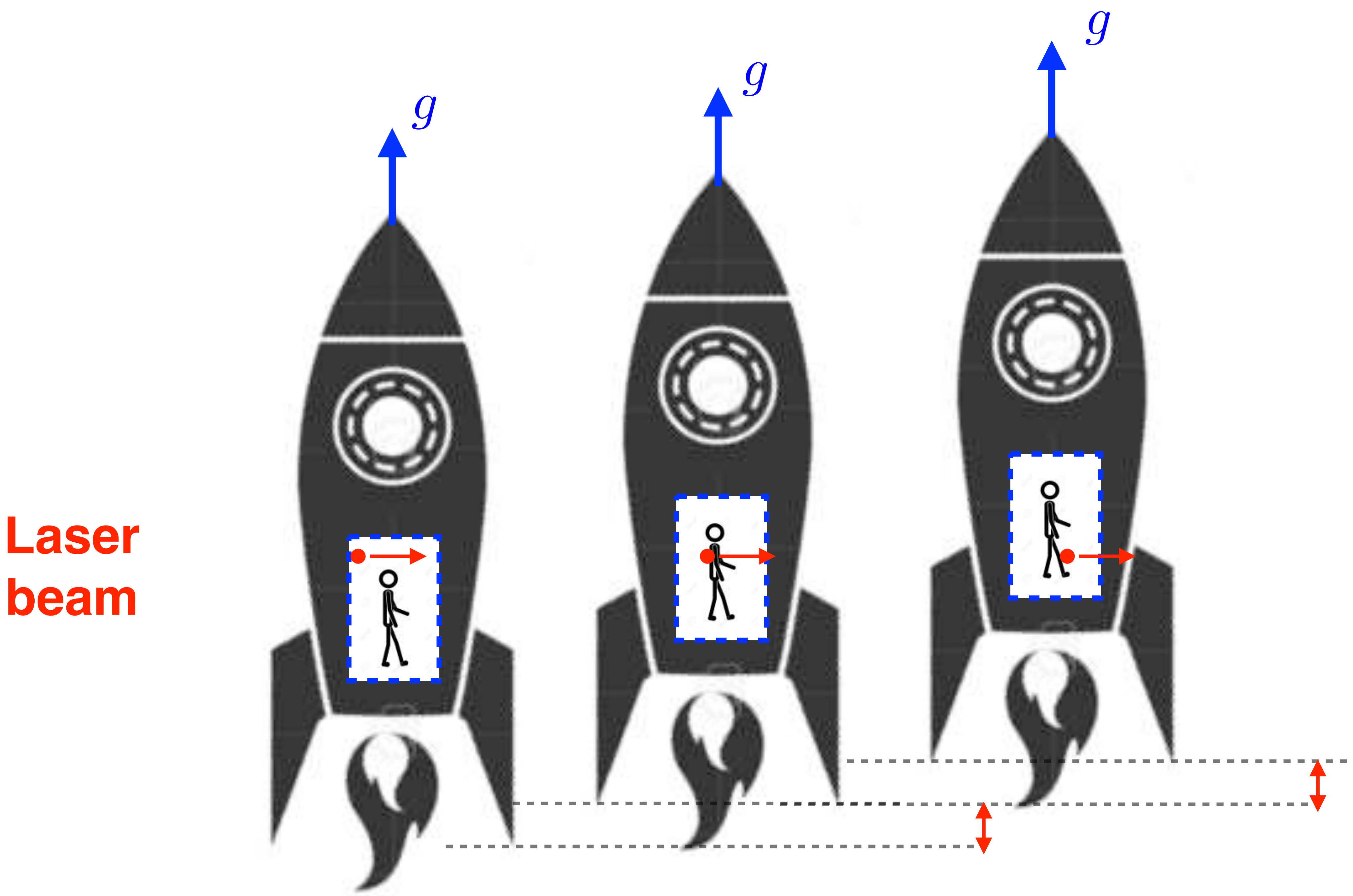

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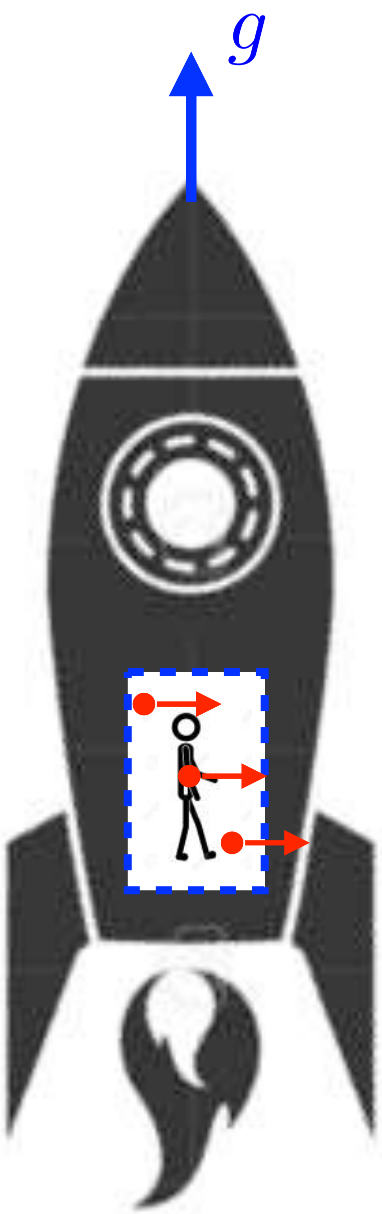
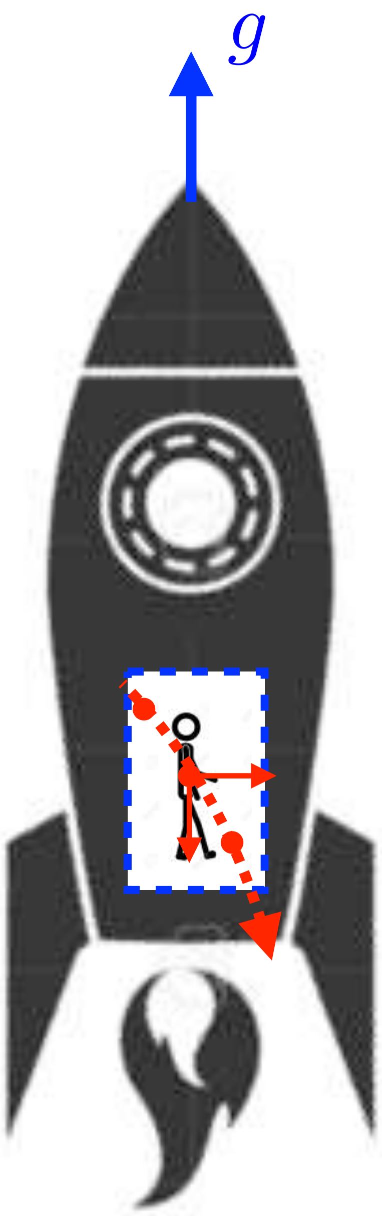


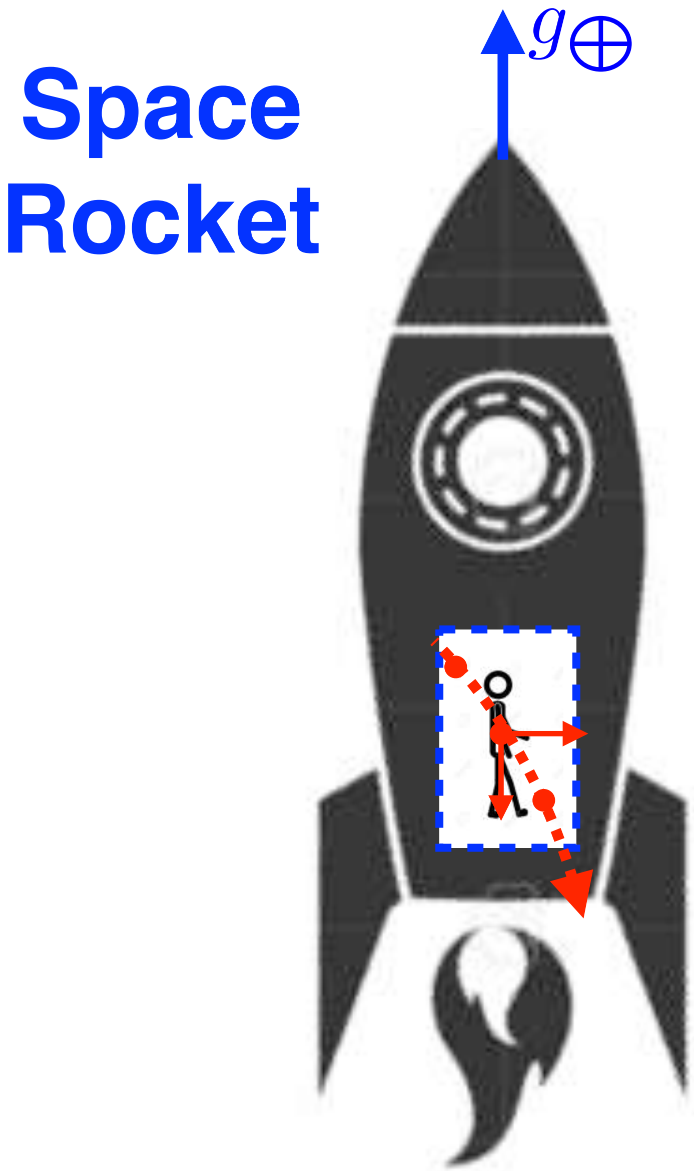
Laser beam

The Equivalence Principle



Laser beam

The Equivalence Principle



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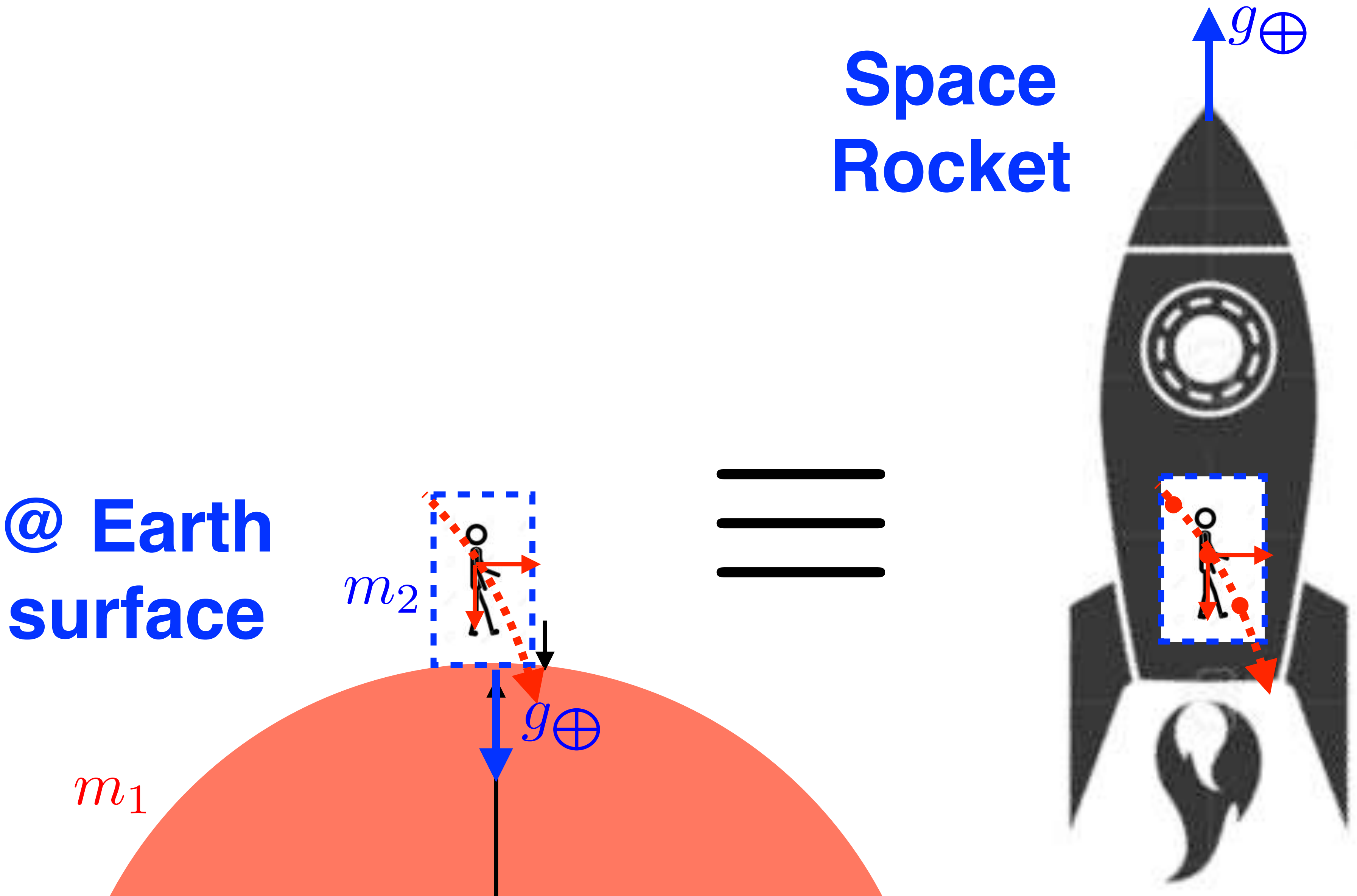
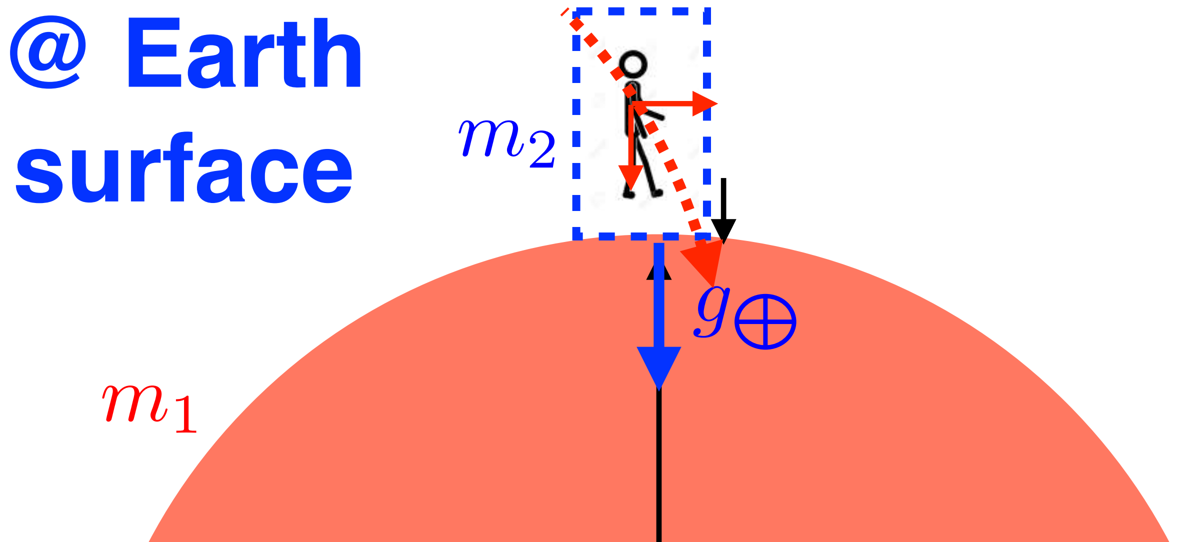


@ Earth surface

Space Rocket

The Equivalence Principle

Gravitational field must bend light !

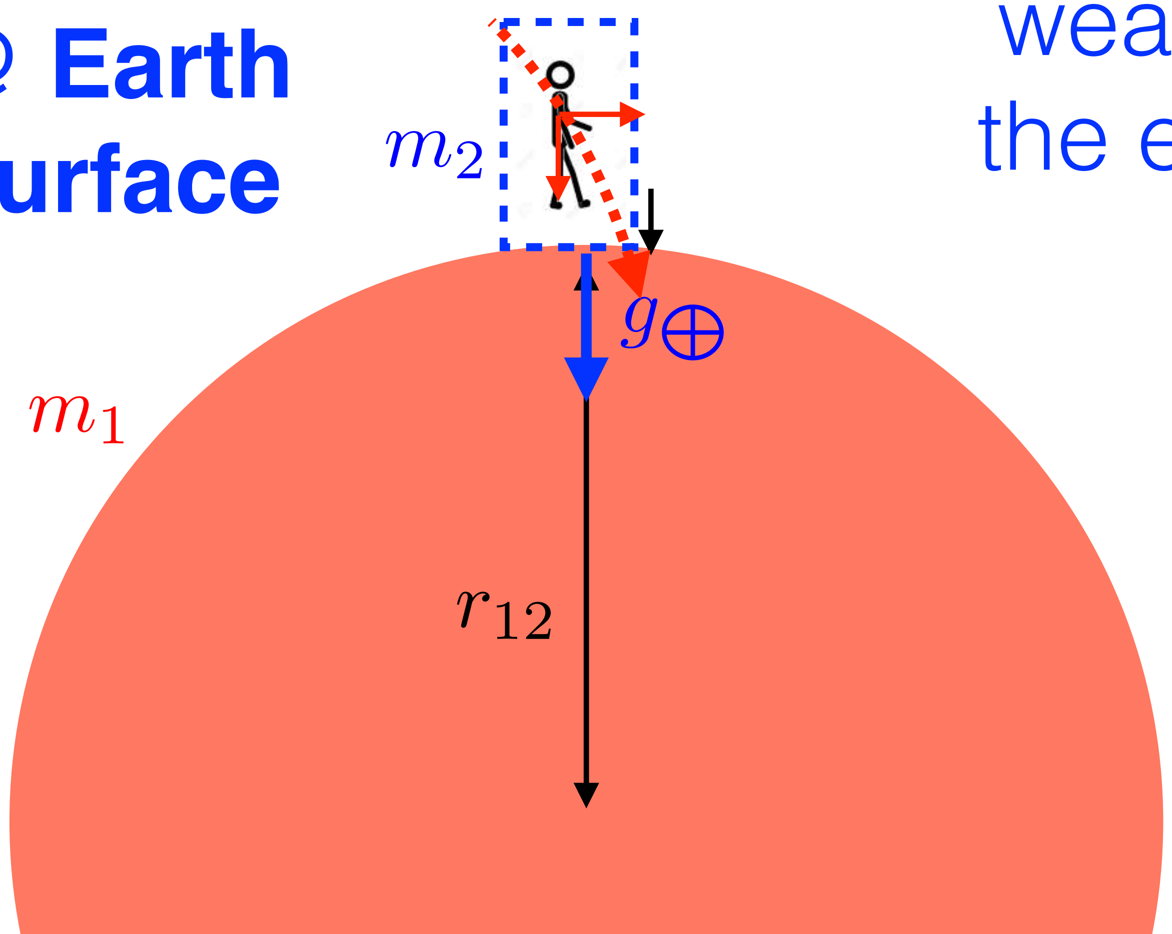


The Equivalence Principle

Gravitational field must bend light !

(Earth gravity too weak to observe the effect, but …)

@ Earth surface



The Equivalence Principle

(Earth gravity too
weak to observe
the effect, but …) →

e.g. the sun does a better job !

Eddington 1919

The Equivalence Principle

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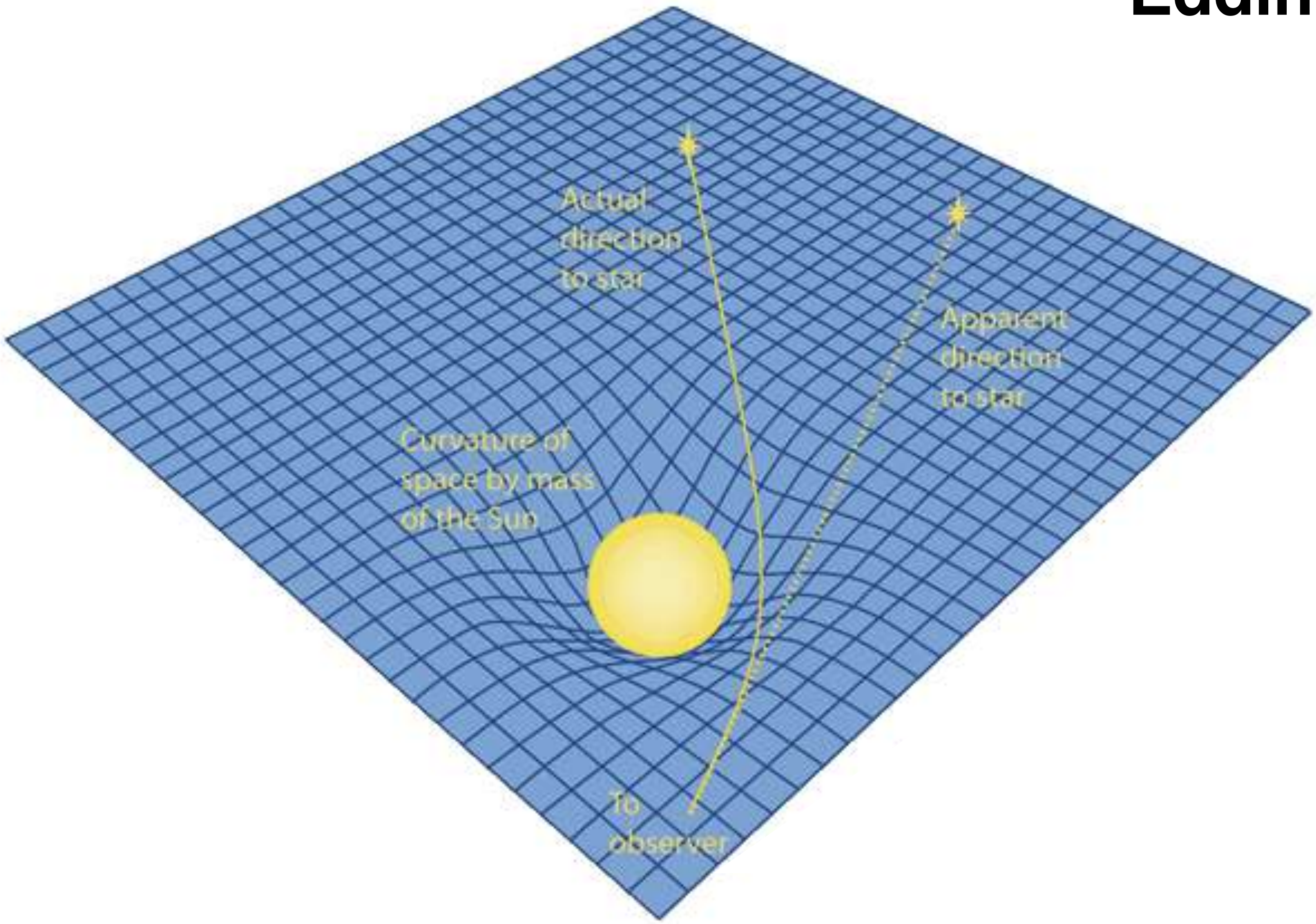

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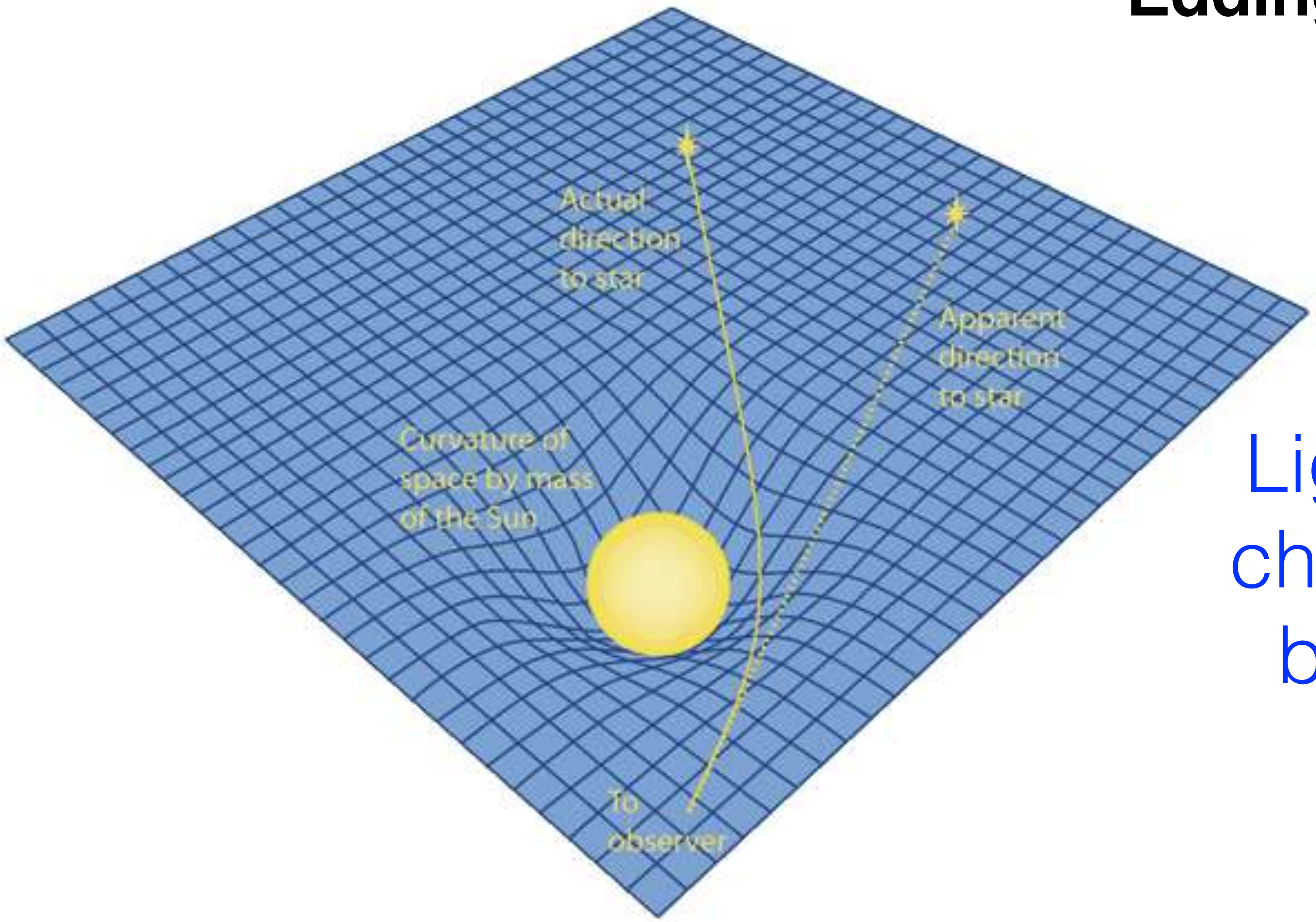


