Scattering in chiral strong backgrounds

Tim Adamo
University of Edinburgh

based on work w/ L. Mason & A. Sharma

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Tree-level gauge theory/gravity in Minkowski space:

We know everything

[Parke-Taylor, Witten, Roiban-Spradlin-Volovich, Hodges, Cachazo-Skinner, Cachazo-He-Yuan,...]
Motivation

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- All multiplicity formulae for tree-level S-matrix
- Important data for massless $g = 0$ sector of string theory
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- Important data for massless $g = 0$ sector of string theory

But what about gauge theory/gravity/string theory in *non-trivial/strong* background fields?
Much less known – even in simplest strong backgrounds

- Tree-level 4-points (AdS, plane waves) [D’Hoker et al., Raju, TA-Casali-Mason-Nekovar]

- Unclear how ‘novel’ methods (unitarity, double copy, scattering equations) extend to strong backgrounds
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**Today:** Can we make all-multiplicity statements on (any) strong backgrounds?
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Today: Can we make all-multiplicity statements on (any) strong backgrounds?

Yes!

Class of chiral, asymp. flat strong backgrounds in 4d gauge theory and gravity
Radiative backgrounds

*Radiative* field is completely characterized by free data at \( \mathcal{I}^\pm \)

Asymp. flat gauge/gravitational fields:

\[
A = \mathcal{A}^0(u, z, \bar{z}) \, dz + \bar{\mathcal{A}}^0(u, z, \bar{z}) \, d\bar{z} + O(r^{-1})
\]

\[
ds^2 = ds^2_M - 2 \frac{m_B(u, z, \bar{z})}{r} \, du^2 + \frac{\sigma^0(u, z, \bar{z})}{r} \, dz^2 + \frac{\bar{\sigma}^0(u, z, \bar{z})}{r} \, d\bar{z}^2 + O(r^{-2})
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$$ds^2 = ds_M^2 - 2m_B(u, z, \bar{z}) \frac{du^2}{r} + \sigma^0(u, z, \bar{z}) \frac{dz^2}{r} + \bar{\sigma}^0(u, z, \bar{z}) \frac{d\bar{z}^2}{r} + O(r^{-2})$$

Free data are spin-weighted functions on $\mathcal{I}^+$:

$$\mathcal{A}^0(u, z, \bar{z}) \leftrightarrow s = 1, \quad \sigma^0(u, z, \bar{z}) \leftrightarrow s = 2$$
SD Radiative fields

Lorentzian-real fields $\mathcal{A}^0$, $\tilde{\mathcal{A}}^0$ and $\sigma^0$, $\tilde{\sigma}^0$ related by complex conjugation

Complexify $\Rightarrow \mathcal{A}^0$, $\tilde{\mathcal{A}}^0$ and $\sigma^0$, $\tilde{\sigma}^0$ independent
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A SD radiative field has $\bar{A}^0 = 0 / \bar{\sigma}^0 = 0$
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A **SD radiative field** has $\tilde{\mathcal{A}}^0 = 0/\tilde{\sigma}^0 = 0$

Idea:

- Exploit integrability of SD sector to compute tree-level S-matrix
Twistor theory

Twistor space: \( Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \) homog. coords. on \( \mathbb{CP}^3 \)

\[ \mathbb{PT} = \mathbb{CP}^3 \setminus \{ \lambda_\alpha = 0 \} \]

\( x \in \mathbb{C}^4 \) given by \( X \cong \mathbb{CP}^1 \subset \mathbb{PT} \) via \( \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha \)

Many applications in flat background:

- Massless free fields \( \leftrightarrow \) cohomology on \( \mathbb{PT} \) [Penrose, Sparling, Eastwood-Penrose-Wells]
- Representation for on-shell scattering kinematics [Hodges]
- Full tree-level S-matrix of \( N = 4 \) SYM [Witten, Berkovits, Roiban-Spradlin-Volovich]
- Full tree-level S-matrix of \( N = 8 \) SUGRA [Cachazo-Skinner]
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Trivializing SD sector

Theorem [Ward 1977]
There is a 1:1 correspondence between:
• SD SU($N$) Yang-Mills fields on $\mathbb{C}^4$, and
• rank $N$ holomorphic vector bundles $E \to \mathbb{P}T$ trivial on every $X \subset \mathbb{P}T$ (+ technical conditions)
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**Theorem [Penrose 1976]**
There is a 1:1 correspondence between:
- SD 4-manifolds $\mathcal{M}$, and
- $\mathbb{P}T$ a complex deformation of $\mathbb{P}T$ with rational curve $X \subset \mathbb{P}T$ with normal bundle $N_X \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$
SD sector encoded by integrable $\mathbb{C}$-structures on twistor space

- **Gauge theory**: $E \to \mathbb{PT}$ partial connection $\bar{D} = \bar{\partial} + A$, $A \in \Omega^{0,1}(\text{End}E)$, $\bar{D}^2 = 0$
- **Gravity**: $T_{\mathbb{P}\mathcal{F}}$ complex structure $\bar{\nabla} = \bar{\partial} + V$, $V \in \Omega^{0,1}(T_{\mathbb{P}\mathcal{F}})$, $\bar{\nabla}^2 = 0$
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For SD radiative fields [Sparling, Newman, Eastwood-Tod]

- $A(Z) = \partial_u A^0(\mu^{\hat{\alpha}} \bar{\lambda}_{\hat{\alpha}}, \lambda, \bar{\lambda}) D\bar{\lambda}$