Light does not
change speed,
but direction

The Equivalence Principle

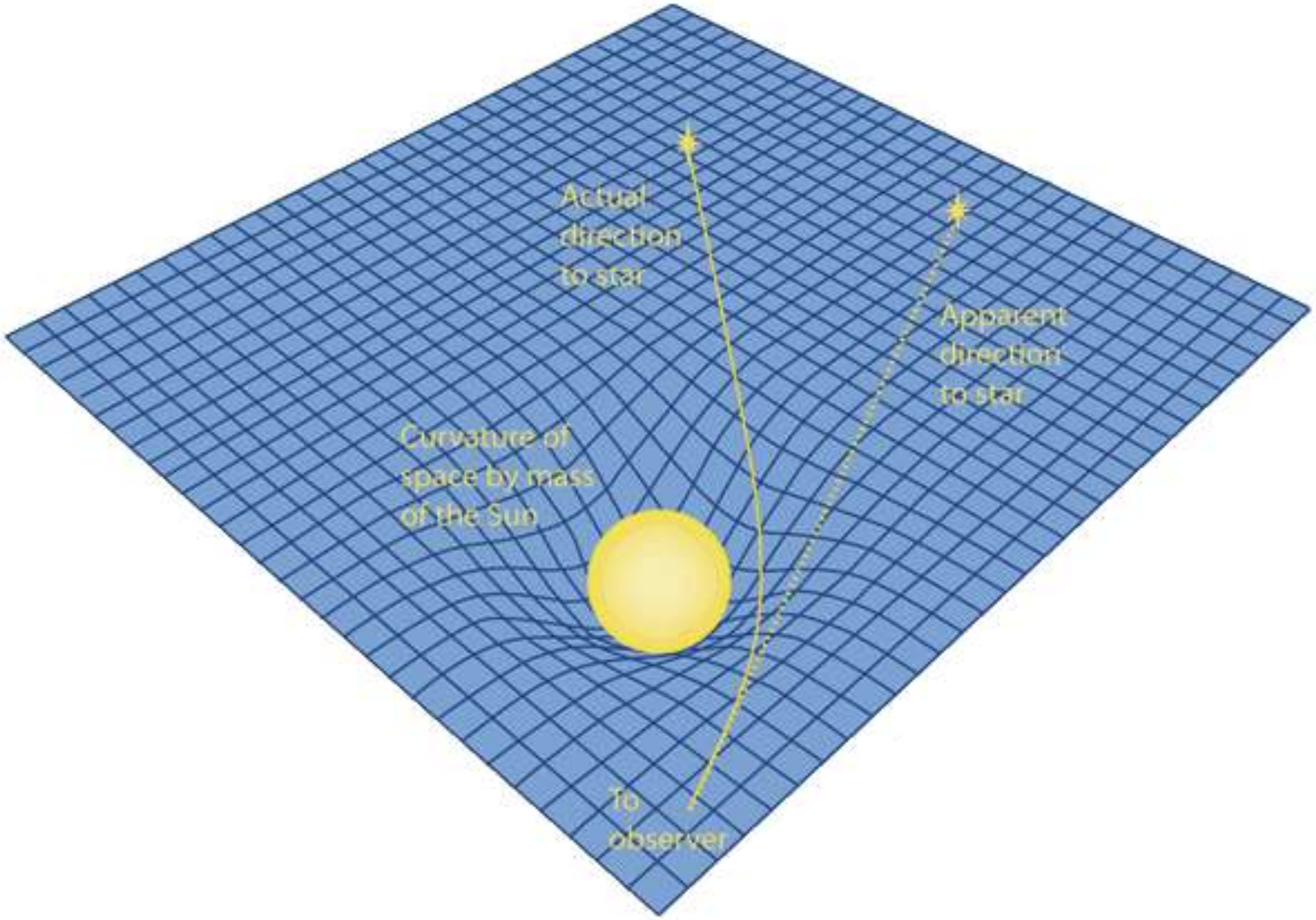
Einstein understood like this…

light bending,

light red/blue-shifting,

gravitational time dilation,

…

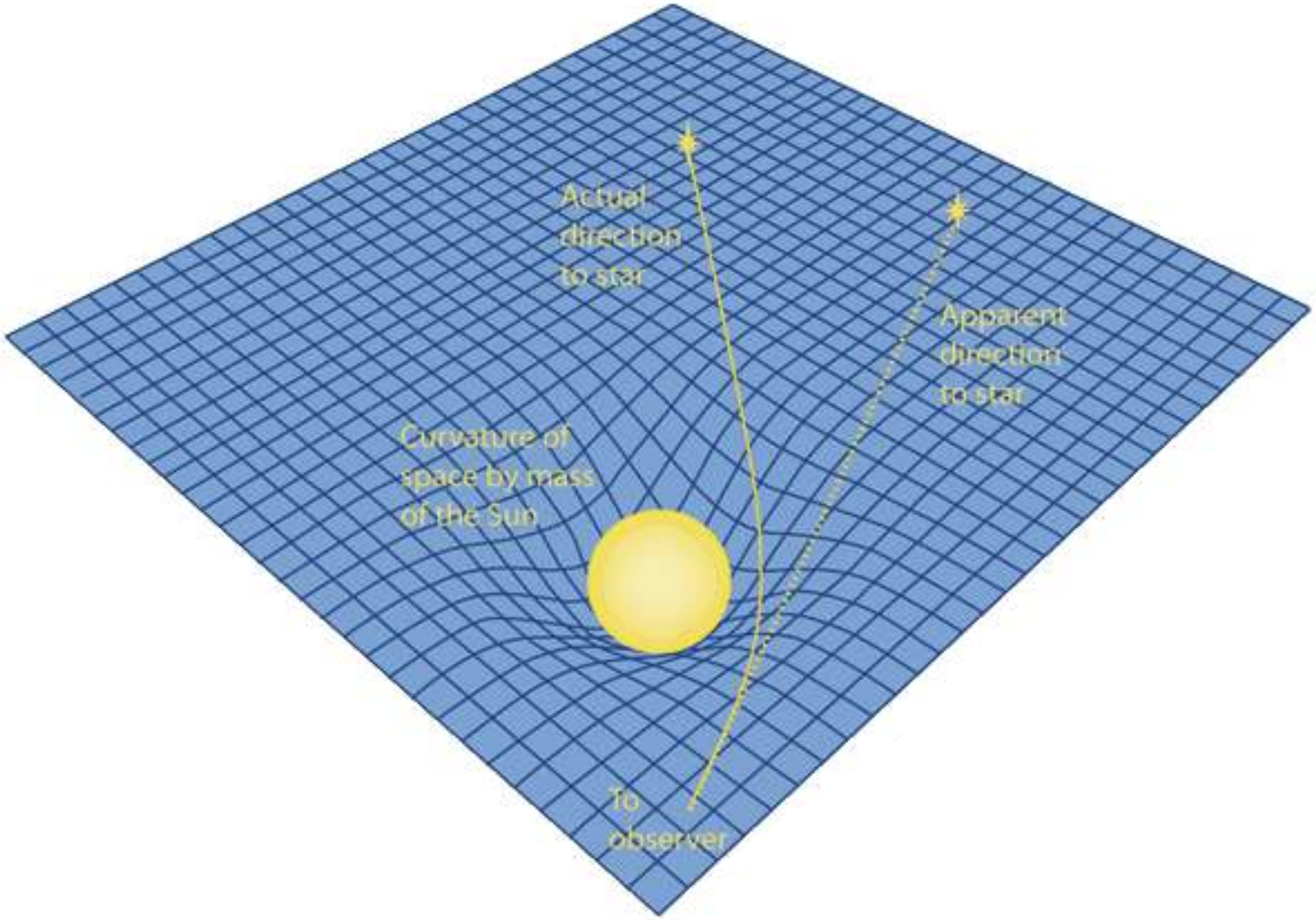


The Equivalence Principle

Einstein understood like this…

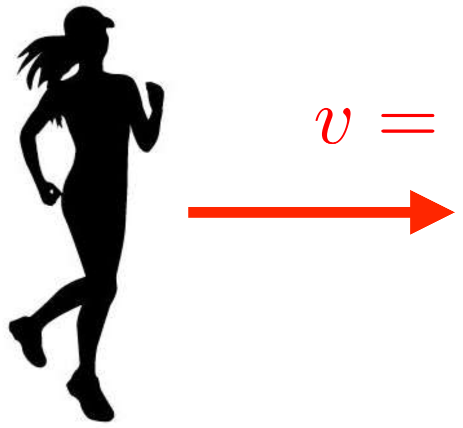
a mathematical formulation was needed !

…

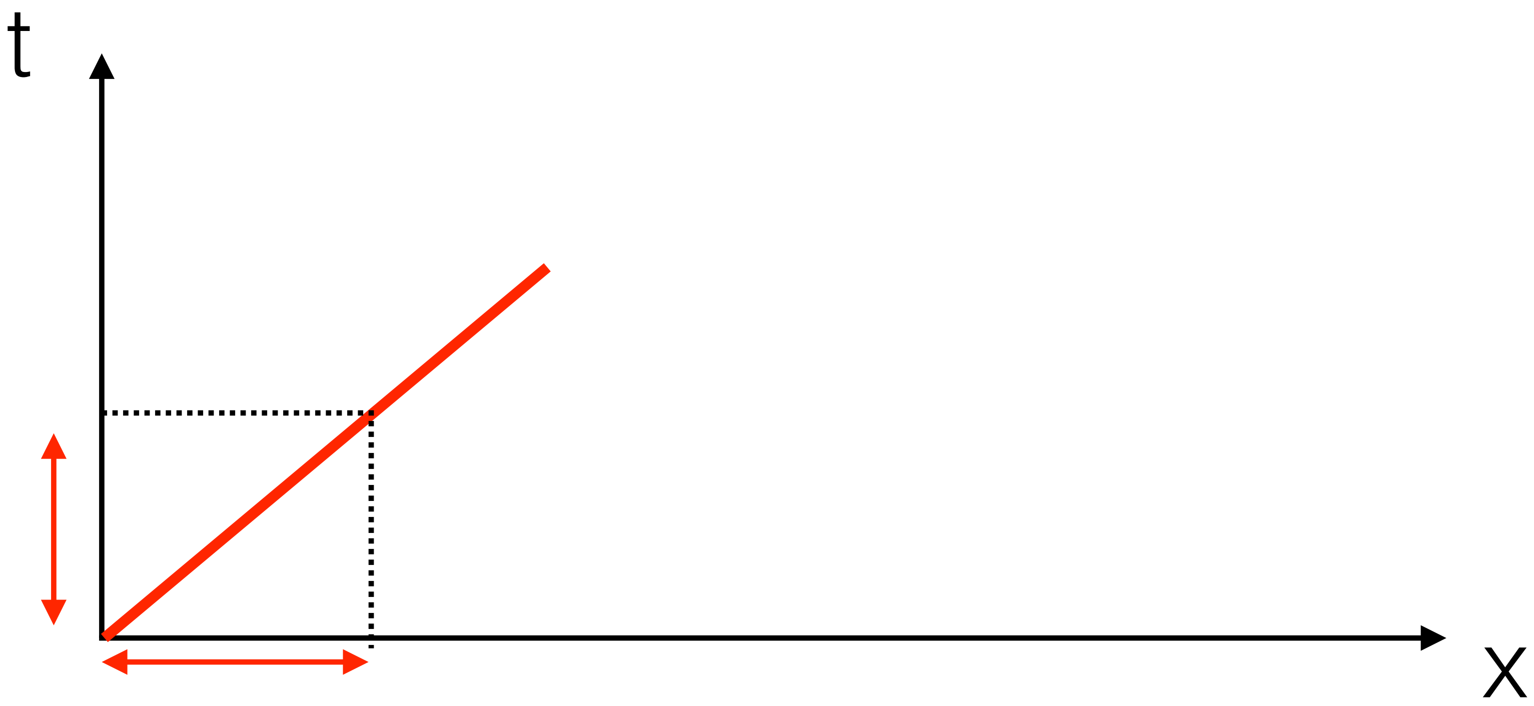


Mathematical formulation of General Relativity (GR)

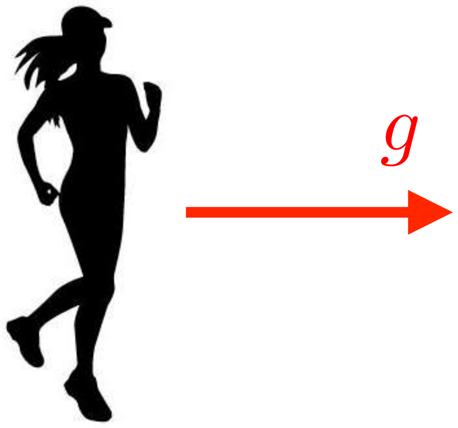
General Relativity Equations

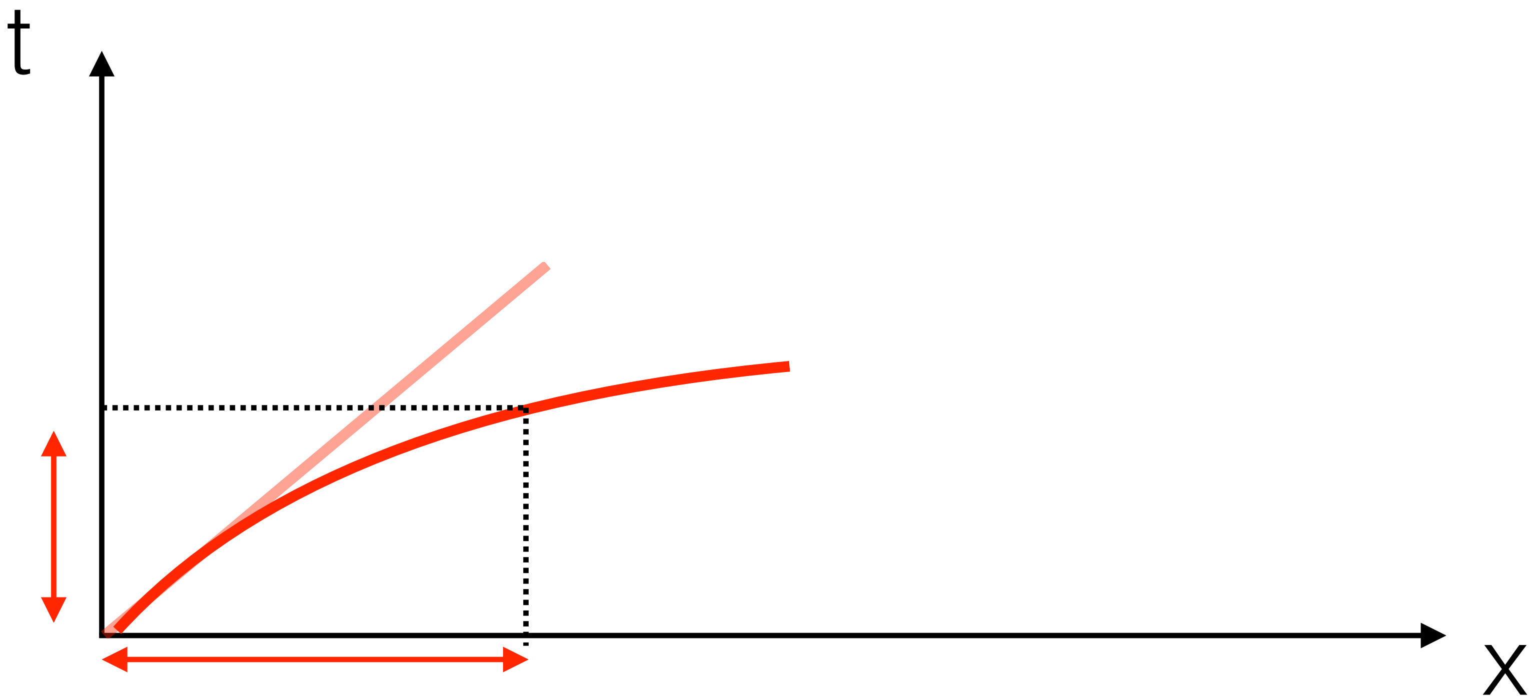


$v = const.$

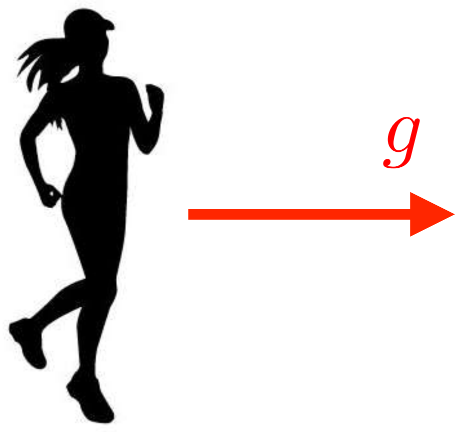


General Relativity Equations

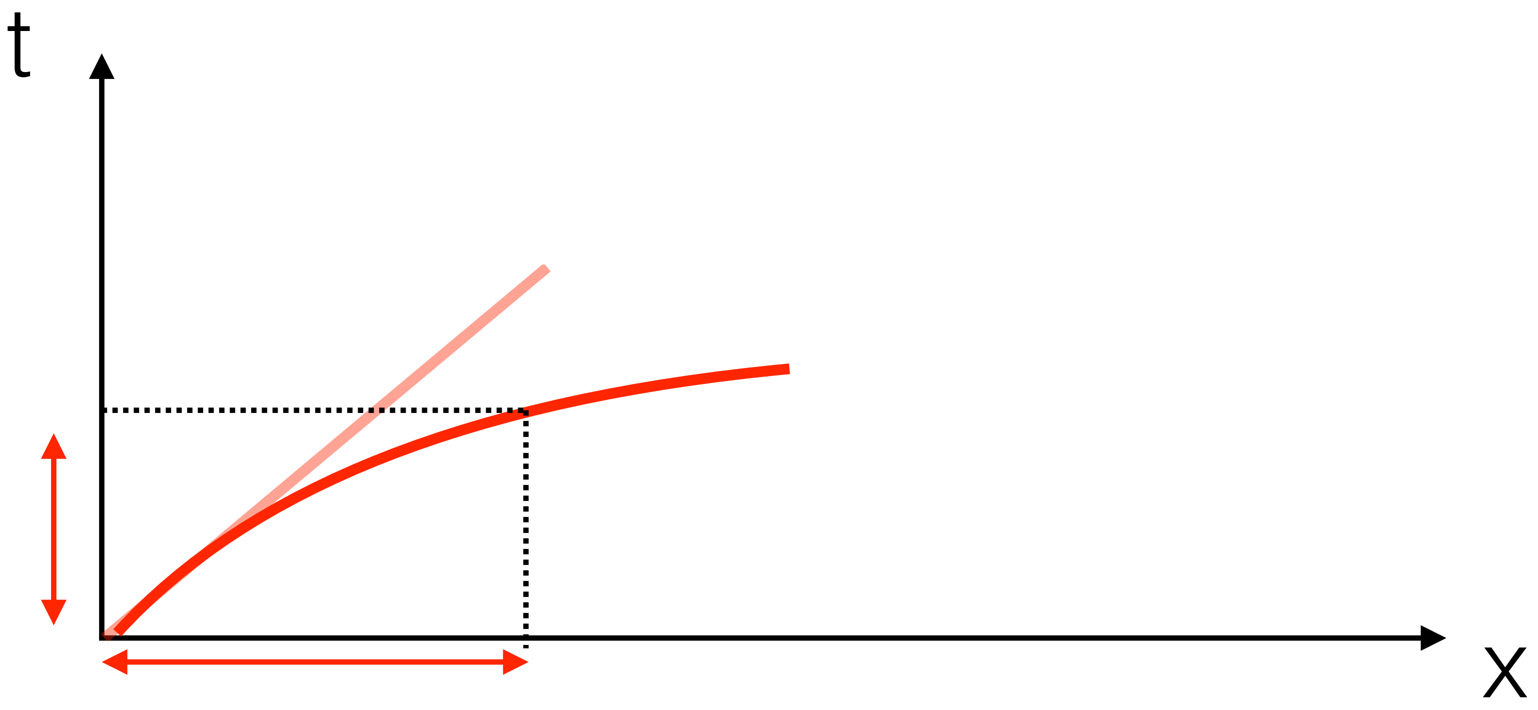




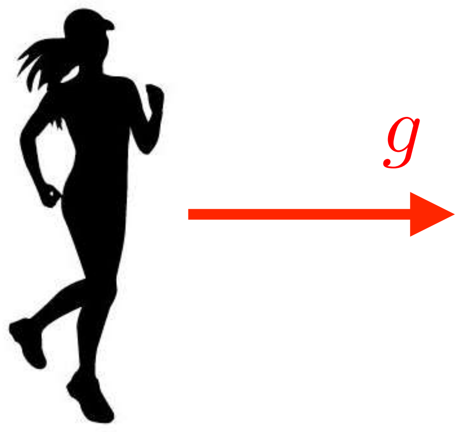
General Relativity Equations



Acceleration $\equiv$ Gravity



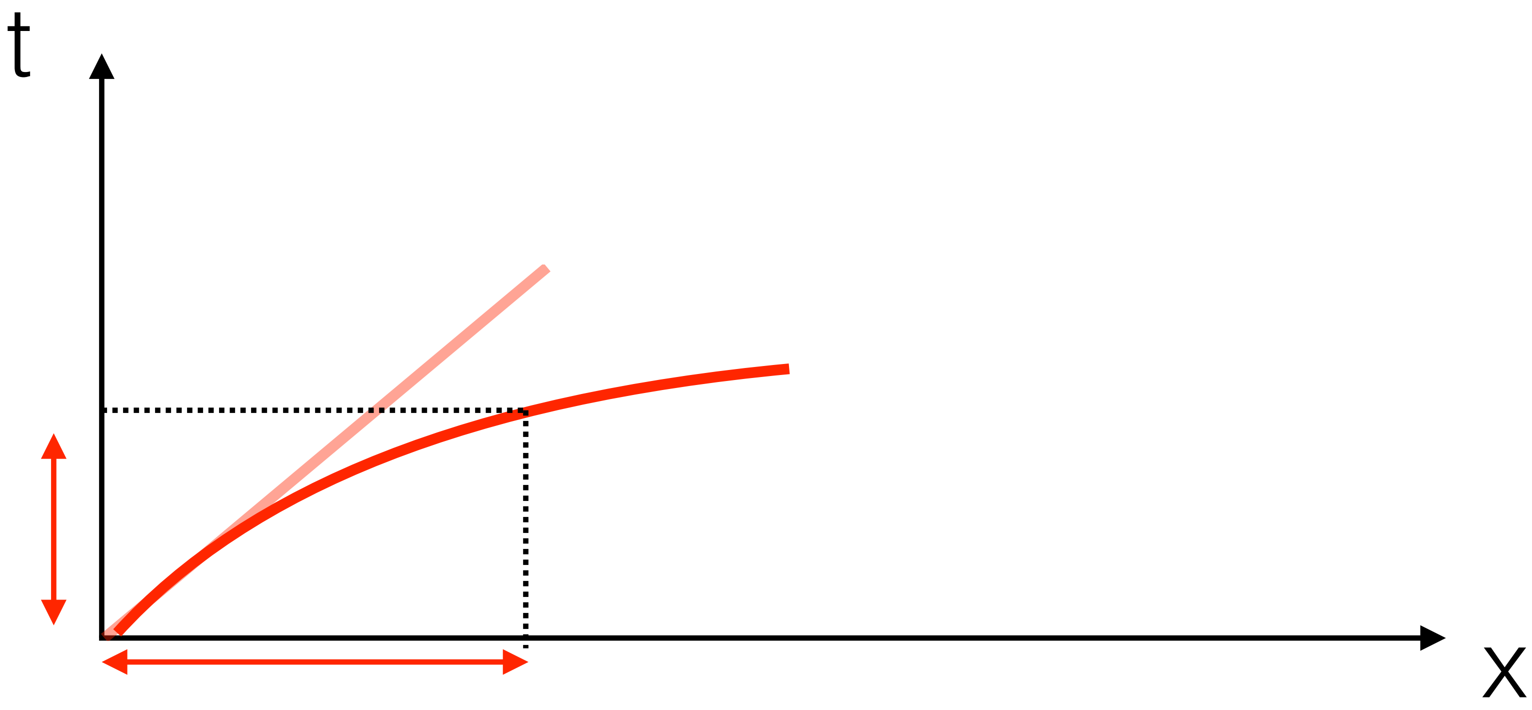
General Relativity Equations



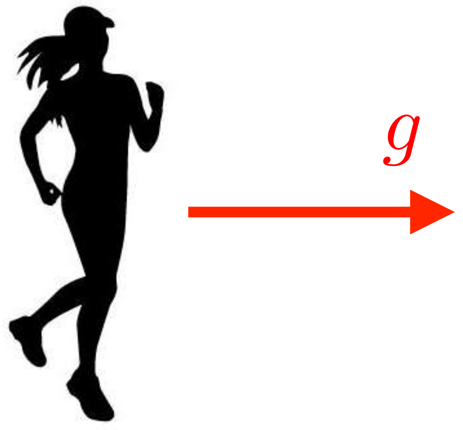
Acceleration $\equiv$ Gravity

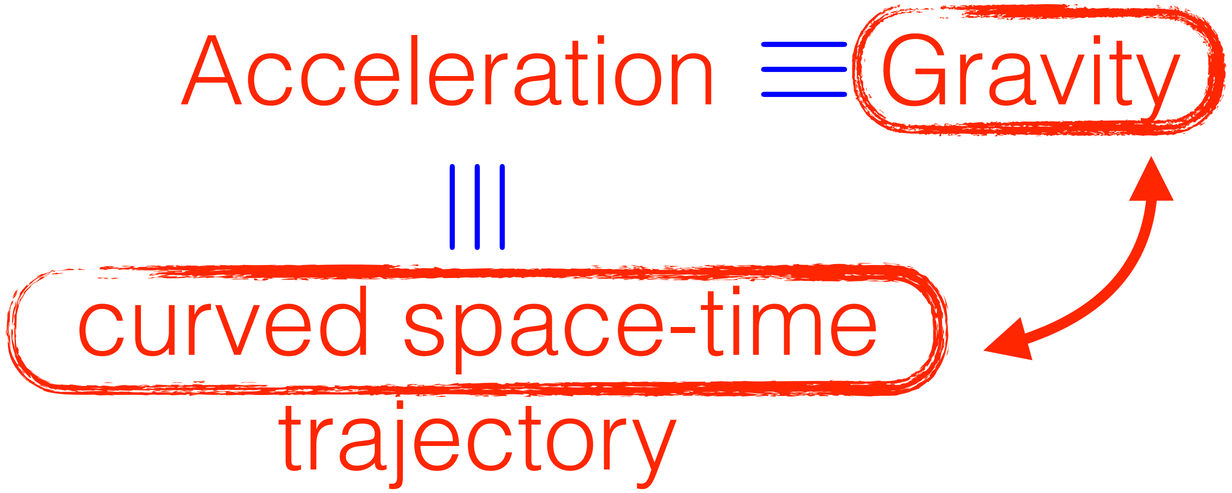
⦀

curved space-time
trajectory



General Relativity Equations



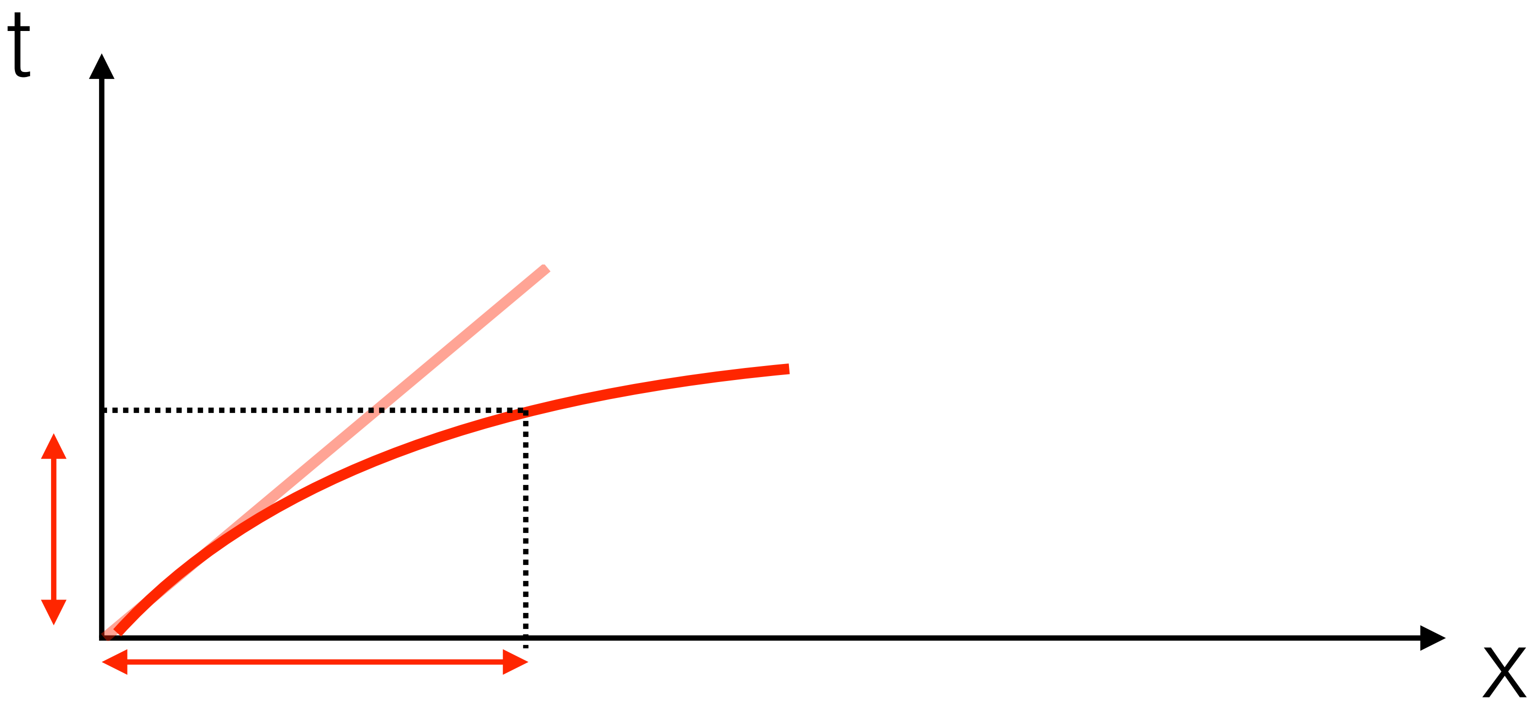


Acceleration ≡ Gravity

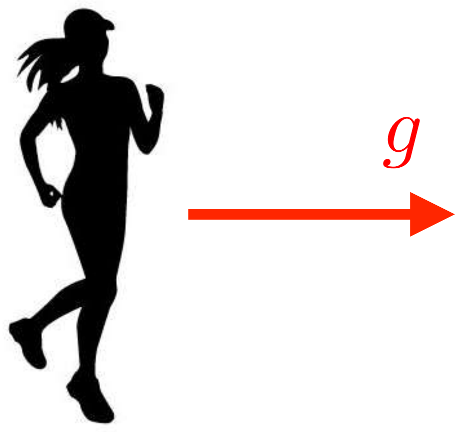
≡

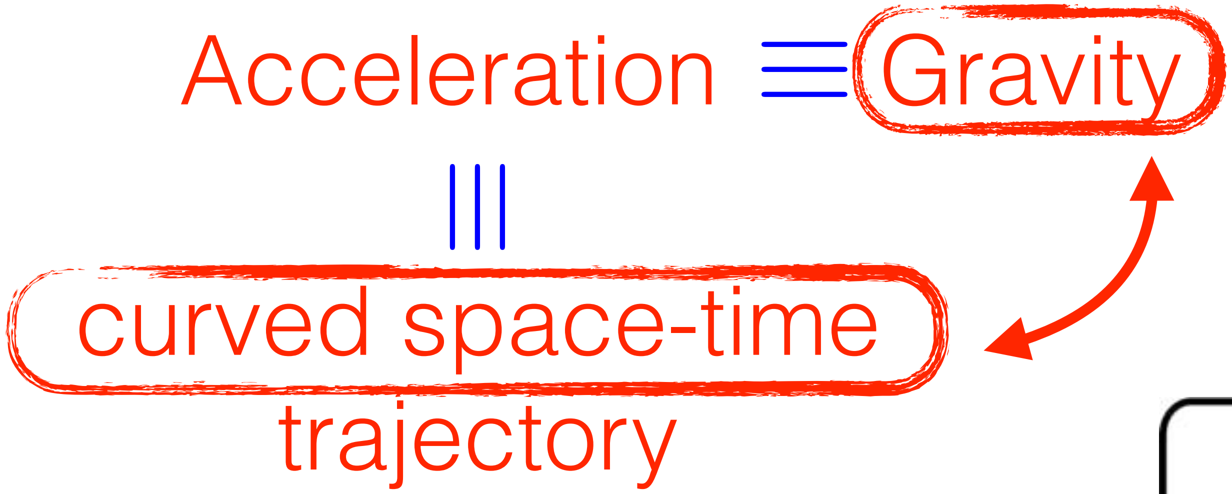
curved space-time

trajectory



General Relativity Equations



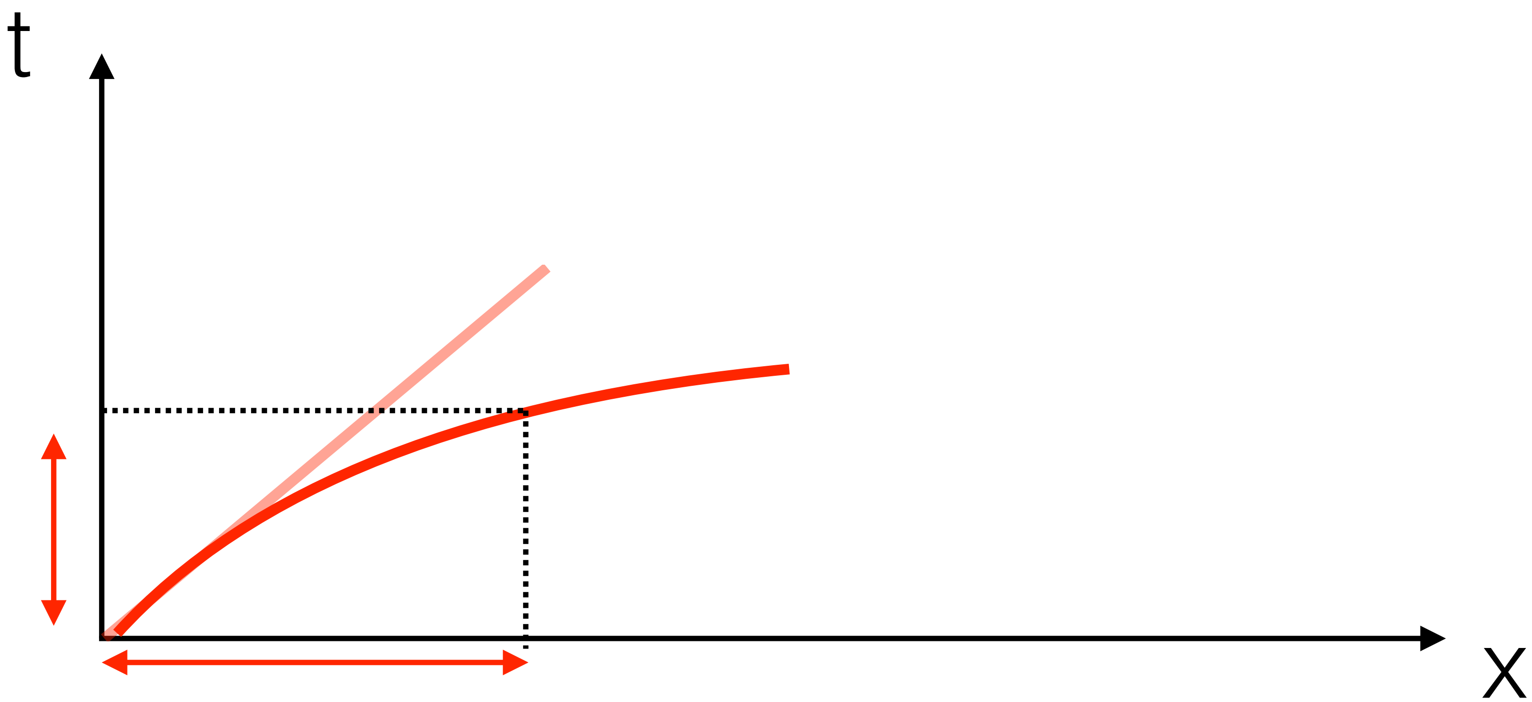


Acceleration $\equiv$ Gravity

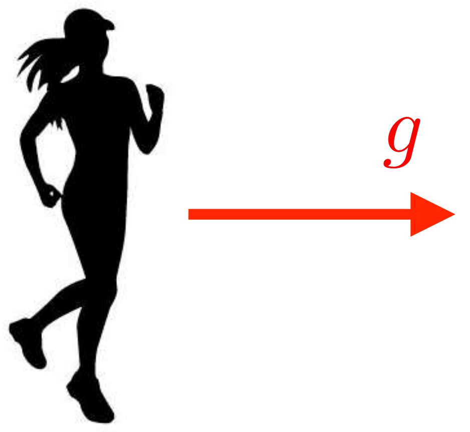
$\equiv$

curved space-time

trajectory



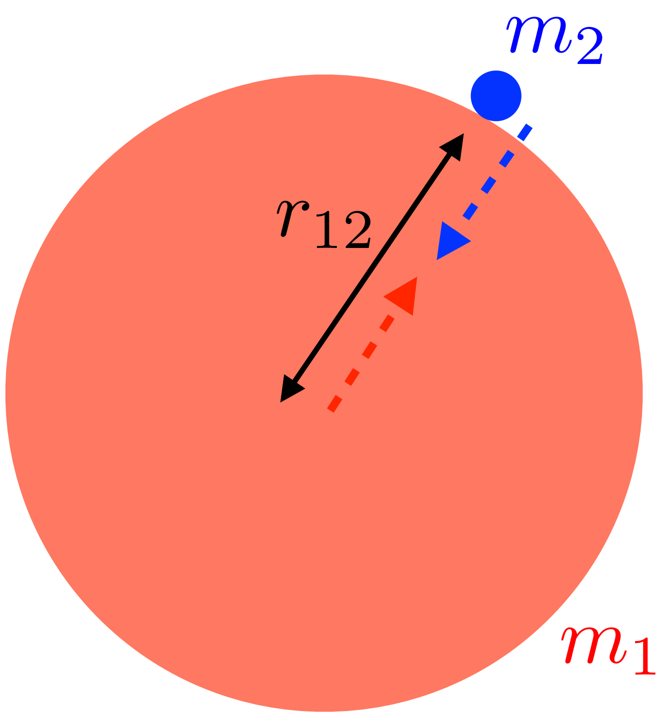
General Relativity Equations



Acceleration $\equiv$ Gravity

$\equiv$

curved space-time trajectory

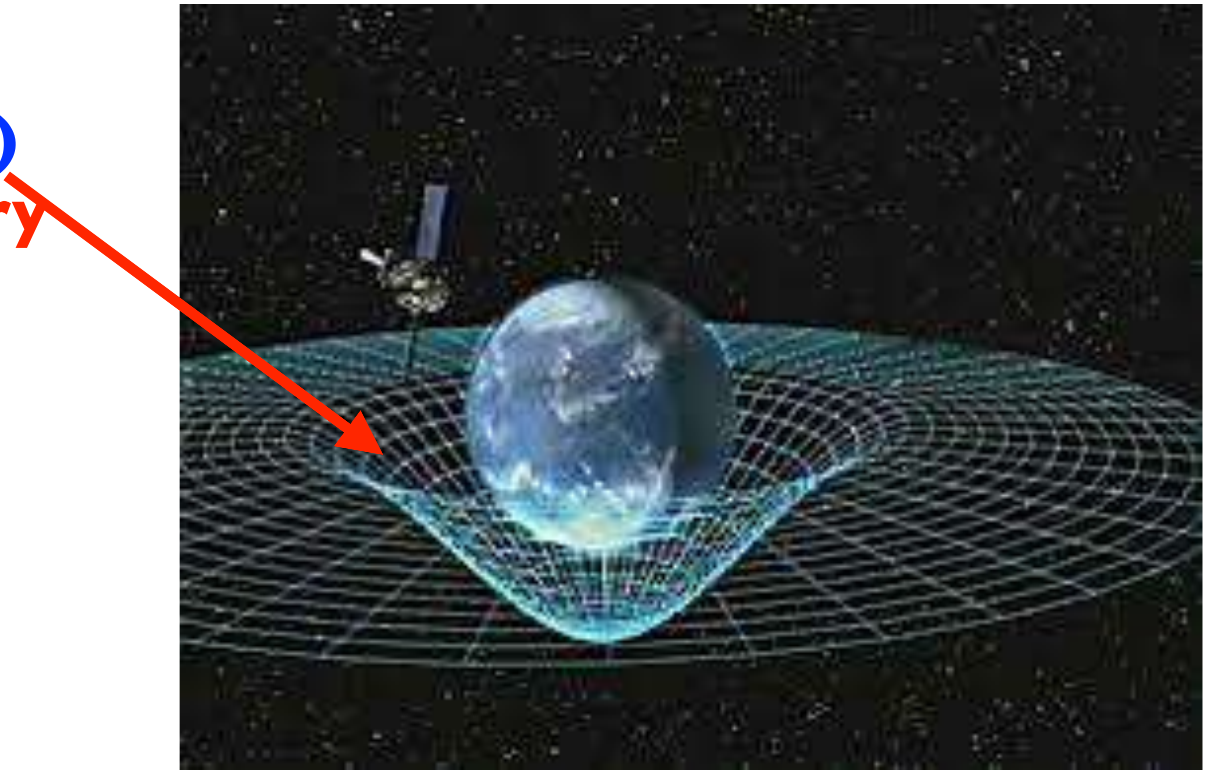


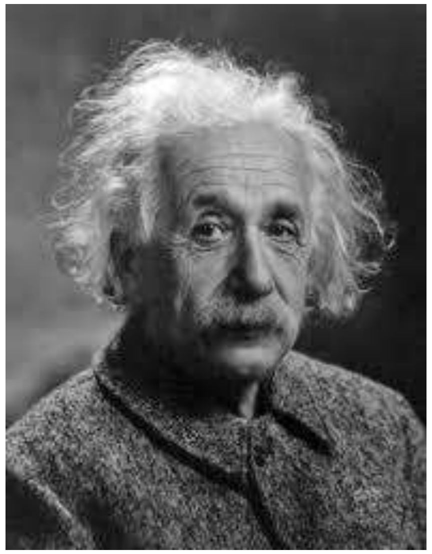
$$F = G\frac{m_1 m_2}{r_{12}^2} = m_2 \cdot g_{\oplus}$$

$$g_{\oplus} = 9.81\ m/s^2$$

General Relativity Equations

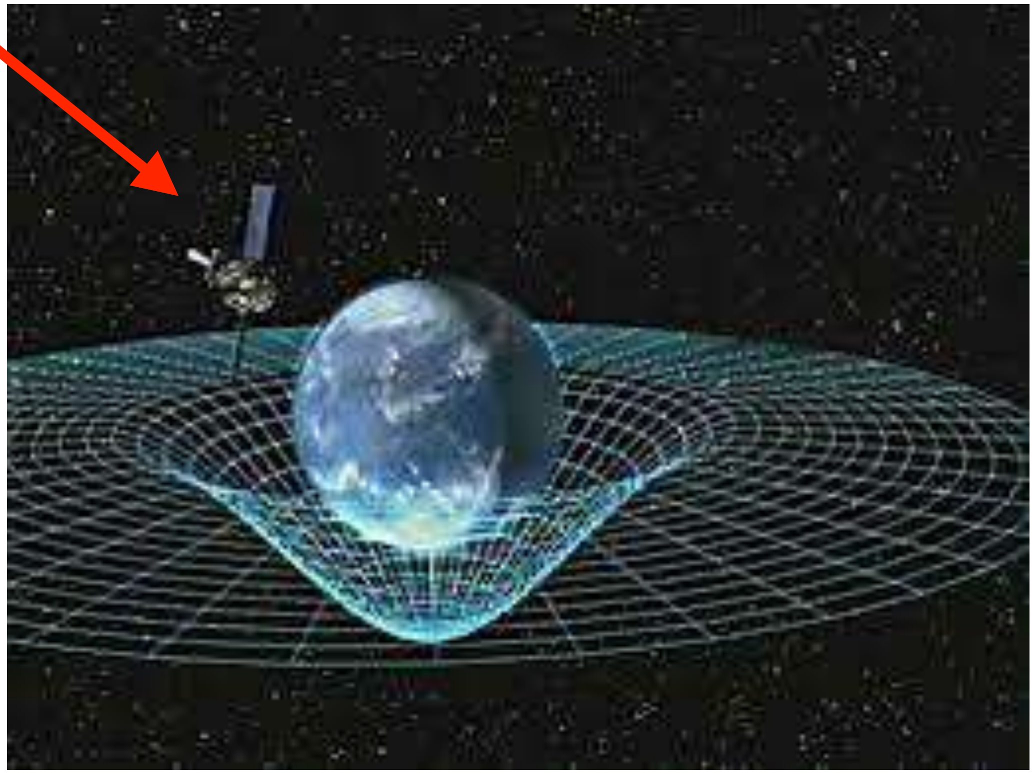
Presence of Matter (Energy/p)
dictates 'Space-Time' Geometry

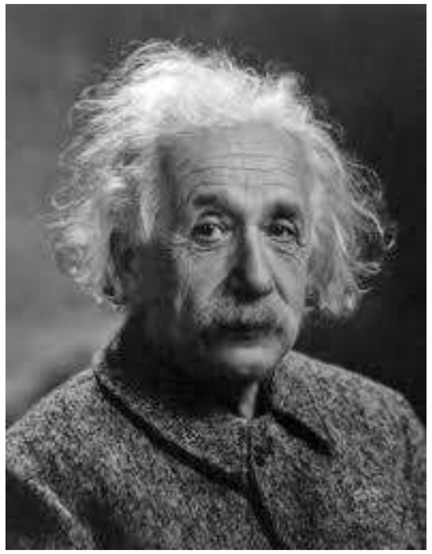




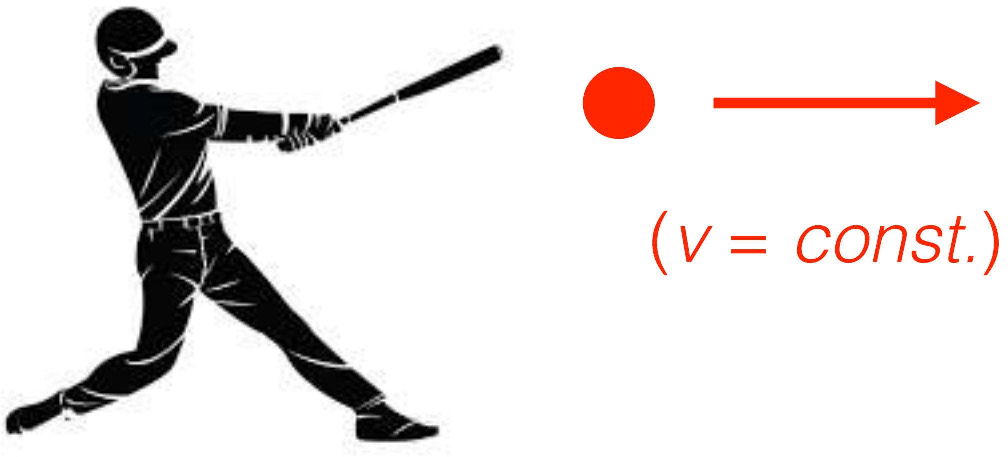
General Relativity Equations

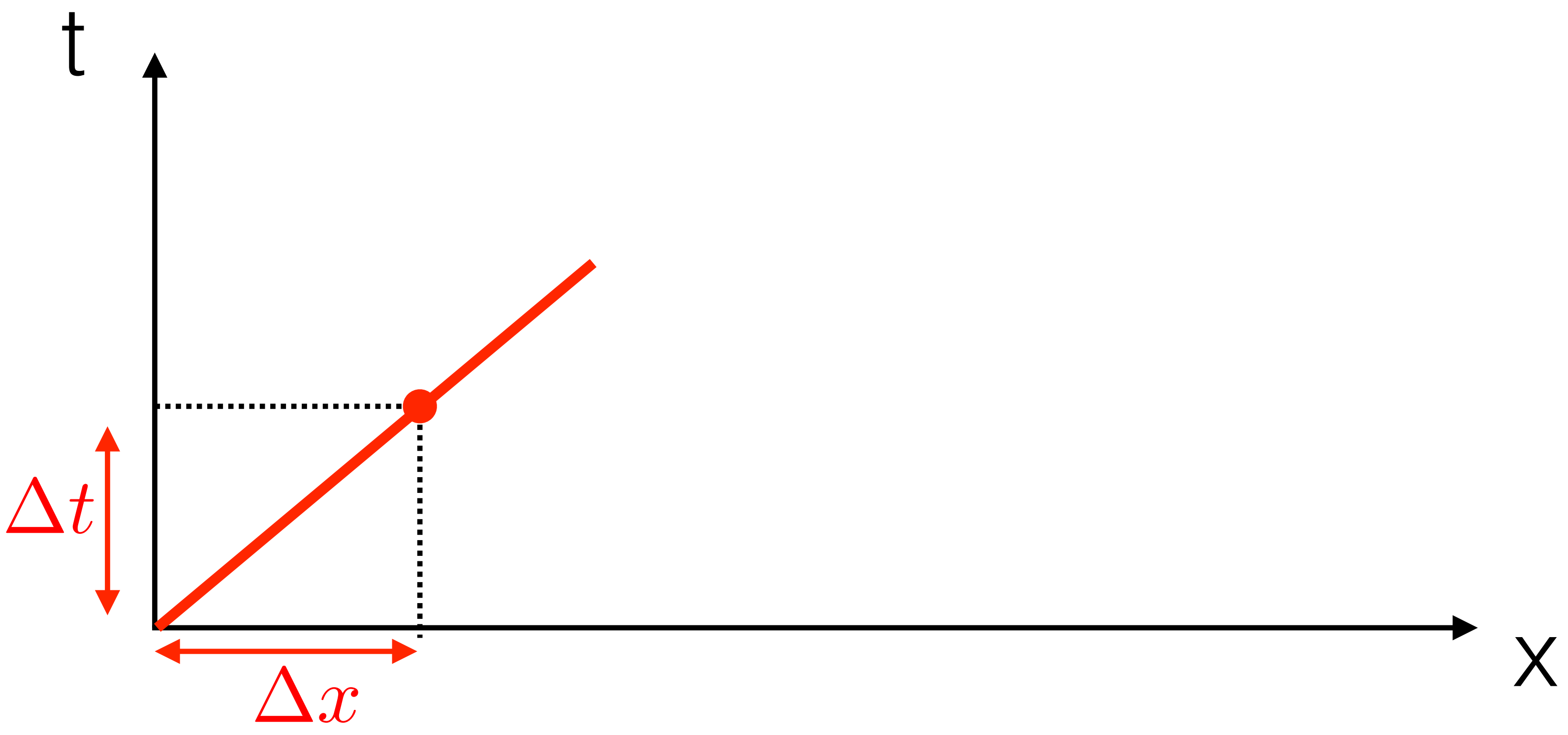
‘Space-Time' Geometry
dictates Movement of Matter