- $V(Z) = \sigma^0(\mu^{\hat{\alpha}} \bar{\lambda}_{\hat{\alpha}}, \lambda, \bar{\lambda}) D\bar{\lambda} \bar{\lambda}^{\hat{\alpha}} \frac{\partial}{\partial \mu^{\hat{\alpha}}}$
Gauge theory

Restrict to SD rad. gauge fields valued in Cartan $E|_X$ trivialized by holomorphic frame

\[ \bar{\partial}|_X H(x, \lambda) = -A|_X H(x, \lambda), \]
\[ A|_X = \bar{\partial} g(x, \lambda), \quad \text{as } H^{0,1}(\mathbb{P}^1, \mathcal{O}) = \emptyset, \]
\[ \Rightarrow \quad H(x, \lambda) = \exp[-g(x, \lambda)] \]
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\]

Recover space-time gauge field from

\[
\partial_{\alpha\dot{\alpha}}g(x, \lambda) = A_{\alpha\dot{\alpha}}(x) + \lambda_{\alpha} g_{\dot{\alpha}}(x, \lambda)
\]
Gluon perturbations

Gluon perturbations in background encoded by cohomology:

\[ + \text{ helicity gluon} \leftrightarrow a \in H^0_D(\mathbb{PT}, \mathcal{O} \otimes \End E) \]

\[ - \text{ helicity gluon} \leftrightarrow b \in H^0_D(\mathbb{PT}, \mathcal{O}(-4) \otimes \End E) \]
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E.g. + helicity with asymp. momentum $k_{\alpha \dot{\alpha}} = \kappa_{\alpha} \tilde{\kappa}_{\dot{\alpha}}$

$$a(Z) = T \frac{\langle \xi \lambda \rangle}{\langle \xi \kappa \rangle} \tilde{\partial} \left( \frac{1}{\langle \lambda \kappa \rangle} \right) e^{i \langle \xi \kappa \rangle [\mu \tilde{\kappa}]}$$

Leads to

$$a^{(\alpha)}_{\alpha \dot{\alpha}}(x) = T \frac{\xi_{\alpha} (\kappa_{\dot{\alpha}} + e g_{\dot{\alpha}}(x, \kappa))}{\langle \xi \kappa \rangle} \exp [i k \cdot x + e g(x, \kappa)]$$
How do we use all this to actually compute amplitudes?
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Lesson from flat background: [Nair, Witten, Berkovits, Roiban-Spradlin-Volovich]

- Tree-level YM amplitudes live on rational holomorphic curves in $\mathbb{PT}$
- $N^k$ MHV degree related to degree of curve $d = k + 1$

\[
\mu^{\hat{\alpha}}(\sigma) = u_{a(d)}^{\hat{\alpha}} \sigma^{a(d)}, \quad \lambda_\alpha(\sigma) = \lambda_\alpha a(d) \sigma^{a(d)}
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Colour-ordered \( N^{d-1} \) MHV amplitude of \( \mathcal{N} = 4 \) SYM:

\[
\int \frac{d^{2d+2} \lambda}{\text{vol GL}(2, \mathbb{C})} \delta^2|4(d+1)\left( \sum_{i=1}^{n} s_i \tilde{\kappa}_i \sigma_i^{a(d)} \right) \\
\times \prod_{j=1}^{n} \frac{ds_j D\sigma_j}{s_j (jj + 1)} \delta^2(\kappa_j - s_j \lambda(\sigma_j))
\]
Central conjecture

Twistor string theory coupled to background field A (trivial extension to $\mathcal{N} = 4$ SYM)

Worldsheet action:

$$ S = \frac{1}{2\pi} \int_{\Sigma} Y_I \partial Z^I + \text{tr}(j A) + S_g + \text{ghosts}, $$

Vertex operators:

$$ V = \int_{\Sigma} \text{tr}(j a(Z)) $$
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Conjecture

$N^k$MHV amplitude on SD rad. background given by worldsheet correlator at degree $d = k + 1$
Amplitude formula

Let $Z^I(\sigma) = U^I_{a(d)} \sigma^{a(d)}$, and

$$A(Z(\sigma)) = \ddbar g(U, \sigma)$$

Result for $N^k$ MHV color-ordered tree amplitude:

$$\int \frac{d^4|4(d+1)U}{\text{vol GL}(2, \mathbb{C})} \prod_{i=1}^n \frac{ds_i \text{D}\sigma_i}{s_j (i \ i + 1)} \delta^2(\kappa_i - s_i \lambda(\sigma_i)) \times \exp \left( [\mu(\sigma_i) \bar{\kappa}_i] + e_i g(U, \sigma_i) \right)$$
Properties

Formula passes many checks
- Explicit 3- and 4-point checks
- Flat & perturbative limits
- Proved for MHV ($d = 1$)

Important differences from RSVW
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Similar – more complicated – story for gravity!
All-multiplicity formulae on some strong backgrounds!

⇒ interesting probe of non-linearities in QFT

Any use/interest for string theory?

• SD *plane waves* \( \subset \) SD radiative backgrounds
• Vacuum plane waves are string backgrounds [Amati-Klimcik, Horowitz-Steif]

• So these formulae provide data for the \( \alpha' \rightarrow 0 \) limit of strings in plane wave backgrounds
Further directions

Many interesting open questions:

• Prove $N^k$MHV formulae for $k > 1$ (analyticity constraints [Ilderton-MacLeod]?)
• Other chiral backgrounds (with sources, not asymptotically flat)
• Non-chiral backgrounds, scattering equations, ambitwistor strings
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Thanks!