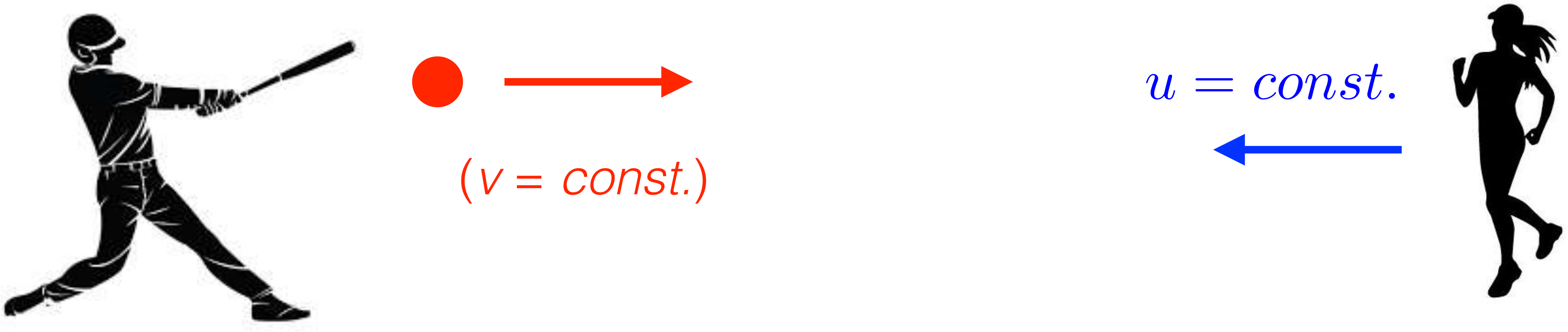


General Relativity Equations



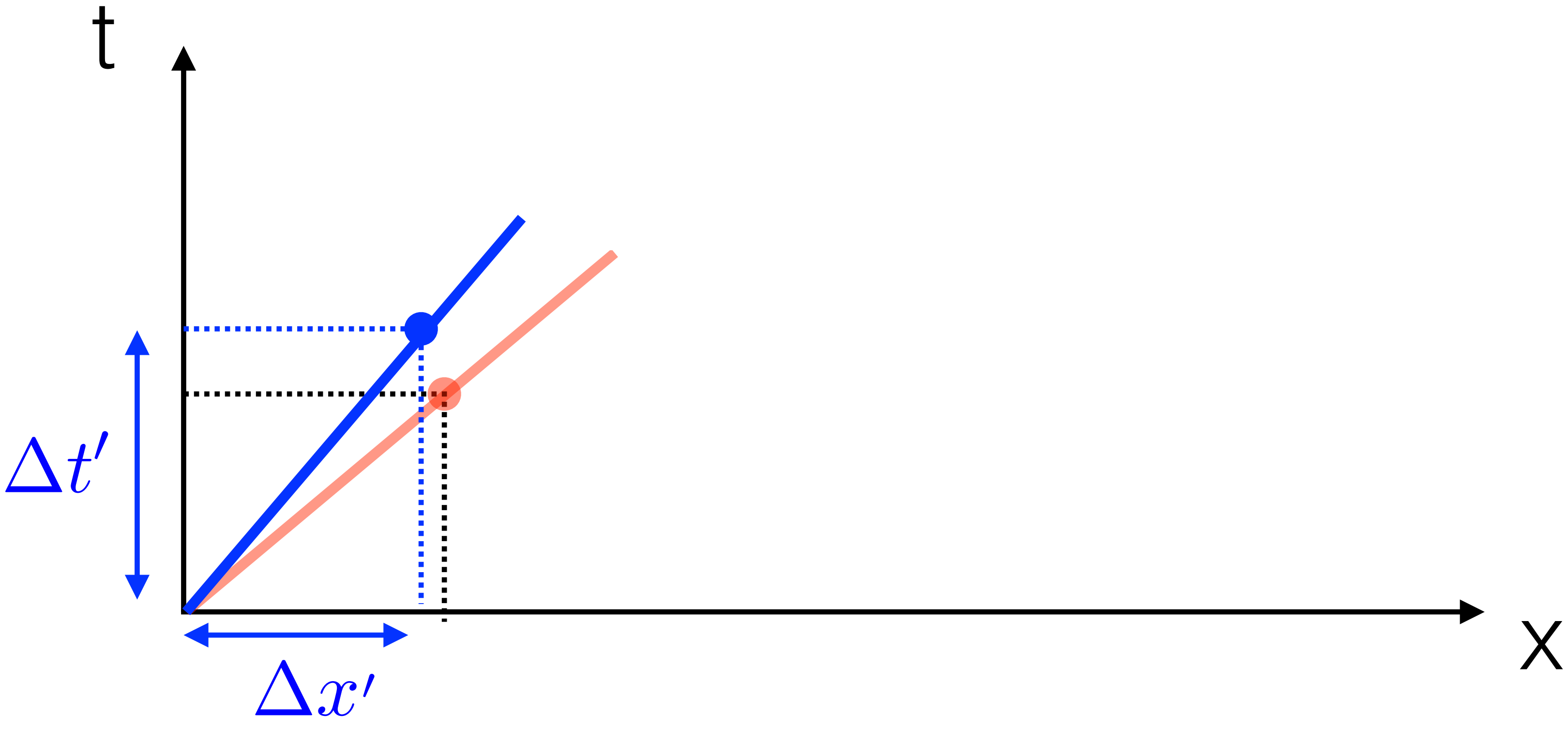


General Relativity Equations

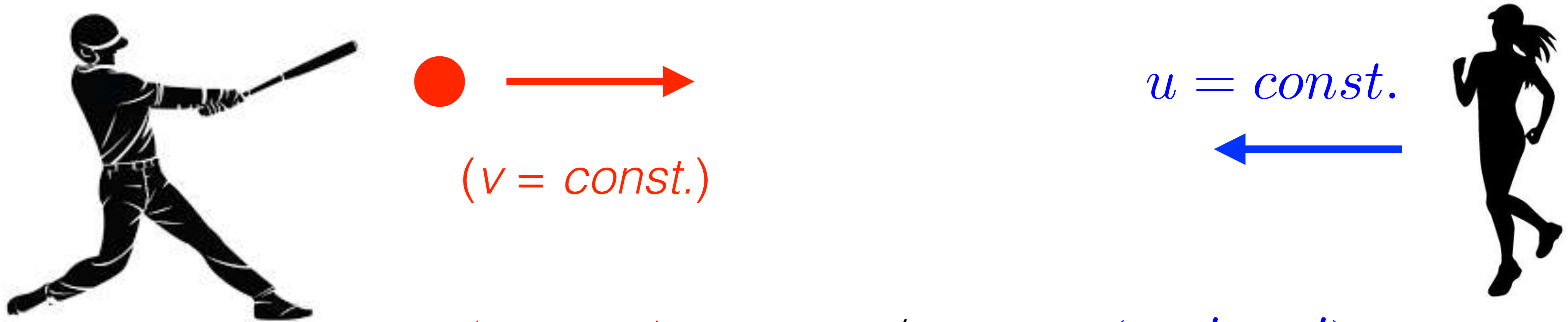


$(v = const.)$

$u = const.$



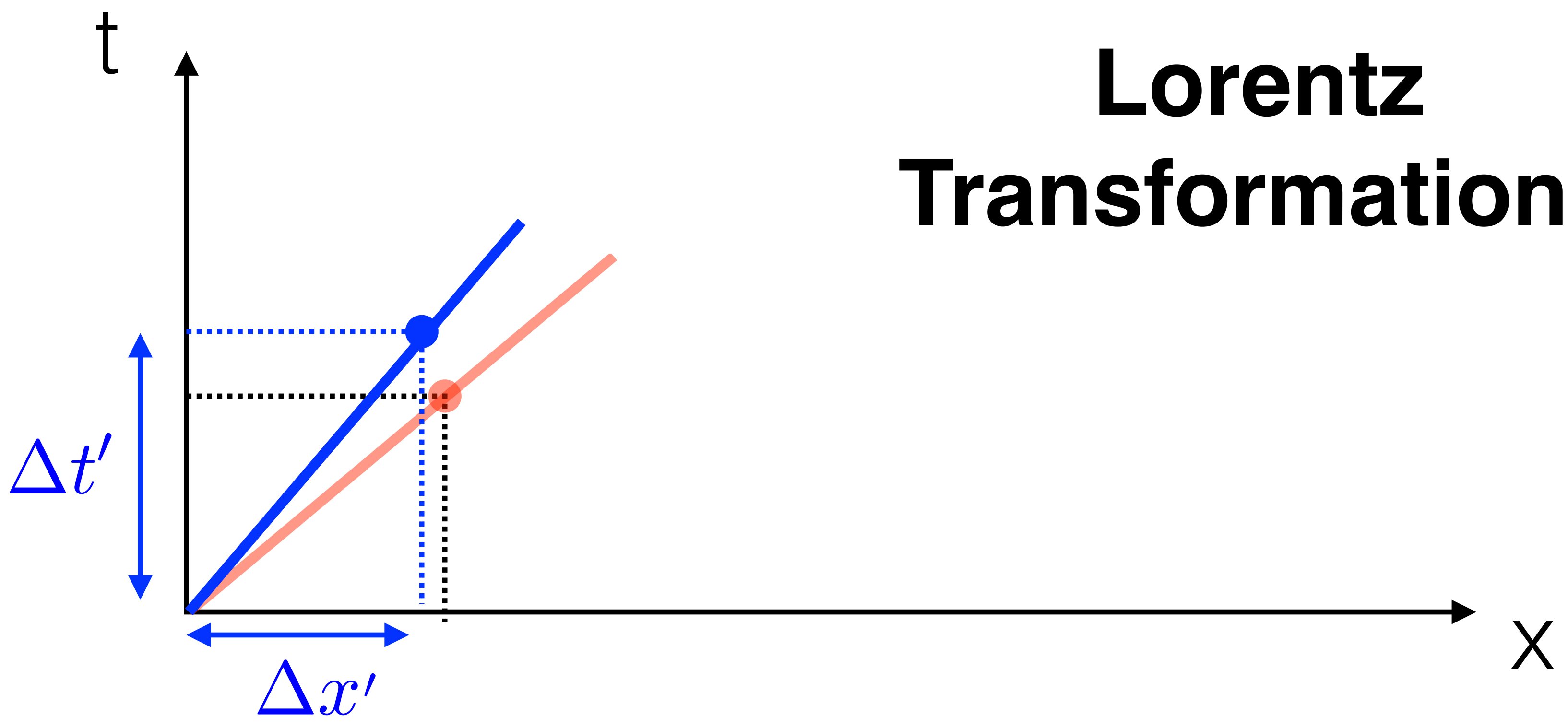
General Relativity Equations



$u = const.$

$(v = const.)$

$$(\Delta x, \Delta t) \neq (\Delta x', \Delta t')$$



Lorentz Transformations

General Relativity Equations

$u = const.$

$$(\Delta x, \Delta t) \qquad \neq \qquad (\Delta x', \Delta t')$$

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 \qquad = \qquad s'^2 = (c\Delta t')^2 - (\Delta x')^2$$

General Relativity Equations


$u = const.$

$(\Delta x, \Delta t) \qquad \neq \qquad (\Delta x', \Delta t')$

$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 \qquad = \qquad s'^2 = (c\Delta t')^2 - (\Delta x')^2$

$$\boxed{ds^2 = c^2 dt^2 - \sum_j dx_j dx^j}$$

Special Relativity

Space-time interval invariant

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j$$

Space-time invariant
interval (**Special Relativity**)

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation
over repeated
indices

Space-time invariant
interval (**Special Relativity**)

$$\eta \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation
over repeated
indices

Space-time invariant
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Minkowski Metric

$\eta \equiv \mathrm{diag}(-,+,+,+)$

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$$\longrightarrow \boxed{ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu}$$

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Space-time invariant interval (**Special Relativity**)

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$$\eta \equiv \text{diag}(-,+,+,+)$$

Space-time invariant interval (**General Relativity**) ⟶

$$\boxed{ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu}$$

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g'_{\alpha\beta}(x')dx'^\alpha dx'^\beta$$

General Relativity Equations

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$$ds^2 = -c^2dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu}$$

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General Relativity Equations

General Relativity: Generalisation of Special Relativity

General Relativity Equations

General Relativity: Generalisation of Special Relativity

* **Equivalence Principle** ⟶ Geodesic motion

$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}[g_{**}]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0}$$

$${\Gamma^\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right)$$

Christoffel Symbol

General Relativity Equations

General Relativity: Generalisation of Special Relativity

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$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}[g_{**}]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0}$$

* **Principle of Relativity** ⟶ $x'^\mu = x'^\mu(\{x^\alpha\})$ Arbitrary change of coordinates

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g'_{\alpha\beta}(x')dx'^\alpha dx'^\beta$$

;

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{dx^\mu}\frac{\partial x'^\beta}{dx^\nu}g'_{\alpha\beta}(x')$$

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space-time geometry

matter (energy/p)

General Relativity Equations

General Relativity: Generalisation of Special Relativity

$$x'^{\mu} = x'^{\mu}(\{x^{\alpha}\})$$

Arbitrary change of coordinates

;

$$g_{\mu\nu}(x)dx^{\mu}dx^{\nu} = g'_{\alpha\beta}(x')dx'^{\alpha}dx'^{\beta}$$

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General Relativity Equations

General Relativity: Generalisation of Special Relativity

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Arbitrary
change of
coordinates

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; $\frac{8\pi G}{c^4} \equiv \frac{1}{m_p^2}$; $m_p = 2.44 \cdot 10^{18}\ \mathrm{GeV}$

$$R_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\lambda\beta}\Gamma^{\lambda}{}_{\alpha\mu}$$

Ricci tensor

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right) \sim (metric)^2$$

Christoffel Symbol

General Relativity Equations

General Relativity (GR)

$$\underset{\text{geometry}}{G_{\mu\nu}} = \frac{1}{m_p^2}\underset{\text{matter}}{T_{\mu\nu}}$$

$$G_{\mu\nu} \equiv \underset{\substack{\downarrow \\ \text{2nd order, }\underline{\text{non-Linear}}}}{\mathcal{D}}[\overset{\substack{\text{metric} \\ \uparrow}}{g_{\alpha\beta}}] \quad = \quad \underset{\text{source}}{m_p^{-2} T_{\mu\nu}(\mathrm{Matt, Rad, Top.Defects, DarkEnergy}, ...)}$$

General Relativity Equations

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↓

Extremely difficult to solve !

General Relativity Equations

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$$\underset{\text{geometry}}{G_{\mu\nu}} = \frac{1}{m_p^2} \underset{\text{matter}}{T_{\mu\nu}}$$

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One example: Cosmology

$$\begin{tikzcd}
 & T^{[U]}_{\mu\nu} \arrow[dd, bend left] \\
\textbf{Universe ?} \arrow[ur] \arrow[dr] & \\
 & g^{[U]}_{\mu\nu}
\end{tikzcd}$$

One example: Cosmology

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry of
the Universe

matter within
the Universe

One example: Cosmology

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu} \longrightarrow$$

geometry of the Universe

matter within the Universe

Principle of Symmetry:

The Universe is Homogeneous & Isotropic

Cosmological Principle

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geometry of the Universe

matter within the Universe

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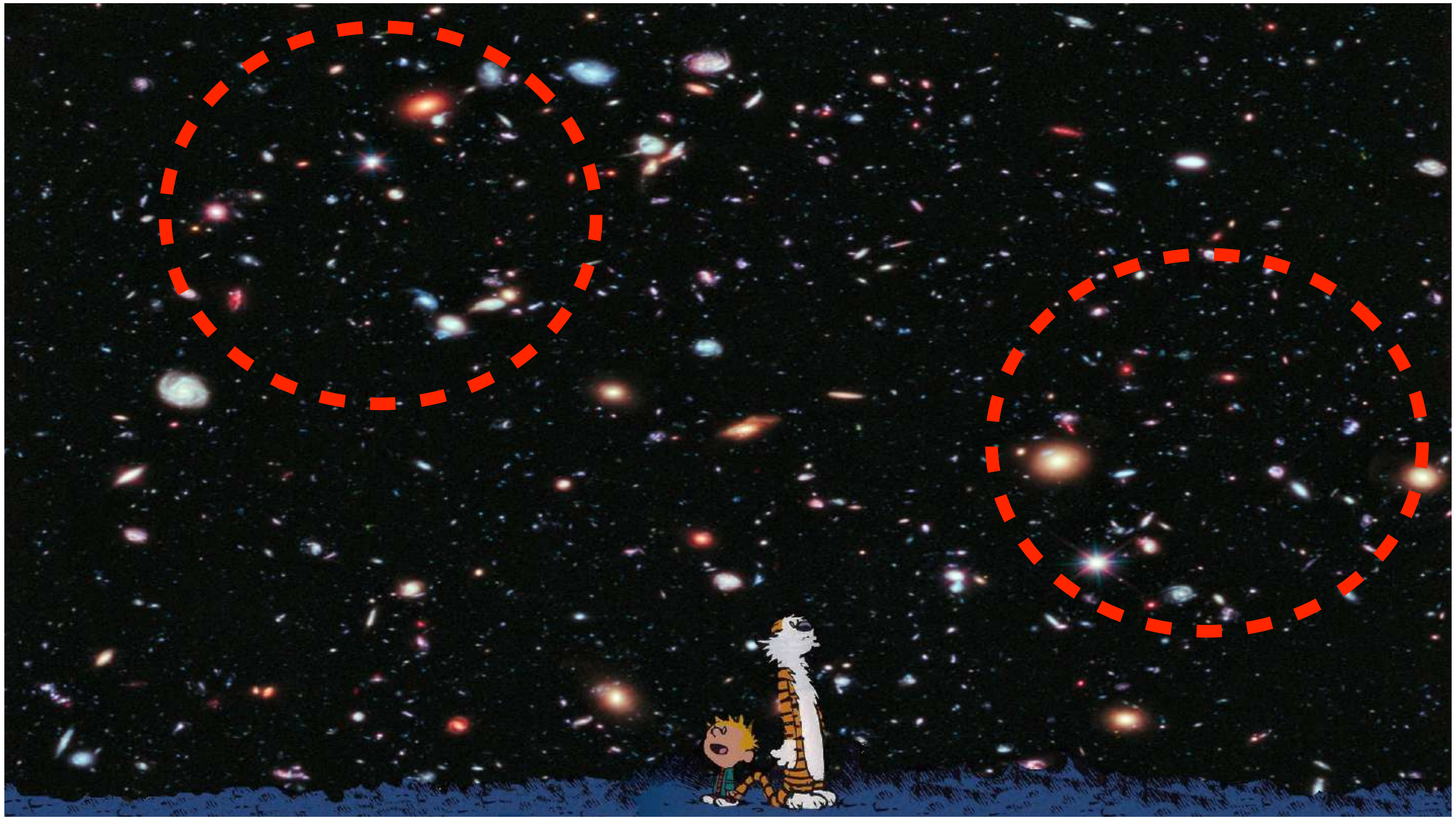

Cosmological Principle

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu} \longrightarrow$$

geometry of the Universe

matter within the Universe

Principle of Symmetry:

The Universe is Homogeneous & Isotropic

$$g_{\mu\nu}^{[U]} \equiv \mathrm{diag}\left(-1, \frac{a^2(t)}{1-kr^2}, a^2(t)r^2, a^2(t)r^2\sin^2\theta\right)$$

FLRW

Friedmann-Lemaître -Robertson-Walker

Cosmological Principle

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry of the Universe

matter within the Universe

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$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \boxed{a^2(t)}\left\{\frac{dr^2}{1-\boxed{k}r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$

Scale Factor
(dynamical)

Curvature
(const.)

$$\begin{cases} k < 0, \mathrm{Open} \\ k = 0, \mathrm{Flat} \\ k > 0, \mathrm{Close} \end{cases}$$

Cosmological Principle

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu} \longrightarrow$$

geometry of the Universe

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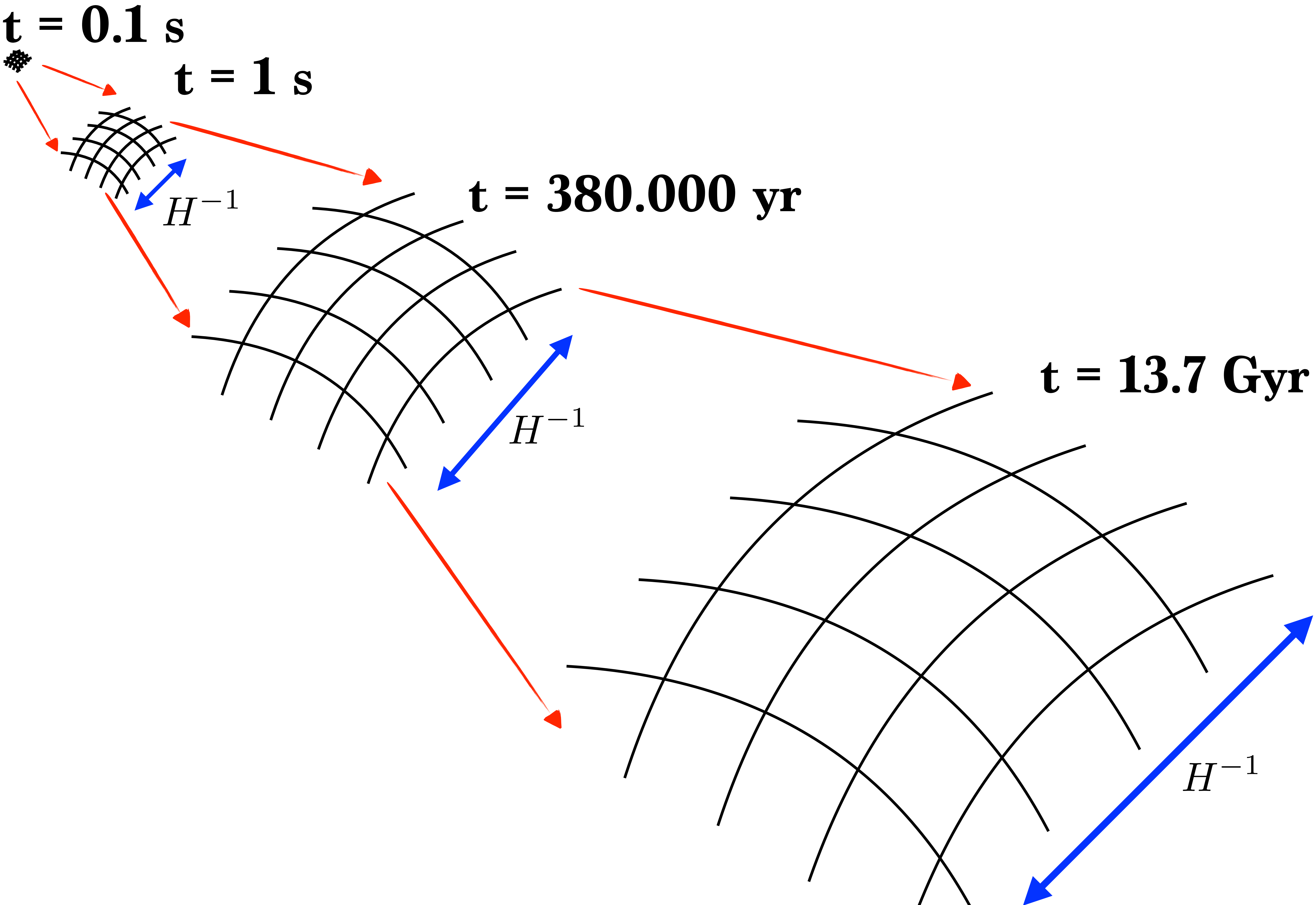
invariant:
$(c = \mathrm{const.})$

$$\left\{\begin{array}{l} k \to k/c^2 \\ r \to c \cdot r \\ a \to a/c \end{array}\right. \Longrightarrow \left\{\begin{array}{ll} a, r, k & \text{unphysical} \\ \frac{k}{a^2},\ a \cdot r,\ kr^2 & \text{physical} \end{array}\right.$$

Expanding Universe



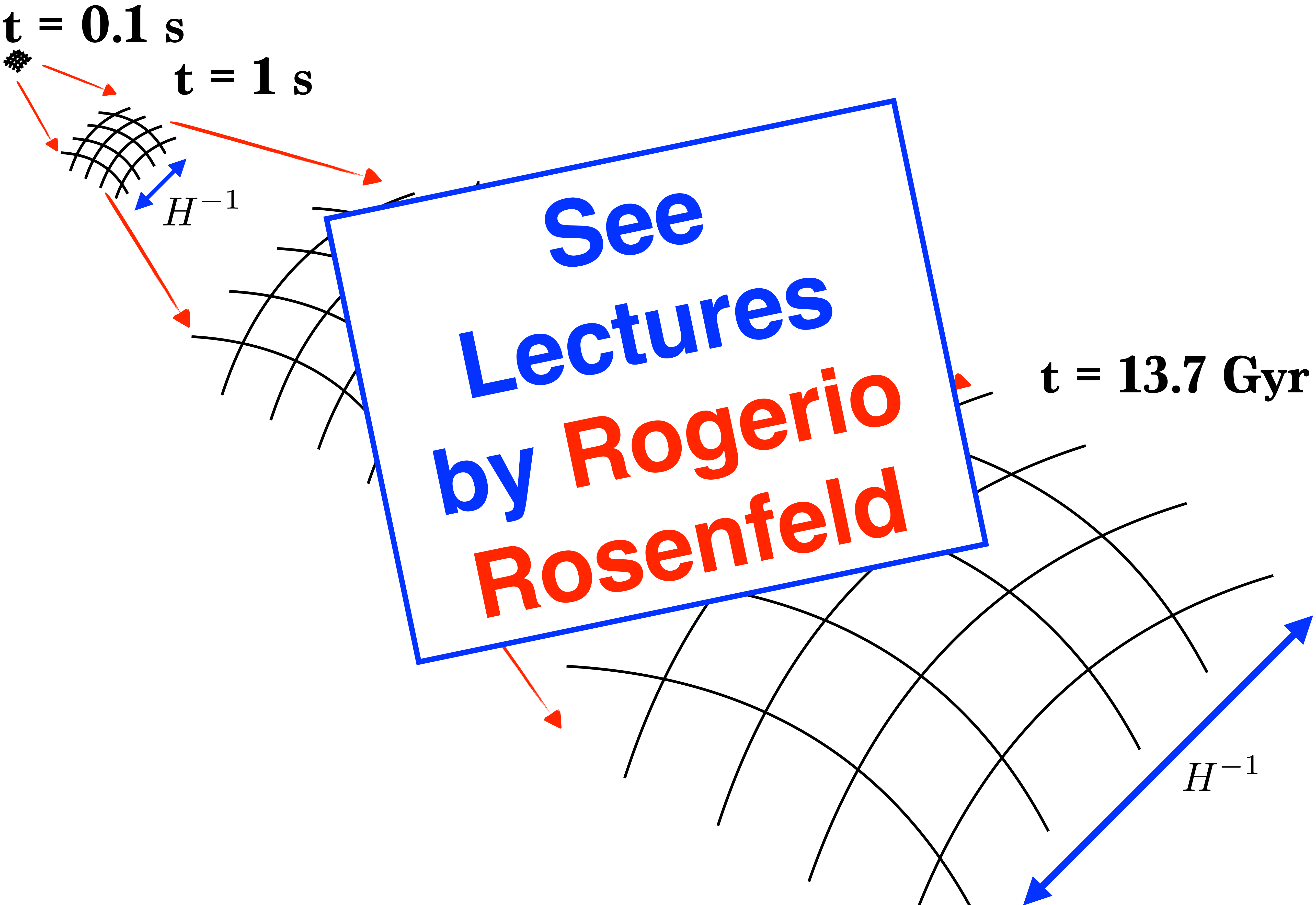
t = 0.1 s

t = 1 s

t = 380.000 yr

t = 13.7 Gyr

Expanding Universe



See Lectures by Rogerio Rosenfeld

Cosmological Principle

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry of the Universe

matter within the Universe

→

Principle of Symmetry:

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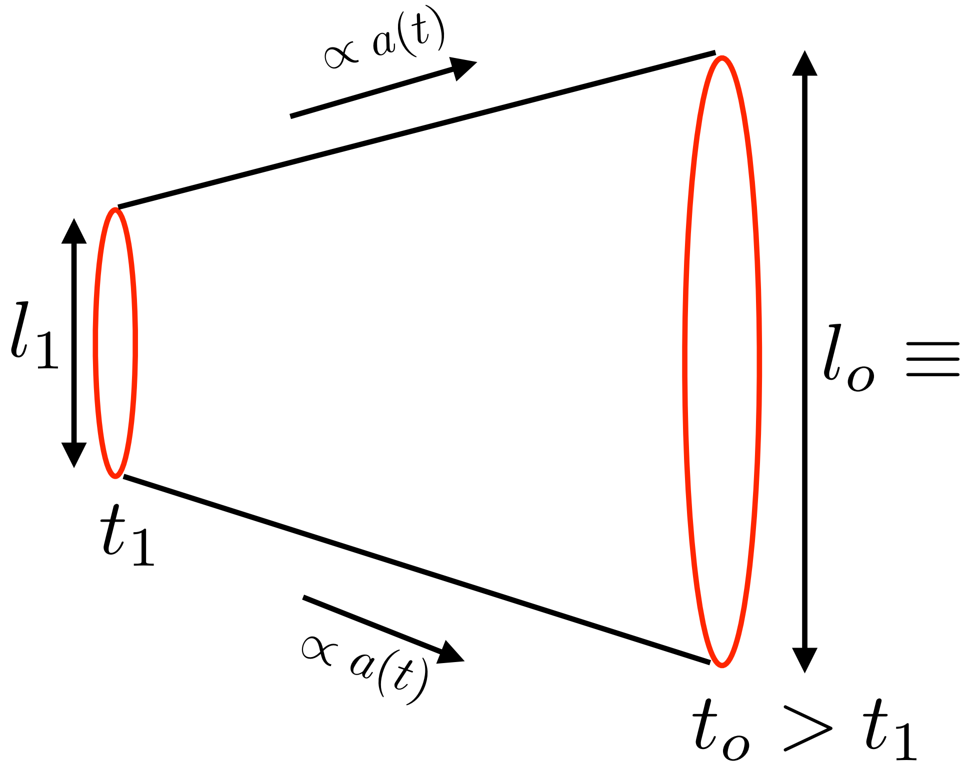


$$l_o \equiv \left(\frac{a(t_o)}{a(t_1)} \right) l_1$$

Redshift

$$z_1 \equiv \frac{a_o - a_1}{a_1}$$

$$\boxed{1 + z \equiv \frac{a(t_o)}{a(t)}}$$

END of digression on GENERAL RELATIVITY

)

Let’s continue with

PRIMER ON GRAVITATIONAL WAVES

Gravitational Framework

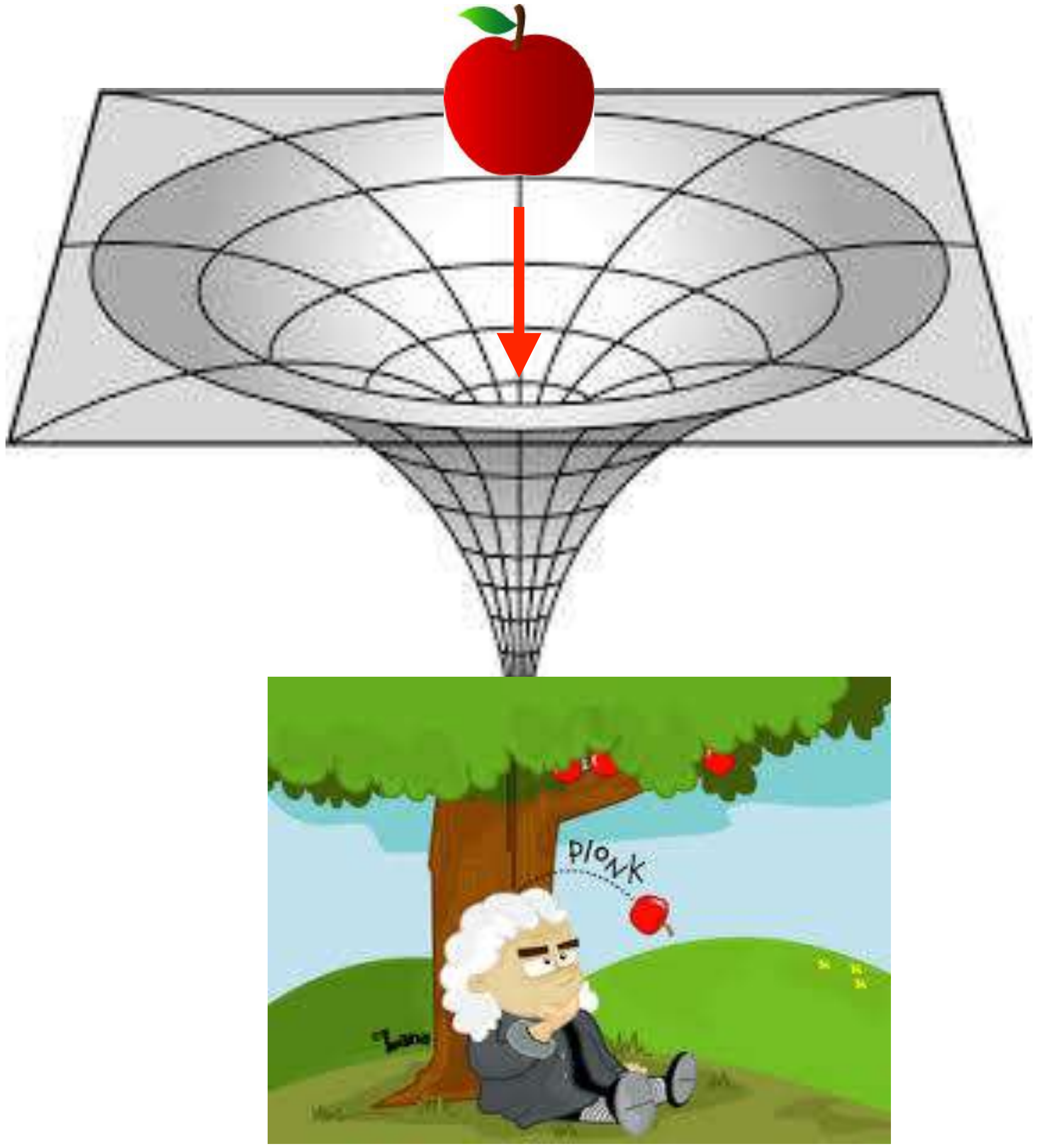
General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass



$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\mathrm{DIFF}: \quad x^\mu \rightarrow x'^\mu(x)$$

symmetry

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How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[\overset{\uparrow}{g_{\alpha\beta}}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, ...)$$

$\downarrow$

source

expand in perturbations

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$\underbrace{G_{\mu\nu}}_{\text{geometry}} = \frac{1}{m_p^2}\underbrace{T_{\mu\nu}}_{\text{matter}}$$

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↓

expand in perturbations

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

source of GWs

How do we define GWs ?

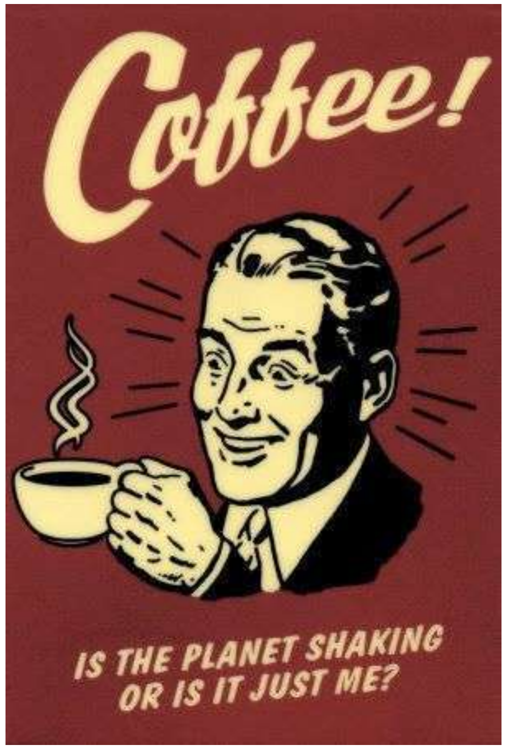
$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

Let’s continue this approach…

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

Let's continue this approach…

But not before … coffee breaking!



Definition of GWs
1st approach

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \underset{\text{Minkowski}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Gravitational Wave Definition

LINEARIZED GRAVITY

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Gravitational Wave Definition

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Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski} \\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

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Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \underset{\uparrow}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

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$$\text{DIFF} : \quad x^\mu \rightarrow x'^\mu(x)$$

symmetry **?**

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1st approach to GWs

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$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$\text{DIFF} : \quad x^\mu \xcancel{\to} x'^\mu(x)$$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$\boxed{(\ |\partial_\mu \xi_\nu(x)| \lesssim |h_{\mu\nu}|\ )}$$

residual symm.

$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{[\mu}\xi_{\nu]}}$$

Gravitational Wave Definition

1st approach to GWs

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Let's expand Einstein Equations !

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski} \\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \underset{\uparrow}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

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$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}} \Rightarrow \partial^\alpha\partial_\alpha\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_{(\mu}\bar{h}_{\alpha\nu)} = -\frac{2}{m_p^2}T_{\mu\nu}$$

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

fixed frame

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$$\overset{\text{Trace-reversed}}{\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \textbf{Einstein tensor expansion}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

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1st approach to GWs

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Einstein tensor expansion

residual symm.

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residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

$$\Rightarrow \quad \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})} = -\frac{2}{m_p^2} T_{\mu\nu}$$

$\mathcal{O}(h_{**})$ **Einstein tensor expansion**

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

Gravitational Wave Definition

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Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{[\mu} \xi_{\nu]}$$

Gravitational Wave Definition

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$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu$$

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \textbf{Einstein tensor expansion}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu = 0 \quad \Longleftrightarrow \quad \Box \xi_\nu = f(x)$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \textbf{Einstein tensor expansion}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

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Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\boxed{\partial'^\mu \bar{h}'_{\mu\nu}(x')} = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu \boxed{= 0} \Longleftrightarrow \Box \xi_\nu = f(x)$$

(solution always!)

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

$$(\; |h_{\mu\nu}| \ll 1 \;)$$

fixed frame

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**}) \ \textbf{Einstein tensor expansion}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

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Lorentz gauge

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski}\\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \underbrace{\eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}}_{= 0} - \underbrace{\partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{= 0} = -\frac{2}{m_p^2} T_{\mu\nu}$$

residual symm.

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Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

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Trace-reversed

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residual symm.

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Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

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Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}}_{} + \cancel{\eta_{\mu\nu}\partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}} - \cancel{\partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}} = \underbrace{-\frac{2}{m_p^2} T_{\mu\nu}}_{}$$

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0} \Rightarrow \partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Lorentz gauge

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \overset{\uparrow}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

fixed frame

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}} + \cancel{\eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}} - \cancel{\partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}} = \underbrace{-\frac{2}{m_p^2} T_{\mu\nu}}$$

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Lorentz gauge

Gravitational Wave Definition

1st approach to GWs

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$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Longrightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu}} + \cancel{\eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}} - \cancel{\partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}} = \underbrace{-\frac{2}{m_p^2} T_{\mu\nu}}$$

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Lorentz gauge

(10 - 4 = 6 d.o.f.)

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski} \\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Is that all ?

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\overset{\text{Minkowski}}{\uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Is that all ? Not really …

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

$$(\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \rightarrow \quad \partial'^\mu \bar{h}'_{\mu\nu} = 0)$$

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski}\\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

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with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

→

IF $T_{\mu\nu} = 0$

Outside Source

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

fixed frame

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

(**transverse-traceless gauge**)

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$(\ |h_{\mu\nu}| \ll 1 \ )$

fixed frame

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

(6 - 4 = 2 d.o.f.)

(**transverse-traceless gauge**)

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} \neq 0$

Inside Source !

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

?

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(transverse-traceless gauge)

6 - 4 = 2 d.o.f. ?

?

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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$$(\ |h_{\mu\nu}| \ll 1\ )$$

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(further residual gauge)

→

IF $T_{\mu\nu} \neq 0$

Inside Source !

$$\boxed{\xcancel{h^{0\mu} = 0, \quad h^i_i = 0, \quad \partial_j h_{ij} = 0}}$$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

6 - 4 = 2 d.o.f. ?

~~(**transverse-traceless gauge**)~~

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

fixed frame

$(\; |h_{\mu\nu}| \ll 1 \;)$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

$$\Longrightarrow$$

IF $T_{\mu\nu} \neq 0$

Inside Source !

Cannot make $h_{*0} = 0$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(6 - 4 = 2 d.o.f.)

Yet there are still only 2 radiative *dof* !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → **Gravitational Waves !**

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ?

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

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Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ? No !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

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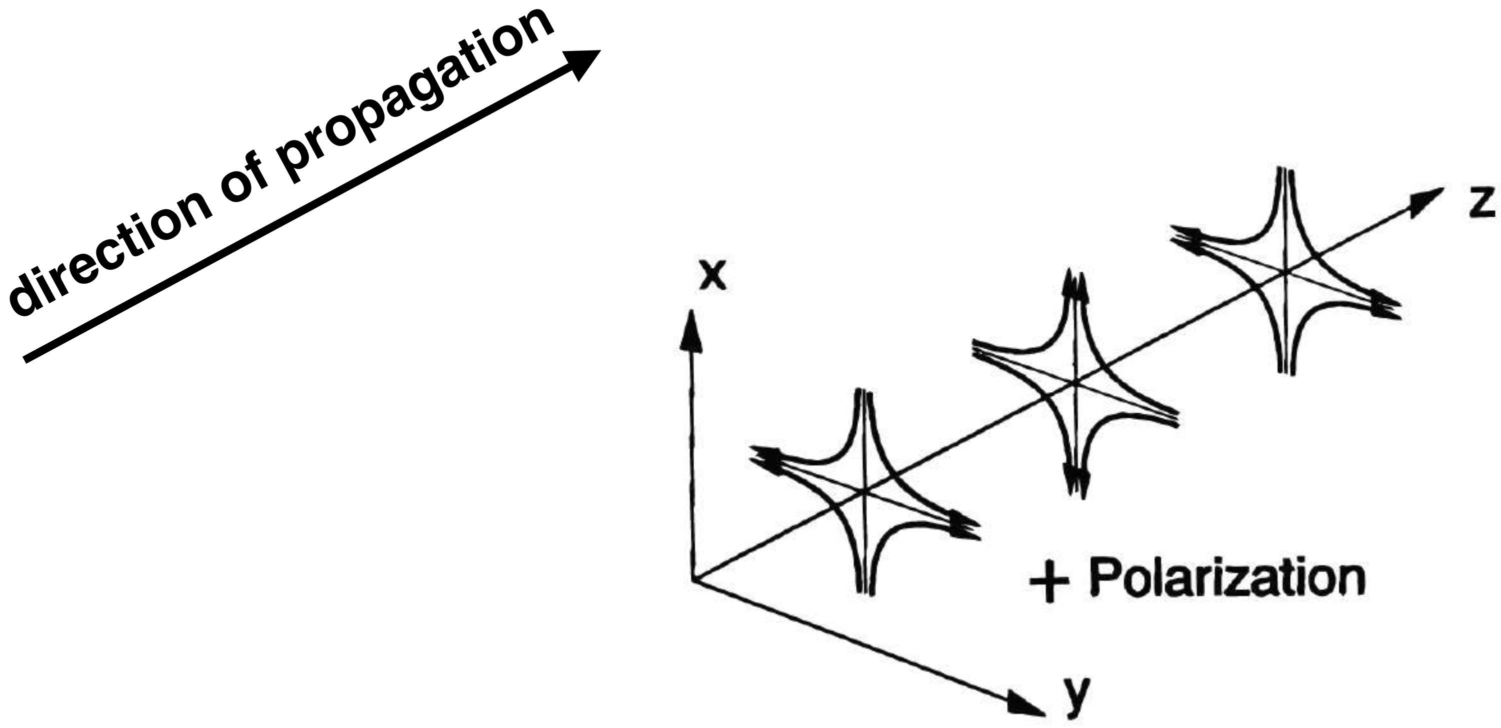
Wave Eq. ⟶ Gravitational Waves !

can GW be 'gauged away' ? No !



Transverse
(& Traceless)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

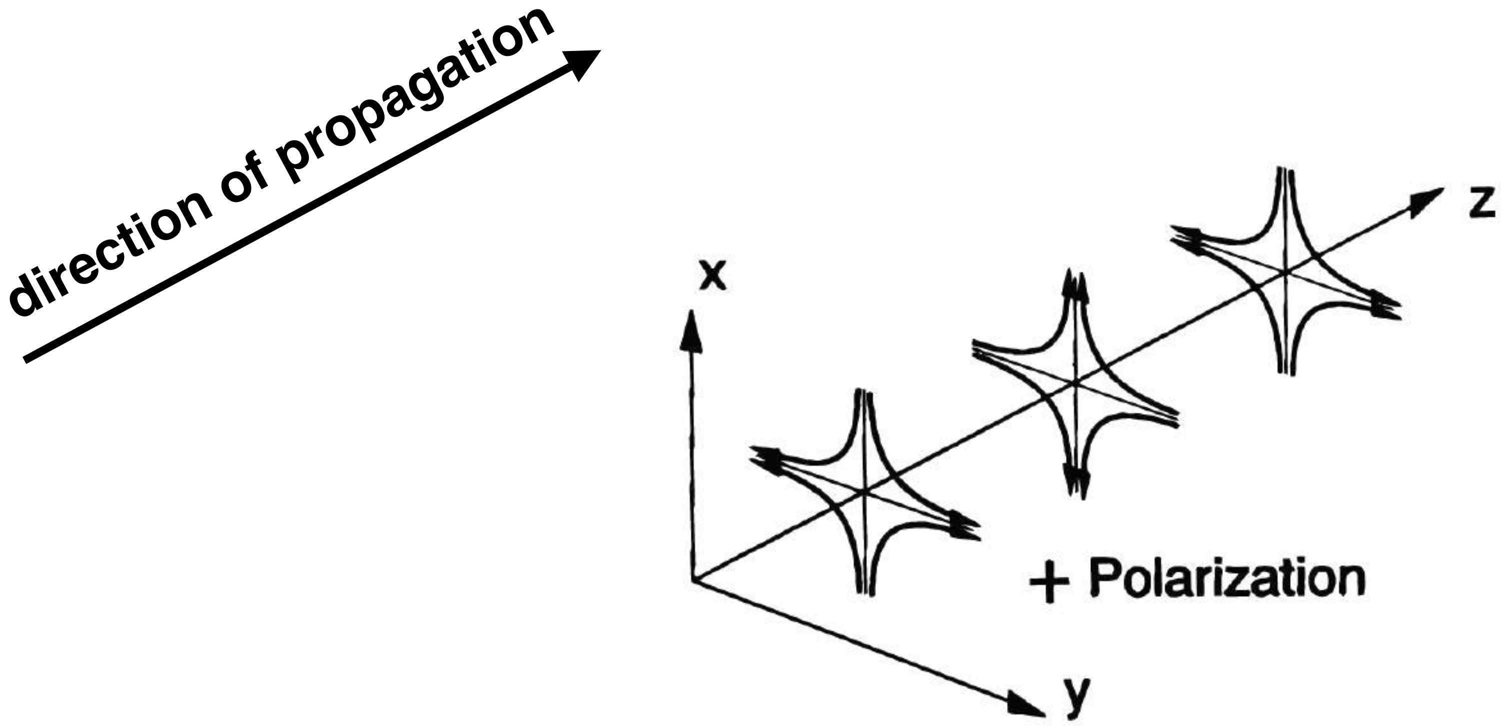
$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. ⟶ Gravitational Waves !

can GW be 'gauged away' ? No !



2 *dof* =
2 polarizations

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ? No !

---

2 *dof* = 2 polarizations

$$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f (t - \hat{n}\mathbf{x})}$$

transverse plane

(plane wave)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ? No !

---

2 *dof* = 2 polarizations

$$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f (t - \hat{n}\mathbf{x})}$$

transverse plane

(plane wave)

$$h_{ab}(f, \hat{n}) = \sum_{A=+,\mathrm{x}} h_A(f, \hat{n}) \epsilon^{(A)}_{ab}(\hat{n}) = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Transverse-Traceless (2 dof)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

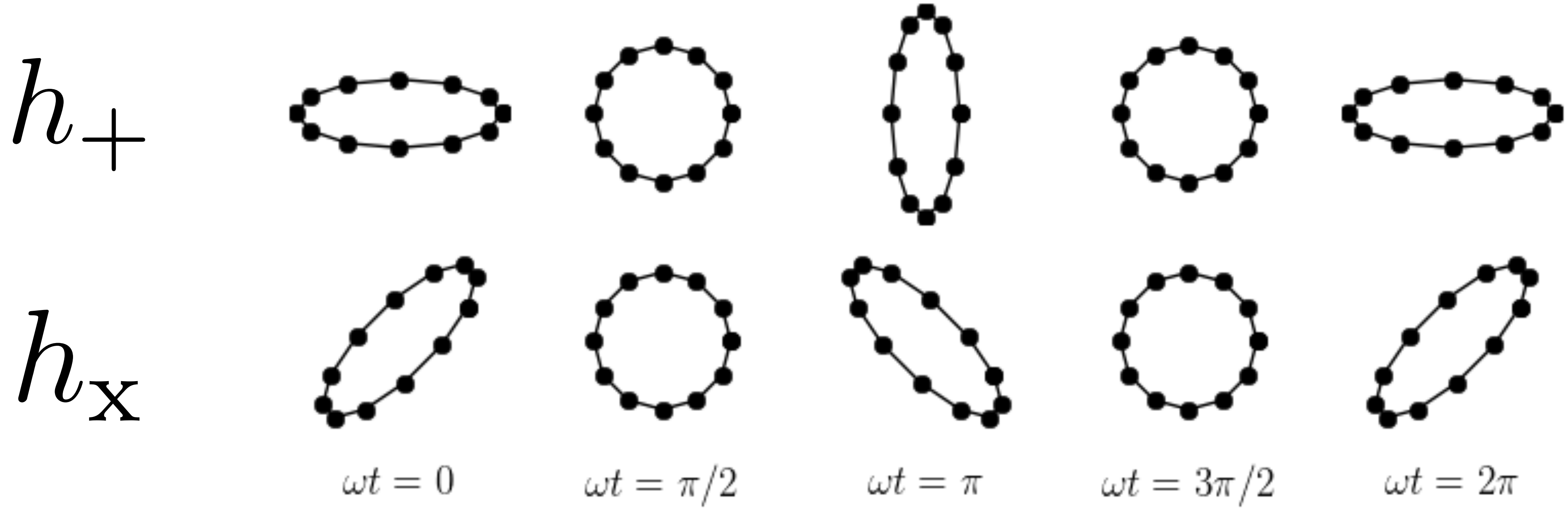
$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ? No !



https://www.youtube.com/watch?v=U_hLM1WPDqM

https://www.youtube.com/watch?v=EtL9UyRx_Us

Definition of GWs
2nd approach

Gravitational Wave Definition

2nd approach to GWs

(gauge invariant def.)

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

Gravitational Wave Definition

Minkowski

$$g_{\mu\nu} = \overset{\uparrow}{\eta_{\mu\nu}} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

2nd approach to GWs

(gauge invariant def.)

(svt decomposition)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

s: scalar
v: vector
t: tensor

Gravitational Wave Definition

2nd approach to GWs

(gauge invariant def.)

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

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s: scalar

v: vector

t: tensor

(svt decomposition)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = \left(\partial_i B + S_i\right),$$

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(svt E/p-tensor components)

$$T_{00} = \rho,$$

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		$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s)		ϕ, B, ψ, E	ρ, u, p, σ
Vector(s)	$\Big\} \in \Re^3$	S_i, F_i	u_i, v_i
Tensor(s)		h_{ij}	Π_{ij}

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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		$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s)	$\in \Re^3$	$\phi,\ B,\ \psi,\ E$	$\rho,\ u,\ p,\ \sigma$
Vector(s)		$S_i,\ F_i$	$u_i,\ v_i$
Tensor(s)		h_{ij}	Π_{ij}

Gravitational Wave Definition

(svt metric perturbations)

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$$\delta g_{00} = -2\phi,$$

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$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

(svt E/p-tensor components)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

		$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s)	$\in \Re^3$	$\phi,\, B,\, \psi,\, E$	$\rho,\, u,\, p,\, \sigma$
Vector(s)		$S_i,\, F_i$	$u_i,\, v_i$
Tensor(s)		h_{ij}	Π_{ij}

Gravitational Wave Definition

(svt metric perturbations)

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		$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s)	$\in \Re^3$	$\phi,\ B,\ \psi,\ E$	$\rho,\ u,\ p,\ \sigma$
Vector(s)		$S_i,\ F_i$	$u_i,\ v_i$
Tensor(s)		h_{ij}	Π_{ij}

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(svt metric perturbations)

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16 degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

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16 degrees of freedom

		$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s)	$\left.\right\} \in \Re^3$	ϕ, B, ψ, E	ρ, u, p, σ
Vector(s)		S_i, F_i	u_i, v_i
Tensor(s)		h_{ij}	Π_{ij}

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16 degrees of freedom

In order NOT to over-count degrees of freedom $\Bigg\{$

Gravitational Wave Definition

(svt metric perturbations)

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In order NOT to over-count degrees of freedom

$$\left\{ \begin{array}{l} \left. \begin{array}{l} \partial_i S_i = 0 \text{ (1 constraint)}, \quad \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, \quad h_{ii} = 0 \text{ (1 constraint)} \end{array} \right\} \\ \\ \\ \end{array} \right.$$

Metric perturbations

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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16 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{l} \left.\begin{array}{l} \partial_i S_i = 0 \text{ (1 constraint)}, \quad \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, \quad h_{ii} = 0 \text{ (1 constraint)} \end{array}\right\} \text{Metric perturbations} \\ \\ \left.\begin{array}{l} \partial_i u_i = 0 \text{ (1 constraint)}, \quad \partial_i v_i = 0 \text{ (1 constraint)}, \\ \partial_i \Pi_{ij} = 0 \text{ (3 constraints)}, \quad \Pi_{ii} = 0 \text{ (1 constraint)}, \end{array}\right\} \text{Energy/Momentum tensor} \end{array}\right.$$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

16 degrees of freedom

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6 constraints for metric perturbations

6 constraints for E/p tensor components

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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10

~~16~~ degrees of freedom

(svt E/p-tensor components)

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10

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In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{ll} \left.\begin{array}{ll} \partial_i S_i = 0 \text{ (1 constraint)}, & \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, & h_{ii} = 0 \text{ (1 constraint)} \end{array}\right\} \\ \\ \left.\begin{array}{ll} \partial_i u_i = 0 \text{ (1 constraint)}, & \partial_i v_i = 0 \text{ (1 constraint)}, \\ \partial_i \Pi_{ij} = 0 \text{ (3 constraints)}, & \Pi_{ii} = 0 \text{ (1 constraint)}, \end{array}\right\} \end{array}\right.$$

6 constraints for metric perturbations

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10 degrees of freedom

In order NOT to over-count degrees of freedom

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\end{aligned}\right\}\\
&\left.\begin{aligned}
&\partial_i u_i = 0 \text{ (1 constraint)}, \quad \partial_i v_i = 0 \text{ (1 constraint)},\\
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\end{aligned}\right\}
\end{aligned}\right.$$

6 constraints for metric perturbations

6 constraints for E/p tensor components

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10 degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0$$

Gravitational Wave Definition

(svt metric perturbations)

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10 degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0. \Rightarrow \left\{ \begin{array}{l} \nabla^2 u = \dot{\rho} \;\; (1 \text{ constraint}), \\ \nabla^2 \sigma = \frac{3}{2}(\dot{u} - p) \;\; (1 \text{ constraint}), \\ \nabla^2 v_i = \dot{u}_i \;\; (2 \text{ constraints}). \end{array} \right\}$$

4 constraints (due to E/p conservation)

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

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10 degrees
of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

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6
~~10~~ degrees
of freedom

Physical
Constraints

$$\partial^\mu T_{\mu\nu} = 0 \;\Rightarrow\; \left\{ \begin{array}{l} \nabla^2 u = \dot\rho \;\; (1 \text{ constraint}), \\ \nabla^2\sigma = \frac{3}{2}(\dot u - p) \;\; (1 \text{ constraint}), \\ \nabla^2 v_i = \dot u_i \;\; (2 \text{ constraints}). \end{array} \right\}$$

4 constraints
(due to E/p
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(svt metric perturbations)

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$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

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10 degrees of freedom

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$$T_{00} = \rho,$$

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6 degrees of freedom

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(svt metric perturbations)

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10 degrees
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6 degrees
of freedom

Physical
Constraints

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6 degrees of freedom

Physical Symmetry

$$\left\{ \begin{aligned} x_\mu &\longrightarrow x_\mu + \xi_\mu \\ \delta g_{\mu\nu} &\longrightarrow \delta g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \end{aligned} \right.$$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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10 degrees of freedom

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i)$$

$$\text{with } \partial_i d_i = 0,$$

Gravitational Wave Definition

(svt metric perturbations)

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$\left\{\begin{aligned} & x_\mu \longrightarrow x_\mu + \xi_\mu \\ & \delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \\ & \xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i) \\ & \quad \text{with } \partial_i d_i = 0, \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} & \phi \longrightarrow \phi - \dot{d}_0, \quad B \longrightarrow B - d_0 - \dot{d}, \\ & \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, \quad E \longrightarrow E - 2d, \\ & S_i \longrightarrow S_i - \dot{d}_i, \quad F_i \longrightarrow F_i - 2d_i, \\ & h_{ij} \longrightarrow h_{ij}. \end{aligned}\right.$$

Gravitational Wave Definition

(svt metric perturbations)

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$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

6 degrees of freedom

Physical Symmetry

(4 d.o.f. spurious)

$$\left\{ \begin{array}{l} x_\mu \longrightarrow x_\mu + \xi_\mu \\ \delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \\ \xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i) \\ \quad \text{with } \partial_i d_i = 0, \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} \phi \longrightarrow \phi - \dot{d}_0, & B \longrightarrow B - d_0 - \dot{d}, \\ \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, & E \longrightarrow E - 2d, \\ S_i \longrightarrow S_i - \dot{d}_i, & F_i \longrightarrow \quad F_i - 2d_i, \\ \multicolumn{2}{c}{h_{ij} \longrightarrow h_{ij}.} \end{array} \right.$$

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$\left\{\begin{array}{ll} \phi \longrightarrow \phi - \dot{d}_0, & B \longrightarrow B - d_0 - \dot{d}, \\ \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, & E \longrightarrow E - 2d, \\ S_i \longrightarrow S_i - \dot{d}_i, & F_i \longrightarrow F_i - 2d_i, \\ & h_{ij} \longrightarrow h_{ij}. \end{array}\right.$$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

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$$T_{00} = \rho,$$

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$\left\{\begin{array}{ll} \phi \longrightarrow \phi - \dot{d}_0, & B \longrightarrow B - d_0 - \dot{d}, \\ \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, & E \longrightarrow E - 2d, \\ S_i \longrightarrow S_i - \dot{d}_i, & F_i \longrightarrow \quad F_i - 2d_i, \\ \boxed{h_{ij} \longrightarrow h_{ij}.} & \end{array}\right\}$$

⇨

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E},$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E,$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i,$$

with $\partial_i \Sigma_i = 0$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

6 degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$\left\{\begin{array}{ll} \phi \longrightarrow \phi - \dot{d}_0, & B \longrightarrow B - d_0 - \dot{d}, \\ \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, & E \longrightarrow E - 2d, \\ S_i \longrightarrow S_i - \dot{d}_i, & F_i \longrightarrow F_i - 2d_i, \\ \boxed{h_{ij} \longrightarrow h_{ij}.} \; (\mathbf{2}) & \end{array}\right\} \Longrightarrow$$

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \quad (\mathbf{1})$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E, \quad (\mathbf{1})$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i, \quad (\mathbf{2})$$

with $\partial_i \Sigma_i = 0$

Gravitational Wave Definition

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \textbf{(1)}$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E, \qquad \textbf{(1)}$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i, \qquad (\partial_i \Sigma_i = 0) \qquad \textbf{(2)}$$

$$h_{ij} \equiv h_{ij}\ , \qquad (h_{ii} = \partial_i h_{ij} = 0) \qquad \textbf{(2)}$$

6 gauge invariant
degrees of freedom

Gravitational Wave Definition

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6 gauge invariant degrees of freedom

Gauge Invariant Einstein Tensor

$$G_{00} = -\nabla^2\Theta,$$

$$G_{0i} = -\frac{1}{2}\nabla^2\Sigma_i - \partial_i\dot{\Theta},$$

$$G_{ij} = -\frac{1}{2}\Box h_{ij} - \partial_{(i}\dot{\Sigma}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi + \Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi + \Theta\right) - \ddot{\Theta}\right].$$

Gravitational Wave Definition

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \mathbf{(1)}$$

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6 gauge invariant
degrees of freedom

Gauge Invariant (perturbed) Einstein Eqs.

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right)$$

$$\nabla^2\Sigma_i = -\frac{2}{m_p^2}S_i, \qquad \Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}.$$

Gravitational Wave Definition

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \textbf{(1)}$$

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Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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Gravitational Wave Definition

6 gauge invariant *d.o.f.*

Gauge Invariant
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$$\boxed{h_{ij}\,, \quad (h_{ii} = \partial_i h_{ij} = 0)}$$

transverse
& traceless
(TT *d.o.f.*)

Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \text{(1)} \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right) \text{ (1)}$$

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transverse & traceless (TT *d.o.f.*)

Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Wave Definition

6 gauge invariant *d.o.f.*

Gauge Invariant (perturbed) Einstein Eqs.

$\nabla^2\Theta = -\frac{1}{m_p^2}\rho,$ (**1**) $\quad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right)$ (**1**)

$\nabla^2\Sigma_i = -\frac{2}{m_p^2}S_i,$(**2**) $\quad \boxed{\Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}.}$ (**2**)

$$\boxed{h_{ij}\,, \quad (h_{ii} = \partial_i h_{ij} = 0)}$$

transverse & traceless (TT *d.o.f.*)

Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Waves (GWs) are TT *d.o.f.* metric perturbations, independently of system of reference

Definition of GWs
3rd approach

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

More subtle problem!

Solution: Separation of scales !

See e.g. Maggiore's 1st Book on GWs

Gravitational Wave Definition

3rd approach to GWs

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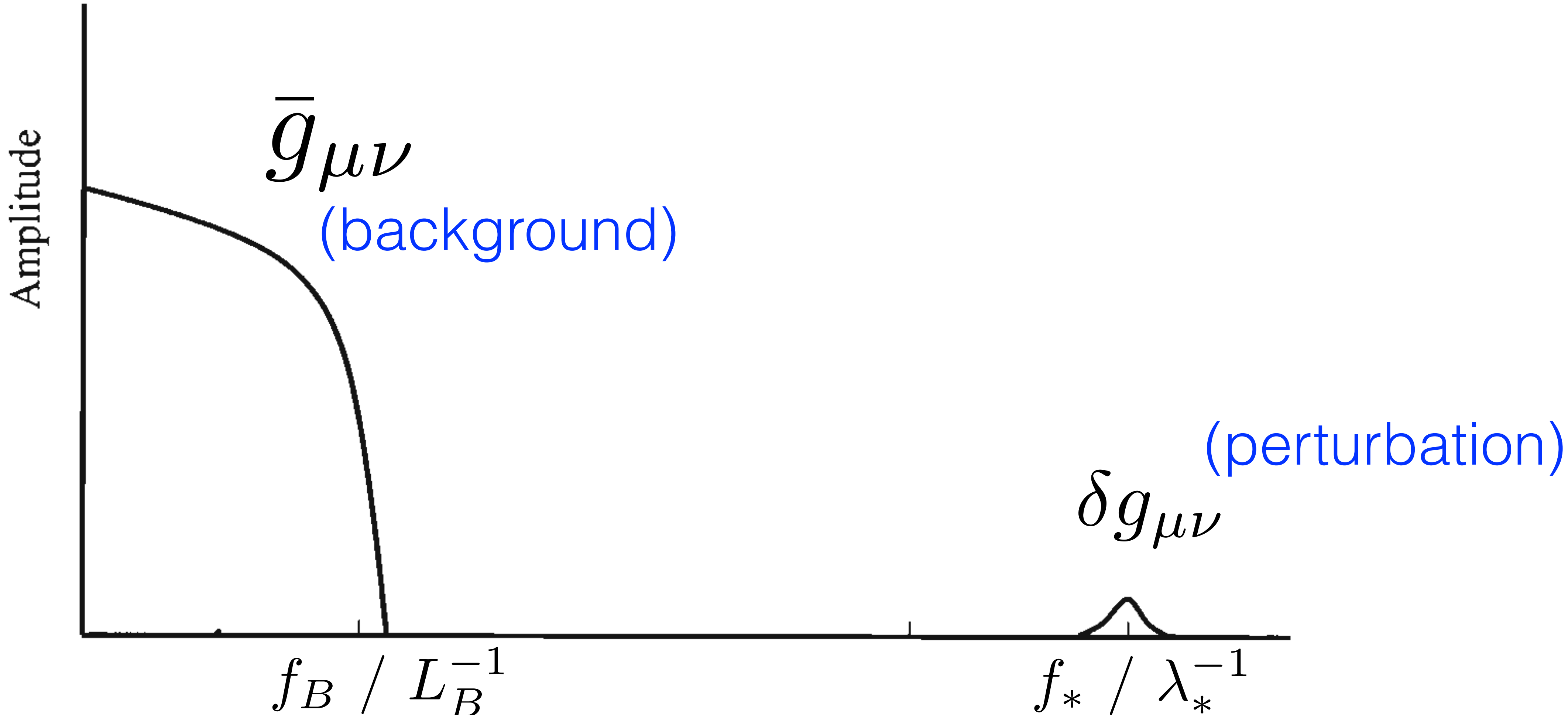

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

(separation not well defined)

More subtle problem!   Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Gravitational Wave Definition

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$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad \longrightarrow \quad R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(\delta g)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(\delta g^2)} + \ldots\,,$$

Gravitational Wave Definition

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Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm Low} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm Low}$

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

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More subtle problem! <u>Solution</u>: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \longrightarrow R_{\mu\nu} = \underset{\text{(background)}}{\bar{R}_{\mu\nu}} + \underset{\mathcal{O}(\delta g)}{R^{(1)}_{\mu\nu}} + \underset{\mathcal{O}(\delta g^2)}{R^{(2)}_{\mu\nu}} + \ldots\,,$$

Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{Low}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{Low}}$

High Freq. / Short Scale: $R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

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Gravitational Wave Definition

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Gravitational Wave Definition

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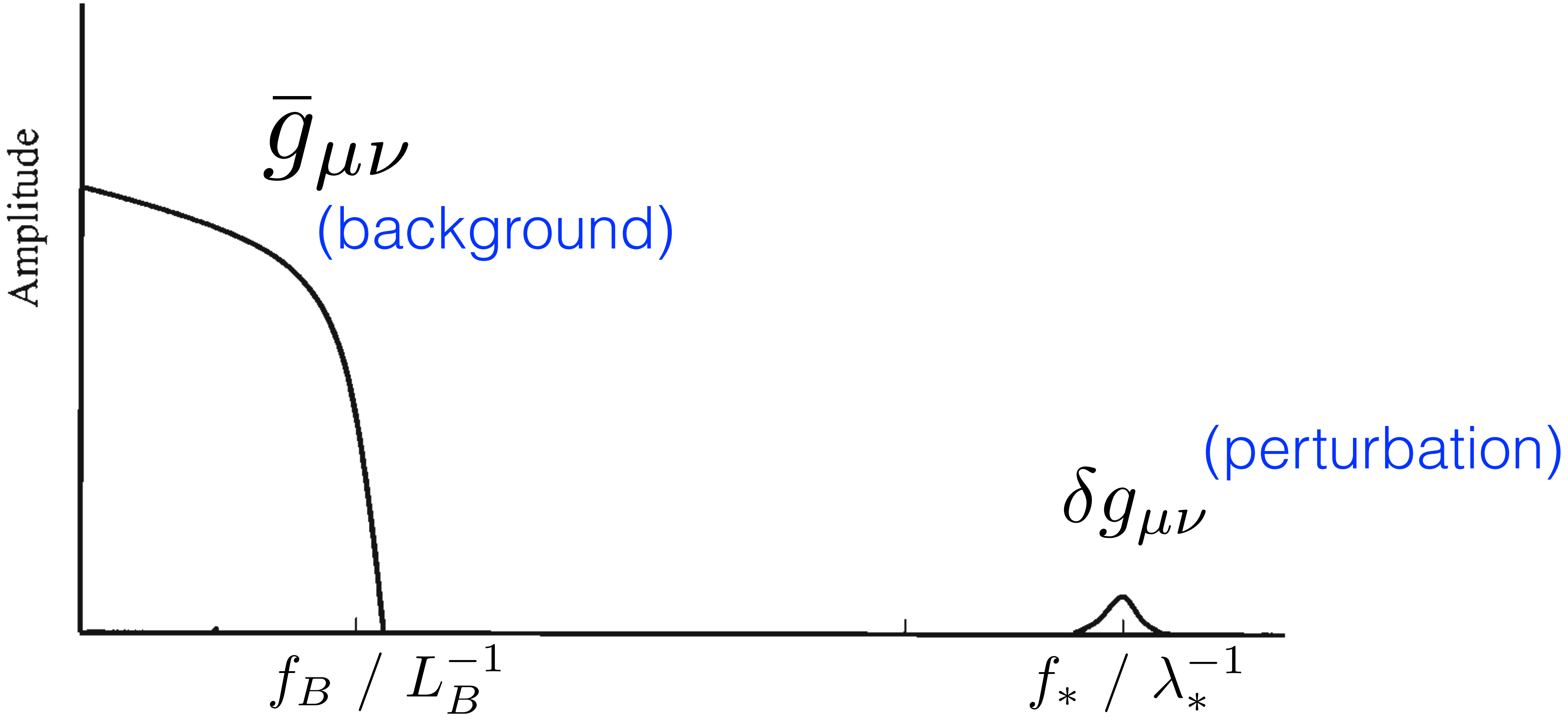
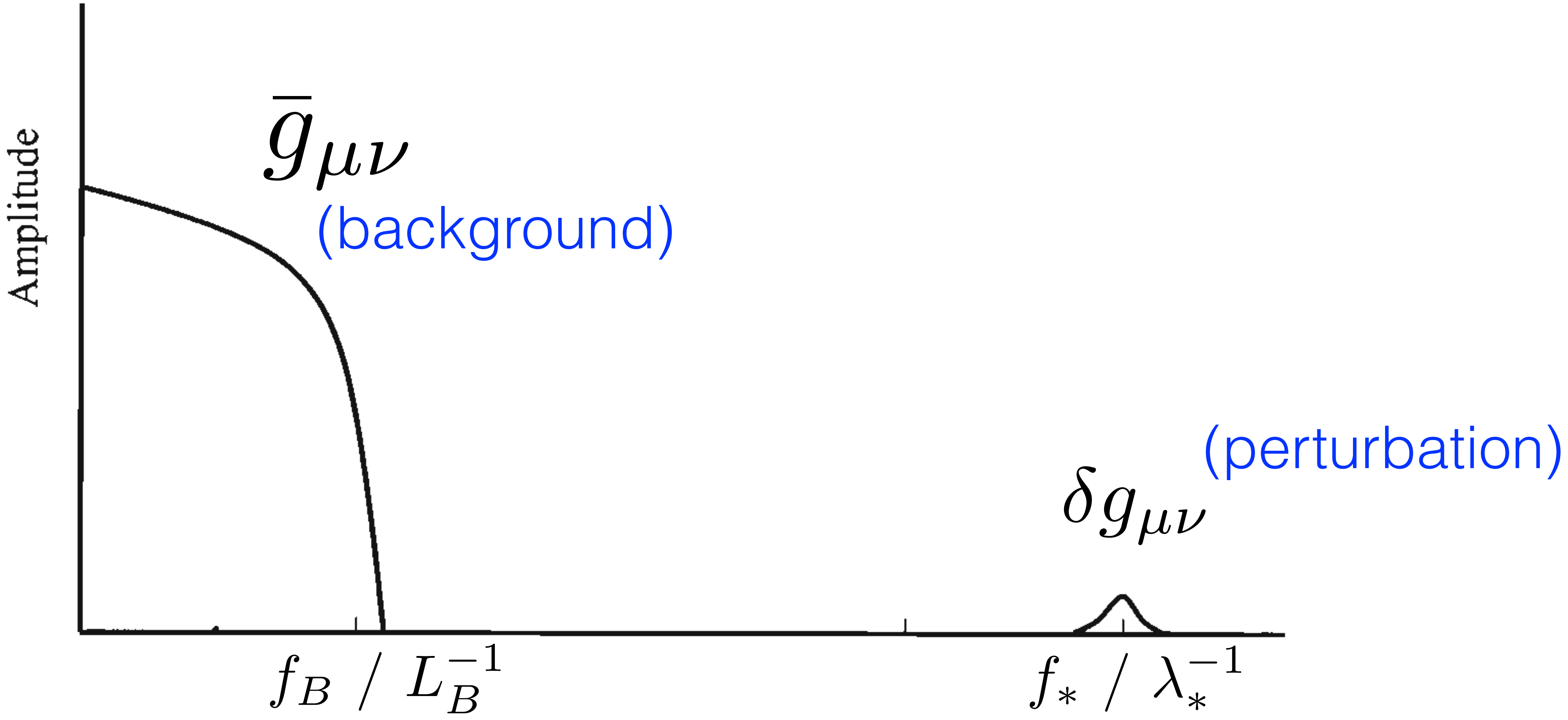


Gravitational Wave Definition

Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{1}{m_p^2} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$ **(space/time average)**



Gravitational Wave Definition

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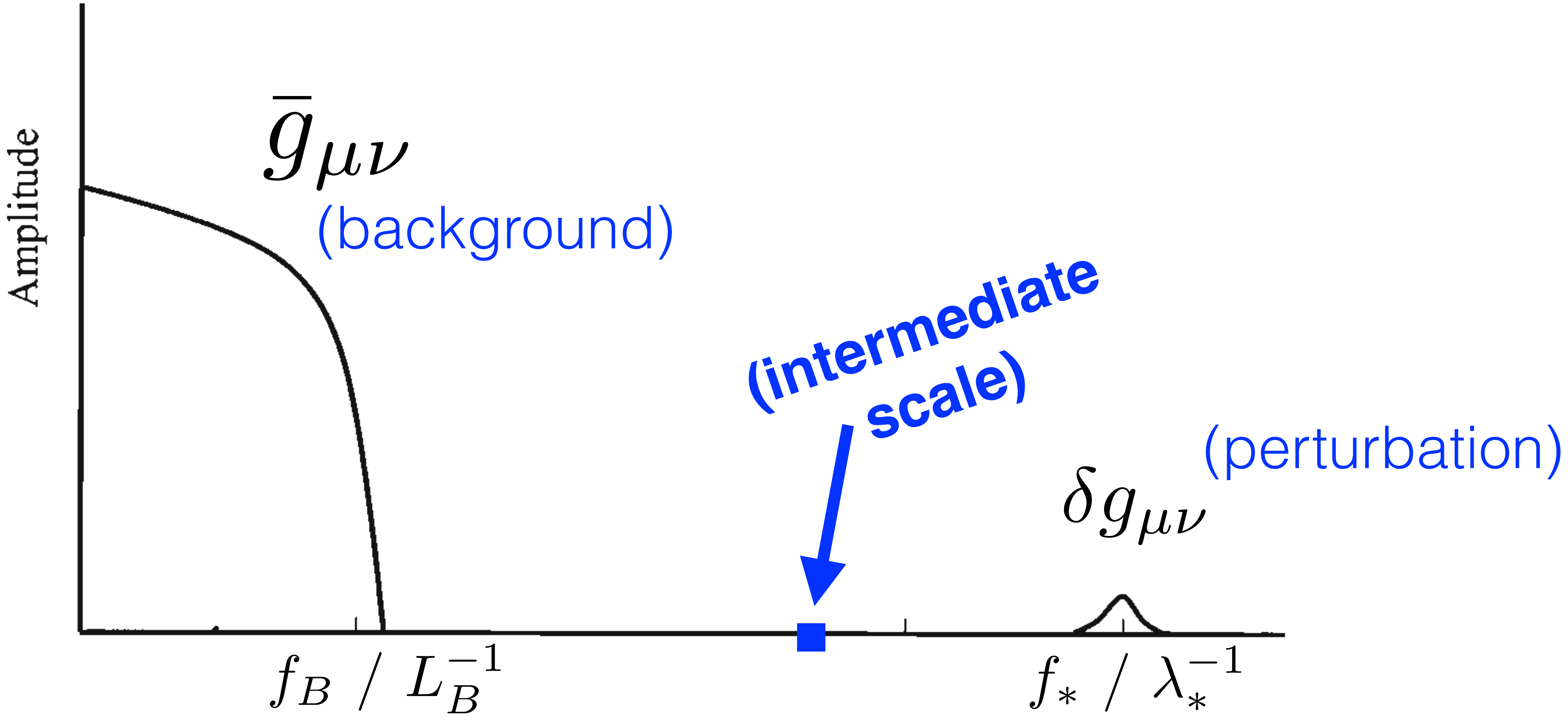
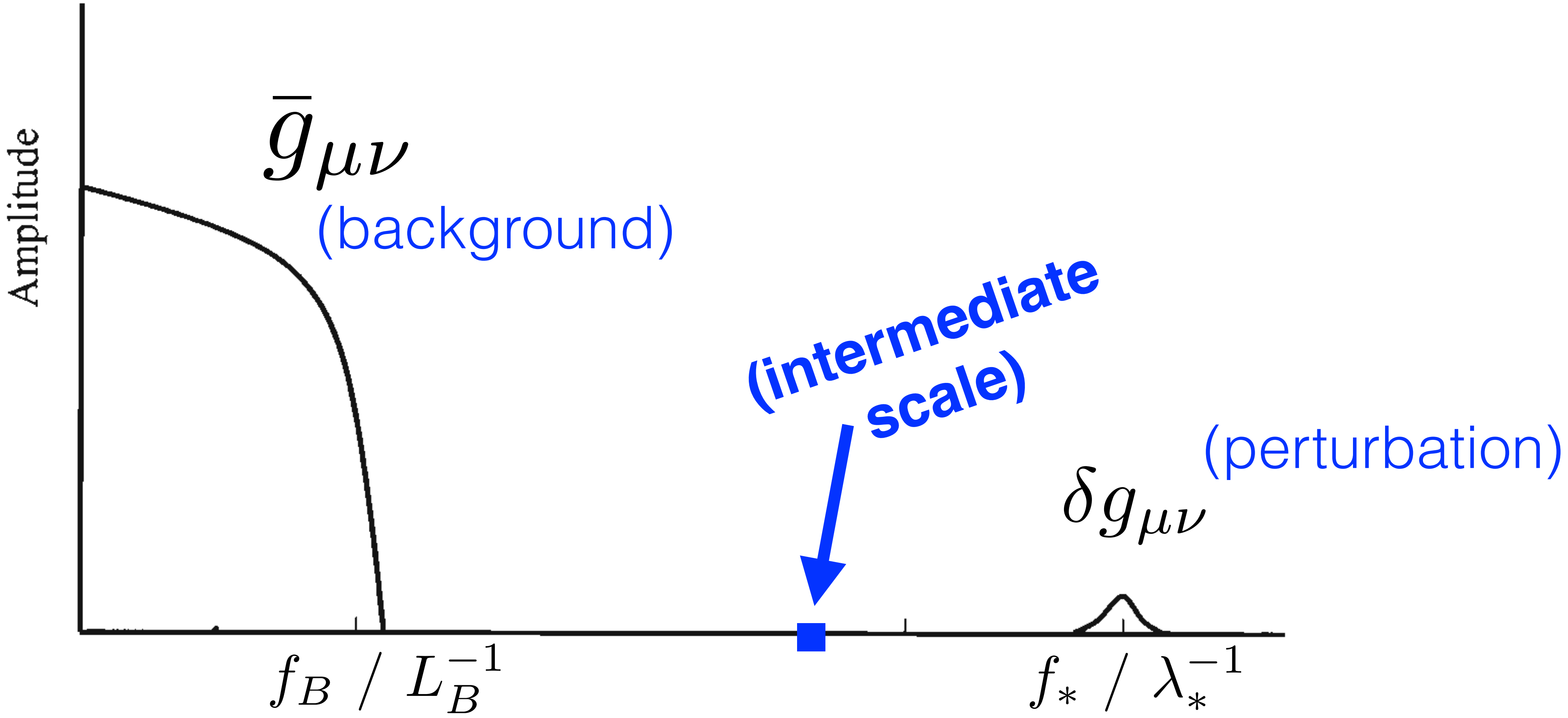


Gravitational Wave Definition

Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{1}{m_p^2} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$ **(space/time average)**

$$t_{\mu\nu} = -\frac{1}{m_p^2} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle$$

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}^{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$



Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

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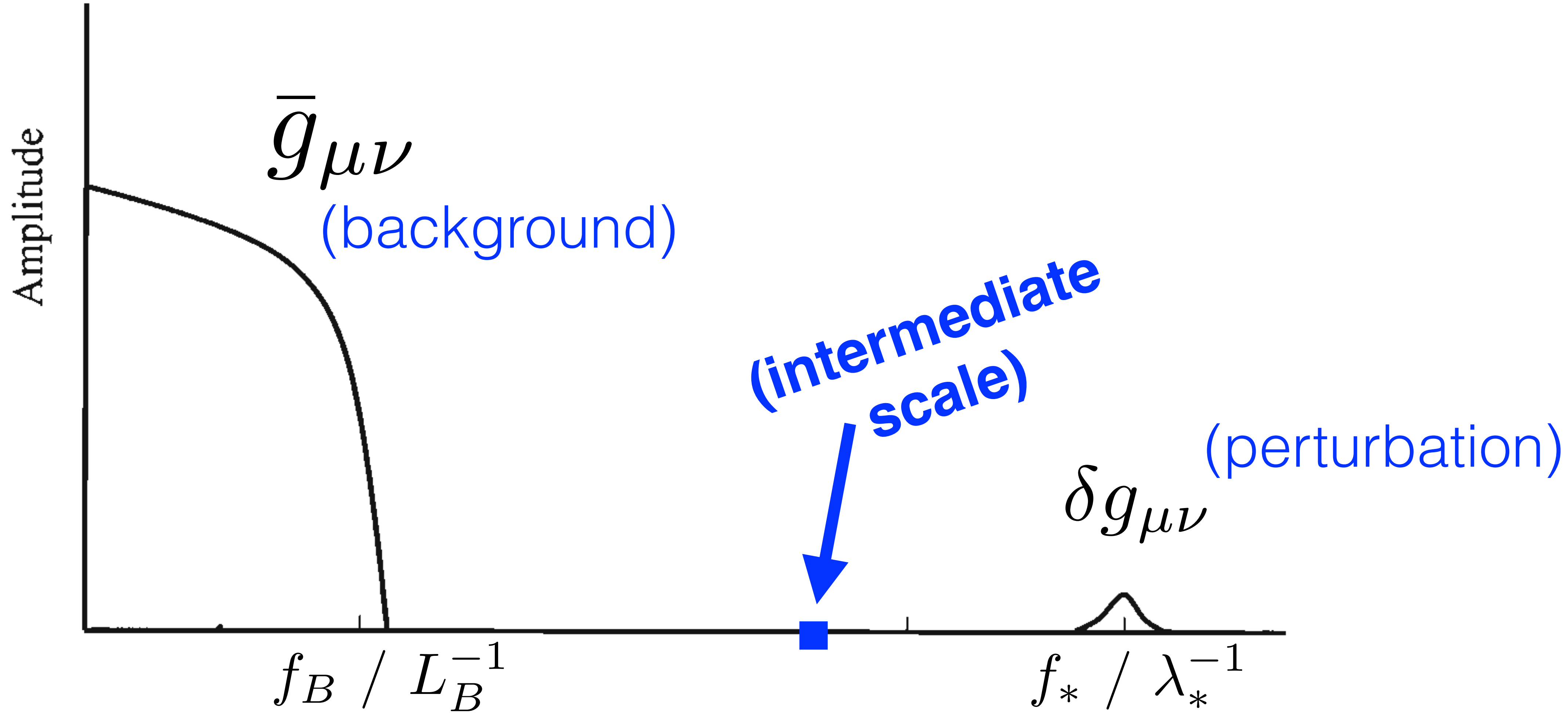

Gravitational Wave Definition

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$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle \longrightarrow t_{\mu\nu} = \frac{m_p^2}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle$$

Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

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$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle \Longrightarrow t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle$$

It can be shown that only TT *dof* contribute to < … >

Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

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$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle \quad \Longrightarrow \quad t_{\mu\nu} = \frac{m_p^2}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle$$

It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\ \langle \partial_\mu \delta g^{\mathrm{TT}}_{ij}\, \partial_\nu \delta g^{\mathrm{TT}}_{ij}\rangle}$$

GW energy-momentum tensor

Gravitational Wave Definition

Low Freq. / Long Scale: $\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \qquad \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}$$

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It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{ij}^{\rm TT}\,\partial_\nu\delta g_{ij}^{\rm TT}\rangle} \xrightarrow[(\delta g_{ij} \equiv h_{ij})]{} \boxed{\frac{dE}{dAdt} = \frac{m_p^2}{4}\langle\dot{h}_{ij}^{\rm TT}\,\dot{h}_{ij}^{\rm TT}\rangle}$$

GW energy-momentum tensor

GW power/area radiated

Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \qquad \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}$$

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle \Longrightarrow t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle$$

It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g^{\rm TT}_{ij}\,\partial_\nu\delta g^{\rm TT}_{ij}\rangle} \xrightarrow[(\delta g_{ij} \equiv h_{ij})]{} \boxed{\rho_{\rm GW} \equiv \frac{m_p^2}{4}\langle\dot{h}^{\rm TT}_{ij}\,\dot{h}^{\rm TT}_{ij}\rangle}$$

GW energy-momentum tensor

GW energy density

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{High}}$$

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

$$\frac{|R^{(2)}_{\mu}|^{\rm High}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\rm High}$$ negligible

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$$

$$\frac{|R^{(2)}_{\mu}|^{\text{High}}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\text{High}}$$

negligible

$$\left.\begin{array}{l} R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right) \\ D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta} \right) \end{array}\right]$$

Lorentz gauge

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

$$\frac{|R^{(2)}_{\mu}|^{\rm High}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\rm High}$$ negligible

$$R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right)$$

$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\, \overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta} \,\right)$$

Lorentz gauge

$\longrightarrow$

vacuum

$$\longrightarrow \quad \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = 0}$$

Propagation of GWs
in curved space-time

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{High}}$$

$$\frac{|R^{(2)}_{\mu}|^{\mathrm{High}}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\mathrm{High}}$$

negligible

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Lorentz gauge

→

vacuum

→ $$\boxed{D_\alpha D^\alpha \delta g^{\mathrm{TT}}_{ij} = 0}$$

Propagation of GWs
in curved space-time

$\left(\ D_i \delta g^{\mathrm{TT}}_{ij} = \bar{g}^{ij}\delta g^{\mathrm{TT}}_{ij} = 0\ \right)$

Gravitational Wave Propagation

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High Freq. / Short Scale?

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$$\left.\begin{array}{l} R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right) \\ D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\, \overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\,\right) \end{array}\right] \longrightarrow$$

Lorentz gauge

matter

$$\longrightarrow \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = \Pi_{\mu\nu}}$$

Creation of GWs
in curved space-time

Gravitational Wave Propagation

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$$\longrightarrow \boxed{D_\alpha D^\alpha \delta g^{\mathrm{TT}}_{\mu\nu} = \overset{\text{matter}}{\Pi^{\mathrm{TT}}_{\mu\nu}}}$$

Creation of GWs
in curved space-time

TT dof = truly radiative !
[no gauge choice]

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j), \qquad \mathrm{TT}: \left\{ \begin{array}{l} h_{ii} = 0 \\ h_{ij,j} = 0 \end{array} \right.$

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j)$, $\quad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij,j} = 0 \end{cases}$

Creation of GWs in curved space-time

Source: Anisotropic Stress

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij}^{\mathrm{TT}},}$ $\quad \boxed{\Pi_{ij} = T_{ij} - \langle T_{ij} \rangle_{\mathrm{FRW}}}$

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j), \qquad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij,j} = 0 \end{cases}$

Creation of GWs in curved space-time

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G\Pi^{\mathrm{TT}}_{ij},}$

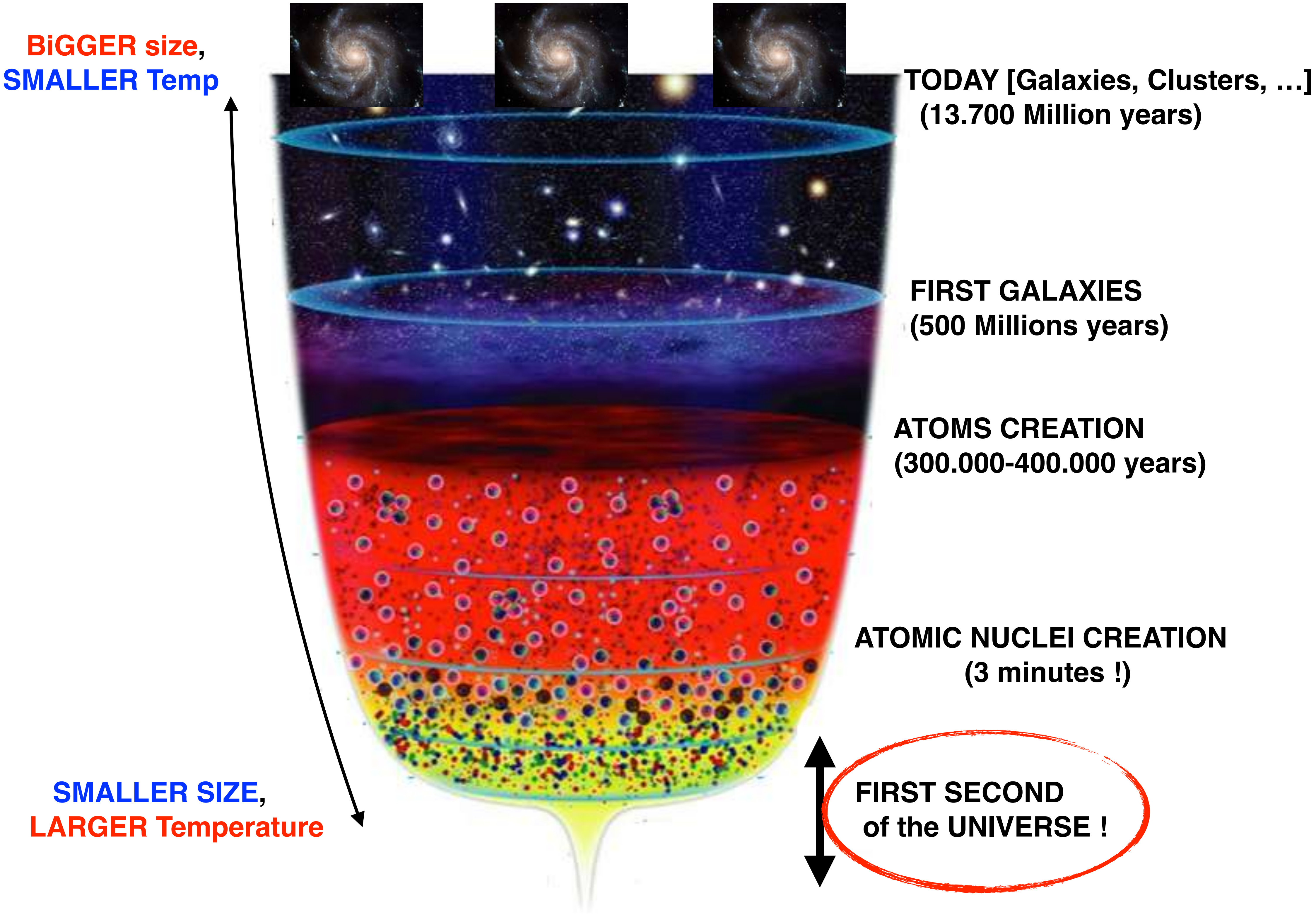
Source: Anisotropic Stress

$\boxed{\Pi_{ij} = T_{ij} - \langle T_{ij}\rangle_{\mathrm{FRW}}}$

GW Source(s): (SCALARS , VECTOR , FERMIONS)

$$\Pi^{TT}_{ij} \quad \propto \quad \{\partial_i\chi^a\partial_j\chi^a\}^{TT}, \quad \{E_iE_j + B_iB_j\}^{TT}, \quad \{\bar{\psi}\gamma_i D_j\psi\}^{TT}$$

Cosmic History



BiGGER size,
SMALLER Temp

TODAY [Galaxies, Clusters, ...]
(13.700 Million years)

FIRST GALAXIES
(500 Millions years)

ATOMS CREATION
(300.000-400.000 years)

ATOMIC NUCLEI CREATION
(3 minutes !)

SMALLER SIZE,
LARGER Temperature

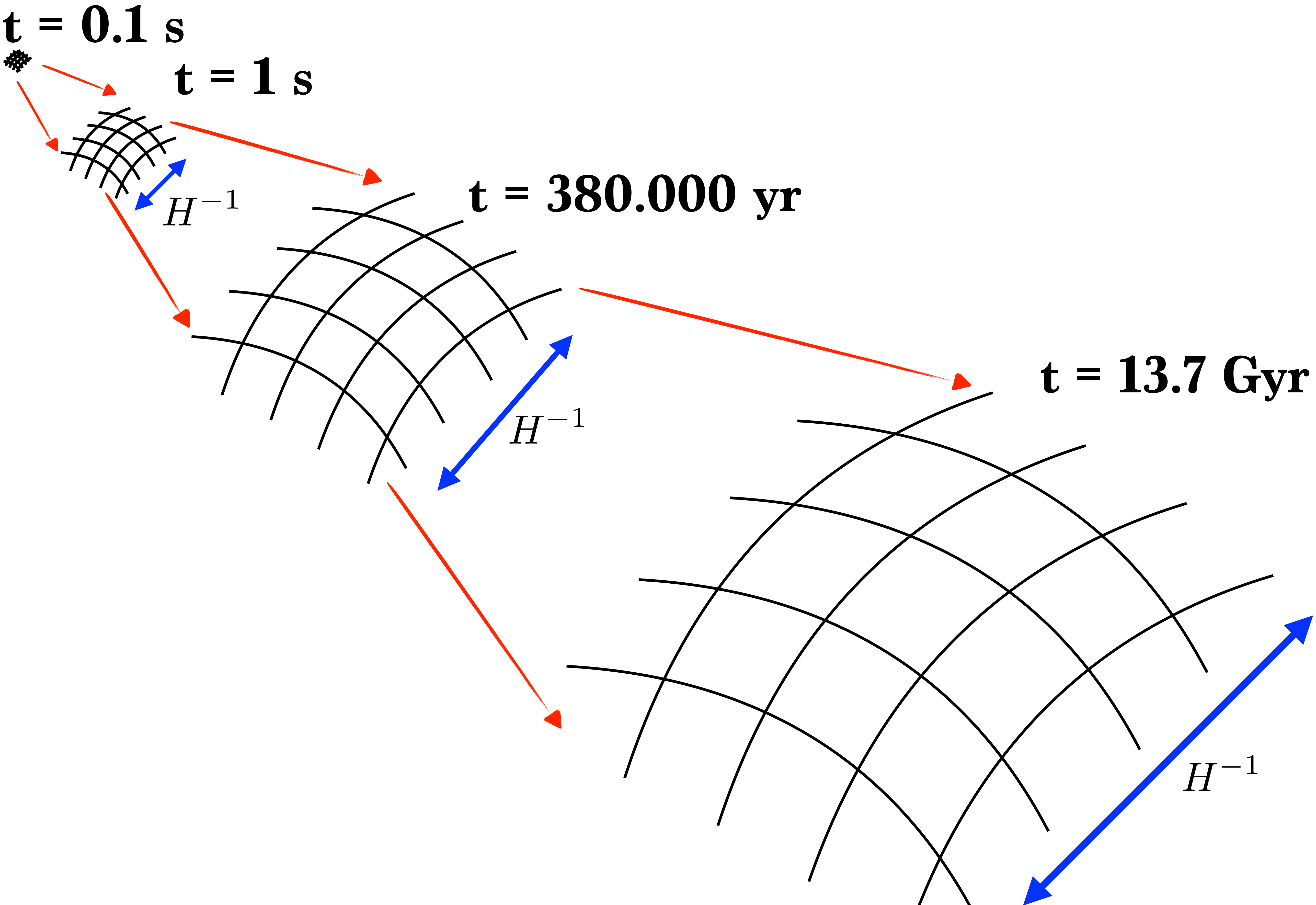
FIRST SECOND
of the UNIVERSE !

To Be Continued …

BACK SLIDES

FLRW

Expanding Universe



t = 0.1 s

t = 1 s

H^{-1}

t = 380.000 yr

H^{-1}

t = 13.7 Gyr

H^{-1}

Expanding Universe

H & I

$$T^{\mu}_{\nu} \equiv \mathrm{diag}(-\rho, p, p, p)$$

$$\Downarrow$$

$$m_p^2 G^{\mu}_{\nu}\left[g_{**}^{(FRW)}\right] = T^{\mu}_{\nu}$$

Expanding Universe

H & I

$$T^{\mu}_{\nu} \equiv \mathrm{diag}(-\rho, p, p, p)$$

$$m_p^2 G^{\mu}_{\nu}\left[g_{**}^{(FRW)}\right] = T^{\mu}_{\nu}$$

Friedmann Equations

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{\rho}{6m_p^2}(1+3w) \qquad \text{(I)}$$

$$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \qquad \text{(II)}$$

$\left(w \equiv \frac{p}{\rho}\right)$ Equation of State (EoS)

Expanding Universe

H & I

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$$\left(w \equiv \frac{p}{\rho}\right)$$ Equation of State (EoS)

(I) + (II)

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w) \qquad \text{(III)}$$

Expanding Universe

UNIVERSE:

1) GR

\+

2) H & I

⇒

Friedmann Equations

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{\rho}{6m_p^2}(1+3w) \qquad (\mathrm{I})$$

$$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \qquad (\mathrm{II})$$

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$$\left(w \equiv \frac{p}{\rho}\right)$$

Equation of
State (EoS)

Expanding Universe

(II) $$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \longrightarrow \boxed{\rho_c \equiv 3m_p^2 H^2}$$

Critical density $(\rho = \rho_c \Leftrightarrow K = 0)$

Expanding Universe

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Critical density $(\rho = \rho_c \Leftrightarrow K = 0)$

$$\rho = \sum_i \rho_i \ ; \quad \Omega_i \equiv \frac{\rho_i}{\rho_c} \Longrightarrow \Omega \equiv \frac{\rho}{\rho_c} = \sum_i \Omega_i \Longrightarrow \boxed{\Omega - 1 \equiv \frac{k}{a^2 H^2}}$$

Cosmic Sum

Expanding Universe

$$\text{(II)}\quad H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \quad \longrightarrow \quad \boxed{\rho_c \equiv 3m_p^2 H^2}$$

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Cosmic Sum

$$\begin{cases} \Omega > 1 \Rightarrow \mathrm{Close}(k > 0) \\ \Omega = 1 \Rightarrow \mathrm{Flat}(k = 0) \\ \Omega < 1 \Rightarrow \mathrm{Open}(k < 0) \end{cases}$$

one-to-one
correlation

Expanding Universe

(II) $$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \longrightarrow \boxed{\rho_c \equiv 3m_p^2 H^2}$$

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Cosmic Sum

(III)

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w) \Rightarrow \rho \propto e^{-3\int \frac{da}{a}(1+w)} = \begin{cases} 1/a^3 & \text{, Mat.}(w=0) \\ 1/a^4 & \text{, Rad.}(w=1/3) \\ \text{const.} & \text{, C.C.}(w=-1) \end{cases}$$

Expanding Universe

(II) $H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \longrightarrow \boxed{\rho_c \equiv 3m_p^2 H^2}$

Critical density $(\rho = \rho_c \Leftrightarrow K = 0)$

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Cosmic Sum

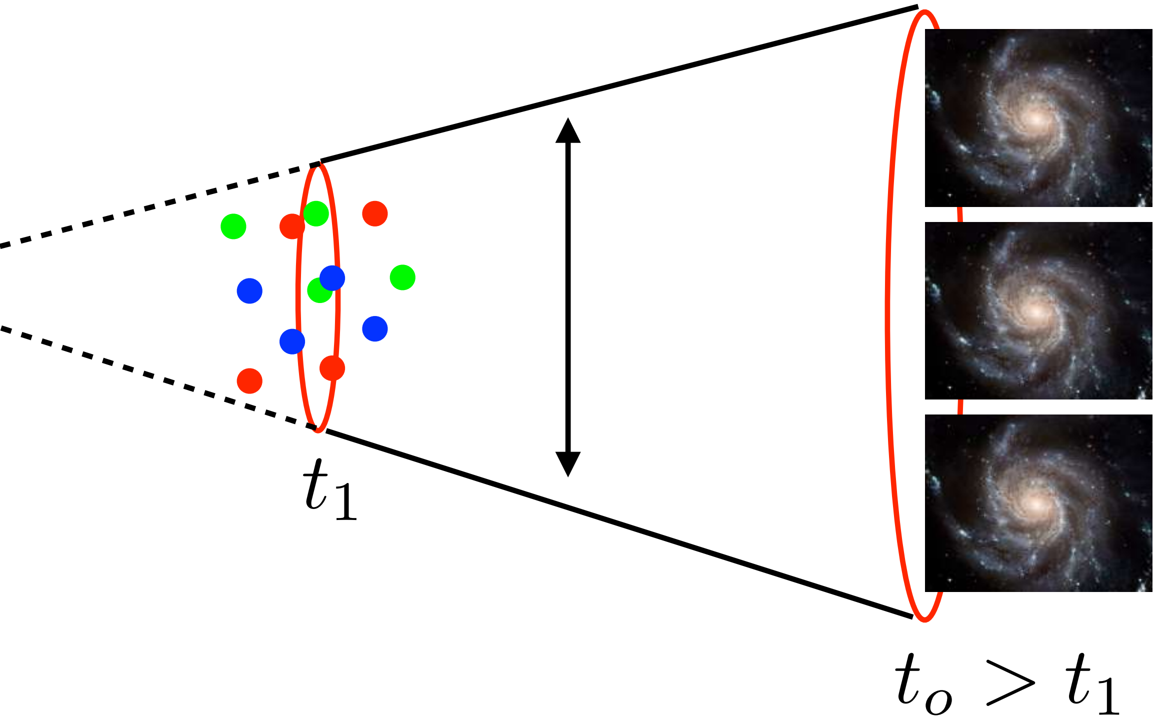
(III) + (II) :

$$H^2(a) = H_o^2 \left\{ \Omega_{\rm R}^{(o)} \left(\frac{a_o}{a}\right)^4 + \Omega_{\rm M}^{(o)} \left(\frac{a_o}{a}\right)^3 + \Omega_{\rm k}^{(o)} \left(\frac{a_o}{a}\right)^2 + \Omega_{\rm DE}^{(o)} e^{-3\int \frac{da}{a}(1+w)} \right\}$$

$$\equiv H_o^2 E^2(a)$$

$$E(a) \equiv \sqrt{\Omega_{\rm R}^{(o)} \left(\frac{a_o}{a}\right)^4 + \Omega_{\rm M}^{(o)} \left(\frac{a_o}{a}\right)^3 + \Omega_{\rm k}^{(o)} \left(\frac{a_o}{a}\right)^2 + \Omega_{\rm DE}^{(o)} e^{-3\int \frac{da}{a}(1+w)}} \qquad \Omega_{\rm k}^{(o)} \equiv -\frac{k}{a_o^2 H_o^2}$$

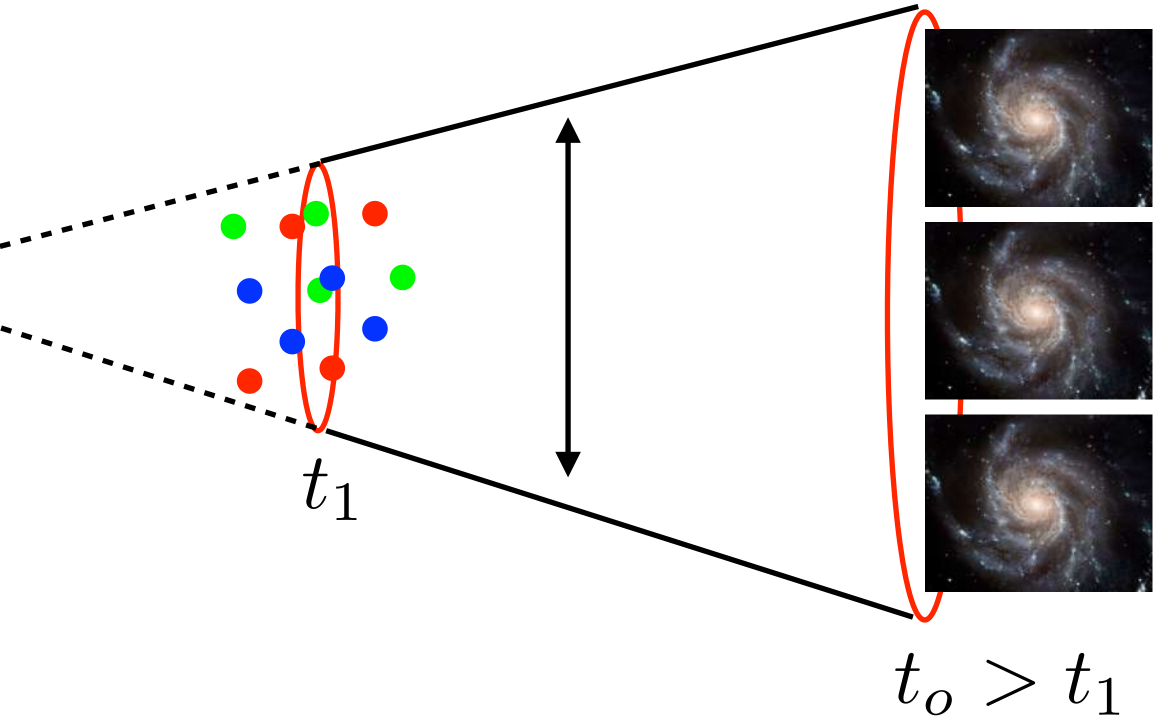
Expansion History



Past: particle ensemble

Statistical Mechanics

Expansion History

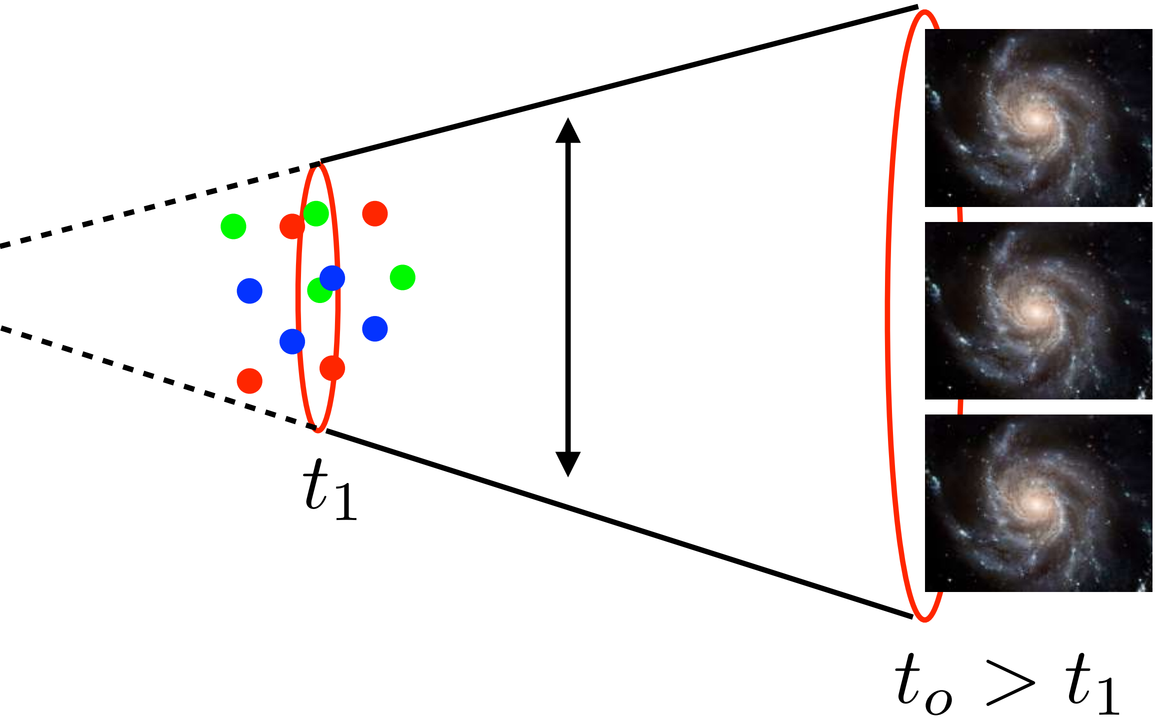


Past: particle ensemble

Statistical Mechanics

$$\text{(III)}\quad \frac{d\rho}{dt} + 3H(\rho + p) = 0 \longrightarrow \quad \frac{dU}{dt} + p\frac{dV}{dt} = 0\,, \qquad \begin{cases} U = a^3\rho\,, \\ V = a^3 \end{cases}$$

Expansion History



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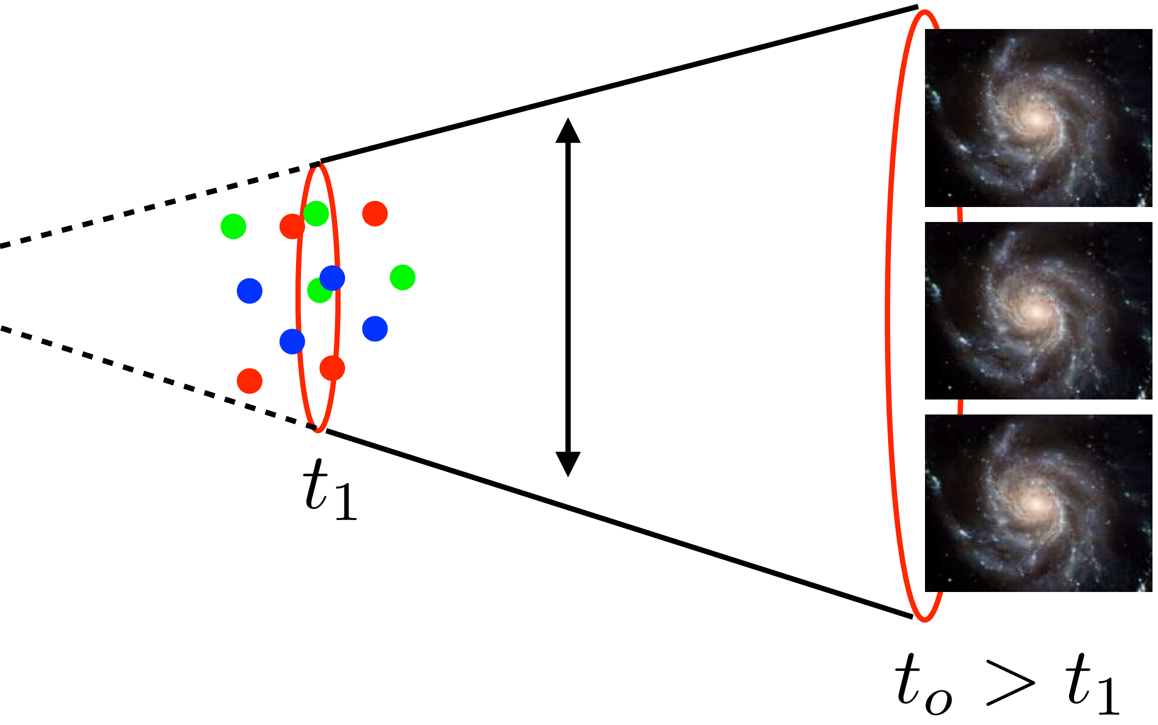
$$\Longrightarrow \quad \begin{cases} \frac{dU}{dt}+p\frac{dV}{dt}=T\frac{dS}{dt}\,, \longrightarrow \textbf{Thermal Eq.} \\ \frac{dS}{dt}=0\,, \longrightarrow \textbf{Adiabatic Exp.} \end{cases}$$

Expansion History

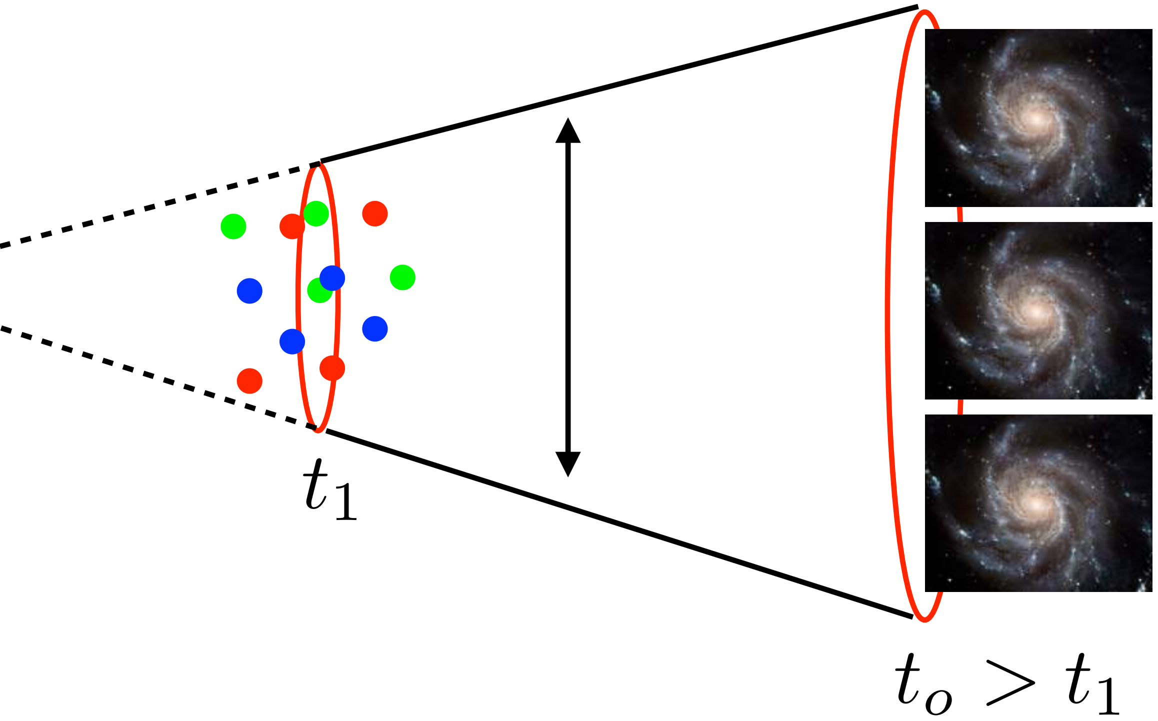


Thermal Eq. **(densities)**

$$\begin{cases} n = g_* \int d\vec{p}\, f(\vec{p})\,, & \textbf{number} \\ \rho = g_* \int d\vec{p}\, E(\vec{p})\, f(\vec{p})\,, & \textbf{energy} \\ p = g_* \int d\vec{p}\, \frac{|\vec{p}|^2}{3E(\vec{p})}\, f(\vec{p})\,, & \textbf{pressure} \end{cases}$$

dof *Dispersion relation* *Statistical Distribution*

Expansion History



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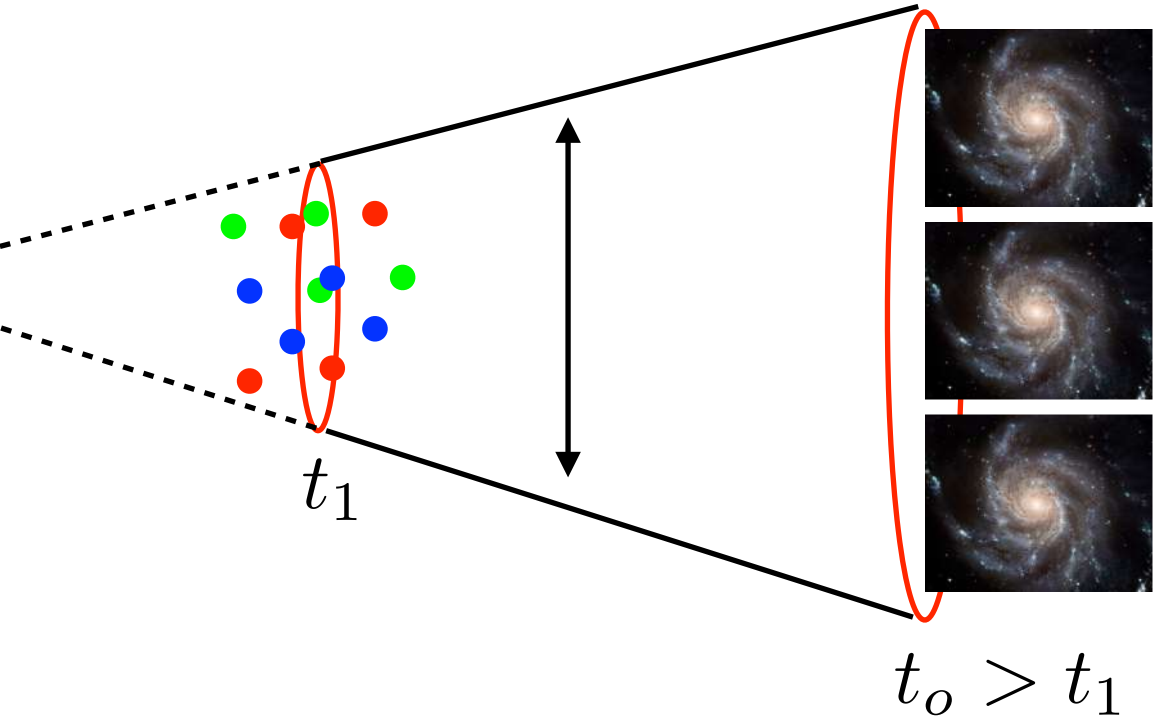
dof *Dispersion relation* *Statistical Distribution*

Bose-Einstein / Fermi-Dirac: $$f(\vec{p}) = \left(e^{E(\vec{p})/T} \pm 1\right)^{-1}, \begin{cases} F(+) & \textbf{[fermions]} \\ B(-) & \textbf{[bosons]} \end{cases}$$

Expansion History



Thermal Eq.

(densities)

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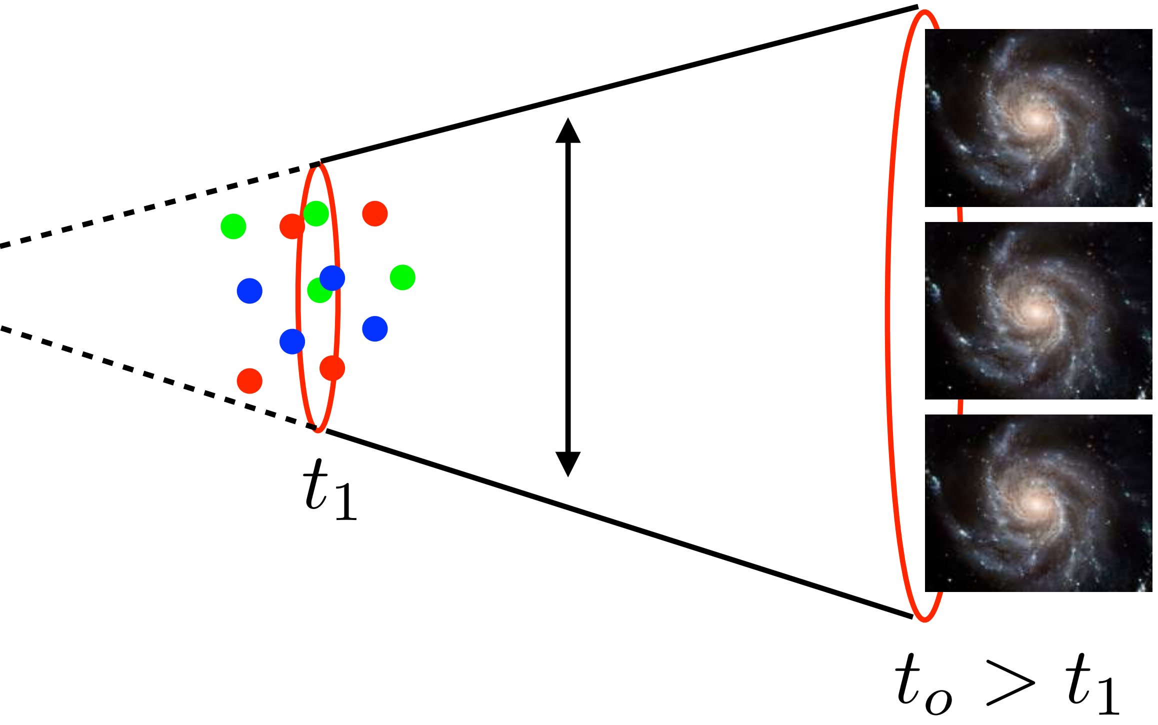
$$\frac{\rho_R \propto 1/a^4}{\rho_M \propto 1/a^3} \propto 1/a\,, \quad \Rightarrow \quad z \geq z_{\rm EQ}\ (t \leq t_{\rm EQ})\,, \quad \rho_R > \rho_M$$

Expansion History



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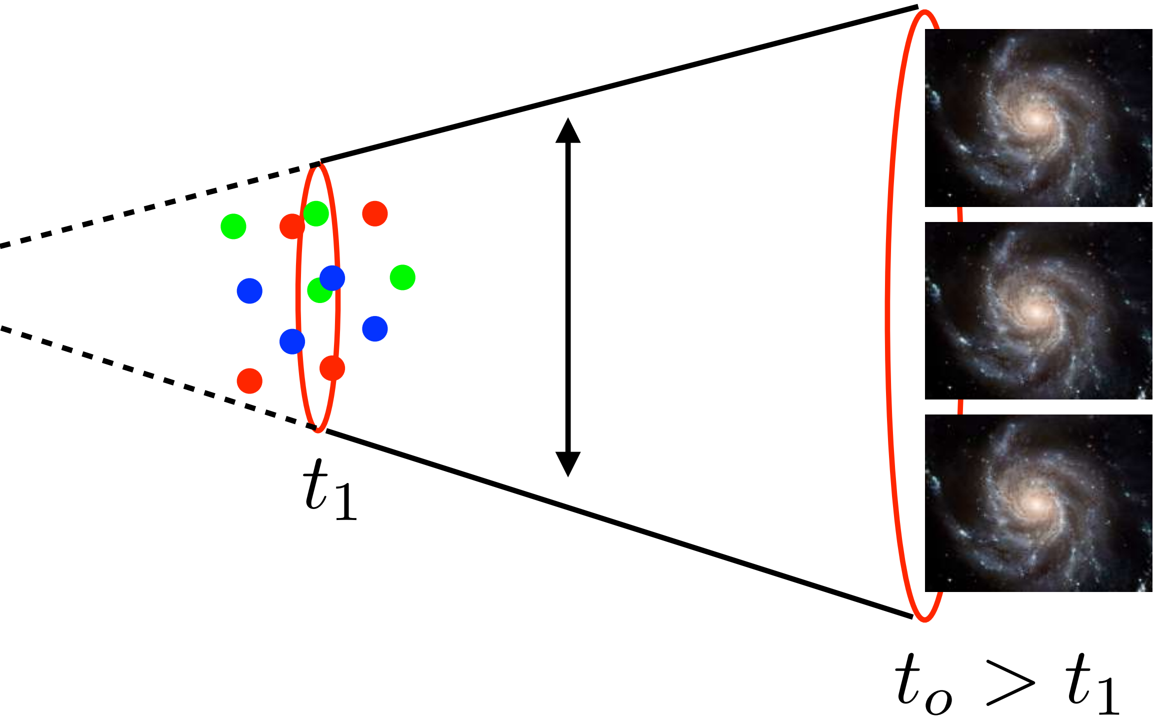
Past: Radiation Domination ! $\boxed{1 + z_{\rm EQ} = \Omega_{\rm M}^{(o)}/\Omega_{\rm Rad}^{(o)} \sim 3400}$

Expansion History



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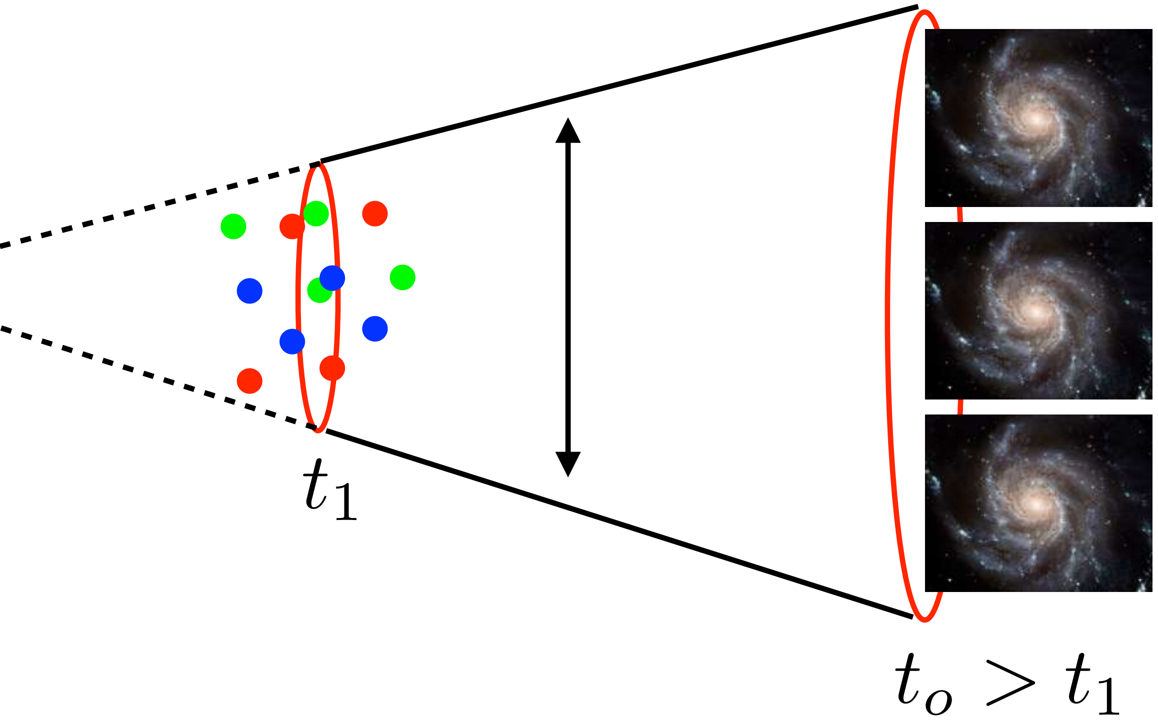
Past: Radiation Domination (RD)

$$\rho_R^{(i)} = f_i g_*^{(i)} \frac{\pi^2}{30} T_i^4\,, \quad f_i = \begin{cases} 1, & \mathrm{B} \\ \frac{7}{8}, & \mathrm{F} \end{cases}$$

Expansion History



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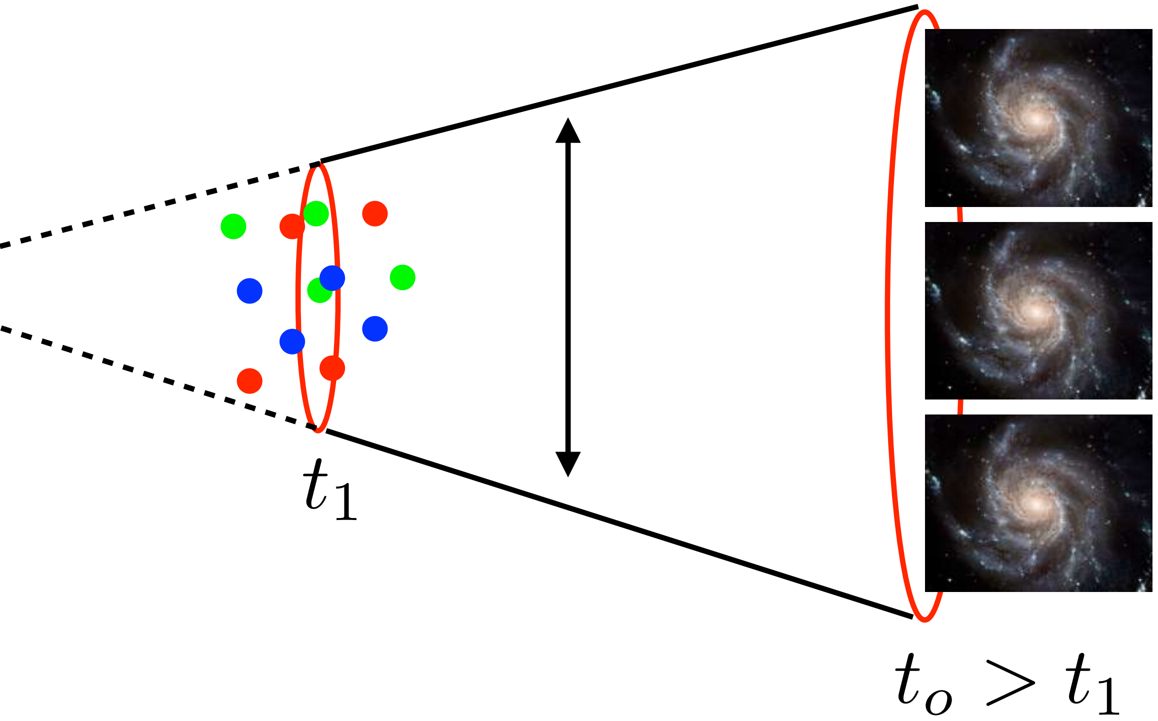

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dof — *Dispersion relation* — *Statistical Distribution*

Bose-Einstein / Fermi-Dirac: $$f(\vec{p}) = \left(e^{E(\vec{p})/T} \pm 1\right)^{-1}\,, \begin{cases} F(+) & \textbf{[fermions]} \\ B(-) & \textbf{[bosons]} \end{cases}$$

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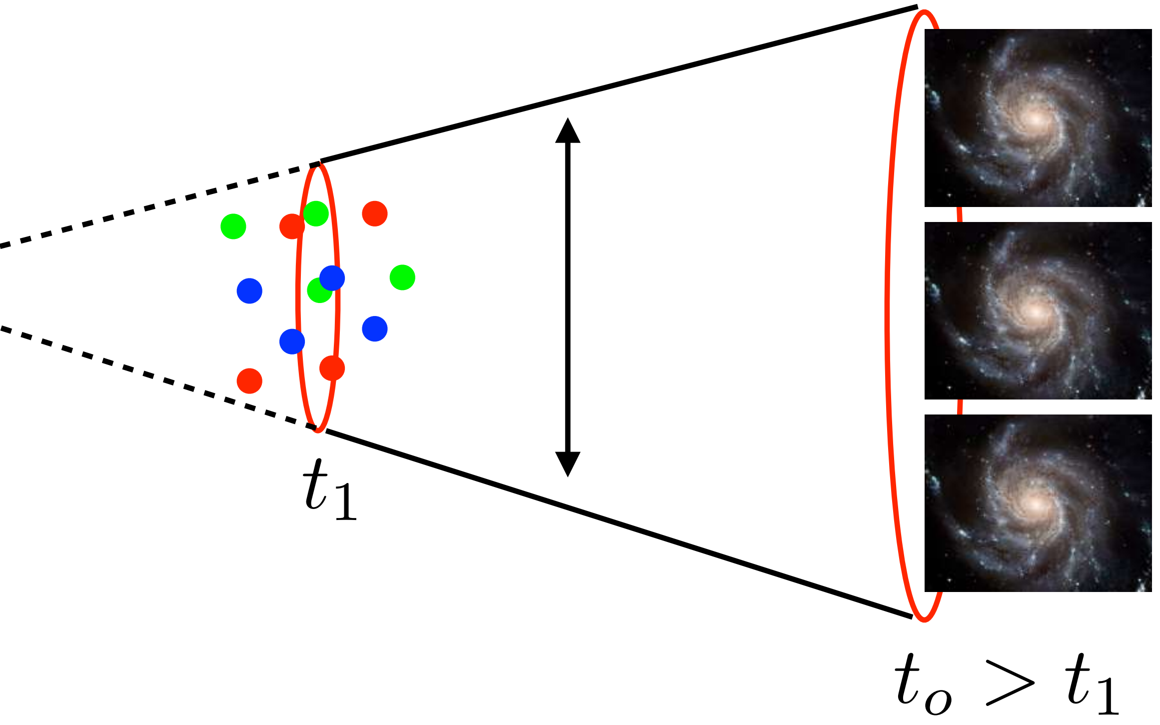
$$\boxed{\rho_R = \sum_i \rho_R^{(i)} \equiv g_*(T) \frac{\pi^2}{30} T^4}$$

$$g_*(T) \equiv \sum_i g_{*,i}^{(B)} \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_i g_{*,i}^{(F)} \left(\frac{T_i}{T}\right)^4$$

Expansion History



Adiabatic Exp:

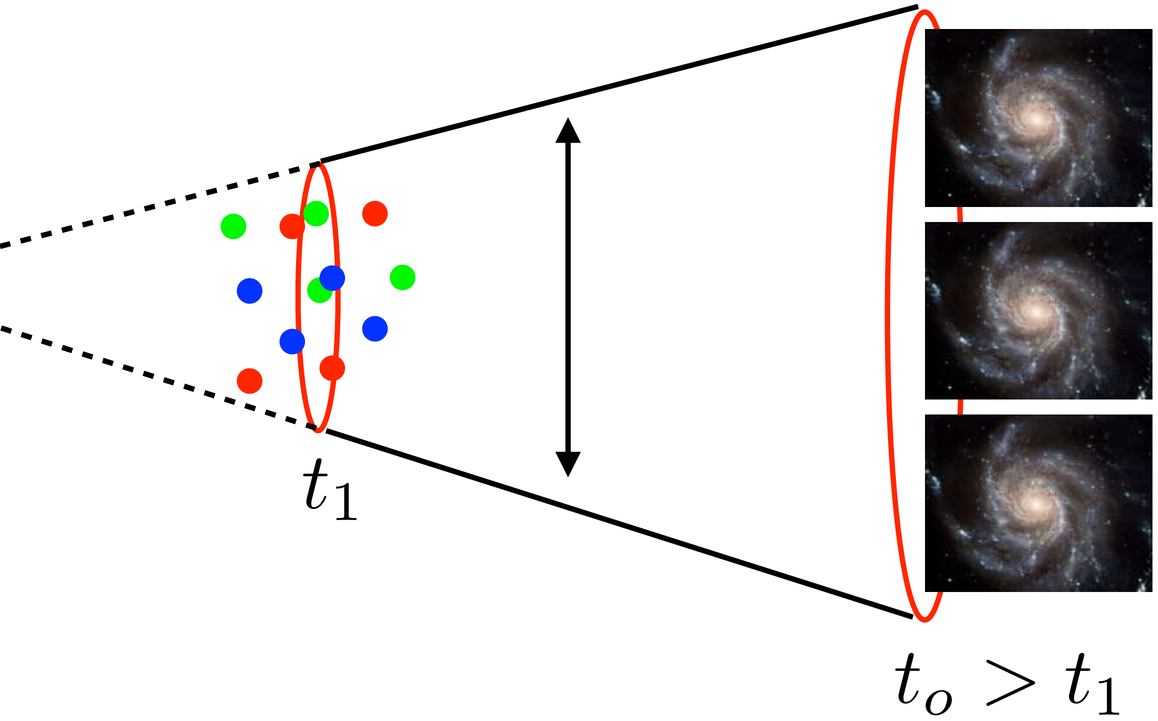
$$S = \frac{a^3(\rho + p)}{T} = const.$$

$$\Downarrow$$

$$\boxed{a^3 T^3 g_*^{(s)}(T) = const.}$$

$$g_*^{(s)}(T) \equiv \sum_i g_{*,i}^{(B)} \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_i g_{*,i}^{(F)} \left(\frac{T_i}{T}\right)^3$$

Expansion History



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$$\Downarrow$$

$$\boxed{a^3 T^3 g_*^{(s)}(T) = const.}$$

$$g_*^{(s)}(T) \equiv \sum_i g_{*,i}^{(B)} \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_i g_{*,i}^{(F)} \left(\frac{T_i}{T}\right)^3$$

When do $g_*(T), g_*^{(s)}(T)$ change ?

- $1)\ \text{Species Decoupling}\,,\ T \to T_i\,,$
- $2)\ \text{Mass threshold}\,, T < 2m_i\,,$