D-instanton Perturbation Theory

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Plan:

1. Overview of the problem and the solution

2. Review of basic aspects of world-sheet string theory and string field theory

3. Some explicit computations

A.S., arXiv:1908.02782, 2002.04043, work in progress

The problem

String theory began with Veneziano amplitude

– tree level scattering amplitude of four tachyons in open string theory

World-sheet expression for the amplitude (in $\alpha' = 1$ unit)

$$\int_0^1 dy\, y^{2p_1.p_2}(1-y)^{2p_2.p_3}$$

– diverges for $2p_1.p_2 \le -1$ or $2p_2.p_3 \le -1$.

Our convention: $a.b \equiv -a^0 b^0 + \vec{a}.\vec{b}$

Conventional viewpoint: Define the amplitude for $2p_1.p_2 > -1$, $2p_2.p_3 > -1$ and then analytically continue to the other kinematic regions.

However, analytic continuation does not always work

It may not be possible to move away from the singularity by changing the external momenta

Examples: Mass renormalization, Vacuum shift

– discussed earlier

In these lectures we shall discuss another situation where analytic continuation fails

– D-instanton contribution to string amplitudes

D-instanton: A D-brane with Dirichlet boundary condition on all non-compact directions including (euclidean) time.

D-instantons give non-perturbative contribution to string amplitudes that are important in many situations

Example: KKLT moduli stabilization uses non-perturbative contribution from D-instanton (euclidean D3-brane)

Systematic computation of string amplitudes in such backgrounds will require us to compute amplitudes in the presence of D-instantons

Problem: Open strings living on the D-instanton do not carry any continuous momenta

$\Rightarrow$ we cannot move away from the singularities by varying the external momenta

Some examples:

Let X be the (euclidean) time direction

Since the D-instanton is localized at some given euclidean time, it has a zero mode that translates it along time direction

4-point function of these zero modes:

$$A = \int_0^1 dy\ \left[y^{-2} + (y-1)^{-2} + 1\right]$$

Derivation of this expression will be discussed later.

$$A = \int_0^1 dy \, \left[y^{-2} + (y-1)^{-2} + 1\right]$$

– diverges from near y=0 and y=1.

In this case no analytic continuation is possible since open strings on D-instantons do not carry momentum.

On physical grounds, we expect this amplitude to vanish since translation along X is an exactly marginal deformation of the world-sheet theory.

In the first example we shall study, we shall see how to get 0 from this divergent integral.

Related work:

Berkovits, Schnabl: hep-th/0307019

Maccaferri, Merlano, arXiv:1801.07607,1905.04958

Erbin, Maccaferri, Vosmera, arXiv:1912.05463, $\cdots$

Another example: Bosonic string theory in two dimensions

World-sheet theory: A free scalar X describing time coordinate and a Liouville field theory with central charge 25

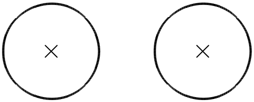
Total central charge adds up to 26, cancelling anomalies on the world-sheet

In this case the closed string ‘tachyon’ is actually a massless state of the theory

In arXiv:1907.07688 Balthazar, Rodriguez and Yin (BRY) computed the D-instanton contribution to the two point amplitude of closed string tachyons

This model is interesting because there is a dual matrix model description that gives the exact results.

The leading contribution comes from the product of two disk one point functions.



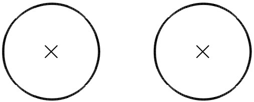
Result:

$$8\,\pi\,\mathbf{N}\,\mathbf{e}^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$

N: An overall normalization constant

g_s: string coupling constant

$-\omega_1, \omega_2$: energies of incoming / outgoing ‘tachyons’



Naively one might have expected this to be proportional to $\delta(\omega_1)\delta(\omega_2)$

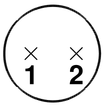
However, for D-instanton boundary conditions, individual disk amplitudes do not conserve energy, since time translation invariance is broken

The energy conservation is restored at the end after integration over the collective coordinates

– will be discussed later.

At the next order, there are ‘two’ contributions.

1. Two point function on the disk.



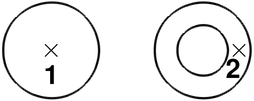
Result:

$$8\,\pi\,N\,e^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$
$$\times\frac{1}{2}\,g_s\int_0^1 dy\,y^{-2}(1+2\omega_1\omega_2 y)+\text{ finite terms}$$

Note the divergences from the $y\to 0$ limit

– cannot be tamed by deforming the ω_i’s.

2. Product of disk one point function and annulus one point function.



$+ (1 \Leftrightarrow 2)$

Result:

$$8\,\pi\,N\,e^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$

$$\times\; g_s\int_0^1 dv\int_0^{1/4} dx\left\{2\,\frac{v^{-2}-v^{-1}}{\sin^2(2\pi x)}+2(\omega_1^2+\omega_2^2)\,v^{-1}\right\}$$

$$+\;\text{Finite terms}$$

Note the divergences from $x\to 0$ and $v\to 0$ that cannot be tamed by adjusting the ω_i's.

Finite terms include divergences that can be tamed by analytic continuation in ω_1,ω_2.

After setting $\omega_2 = -\omega_1$, the total divergent factor is:

$$\frac{1}{2} \int_0^1 dy\, y^{-2}(1 - 2\omega_1^2 y) + \int_0^1 dv \int_0^{1/4} dx \left\{ 2\, \frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 4\, \omega_1^2\, v^{-1} \right\}$$

BRY replaced this by

$$A + B\, \omega_1^2$$

with unknown constants A and B.

They then numerically compared the result with matrix model results as function of ω_1.

The best fit results:

$$A = -0.496, \qquad B = -1.399$$

Question: Can we get these results from string theory without invoking the matrix model?

Answer: $B = -\ln 4 \simeq -1.386...$, $\qquad A = ?$

The solution

We shall use string field theory (SFT) to deal with the divergences arising in the world-sheet theory.

SFT is a regular quantum field theory (QFT) with infinite number of fields

Perturbative amplitudes: sum of Feynman diagrams

Each diagram covers part of the integration region over the world-sheet variables (moduli space of Riemann surfaces and locations of vertex operators)

Sum of the diagrams covers the full integration region.

How do we get integral over world-sheet variables from a Feynman diagram?

Express internal propagator as

$$(k^2+m^2)^{-1} = \int_0^\infty ds\, e^{-s(k^2+m^2)} = \int_0^1 dq\, q^{k^2+m^2-1}, \quad q \equiv e^{-s}$$

The integration over q gives integration over world-sheet variables after a change of variable.

Divergences come from the $q \to 0$ region for $k^2+m^2 \leq 0$.

All divergences in string theory are of this kind.

For D-instantons k=0, and we cannot analytically continue in momenta to make $k^2+m^2 > 0$.

$$(\mathbf{m}^2)^{-1} = \int_0^1 \mathbf{dq}\,\mathbf{q}^{\mathbf{m}^2-1}$$

This equation is:

1. An identity for $\mathbf{m}^2 > 0$.

2. For $\mathbf{m}^2 < 0$ the lhs is finite but the rhs is divergent

$\Rightarrow$ use lhs to define the integral.

– Change variables from the moduli of Riemann surfaces to the variables $\mathbf{q}_1, \mathbf{q}_2, \cdots$ associated with the propagators

– Replace $\int_0^1 \mathbf{dq}\,\mathbf{q}^{\beta-1}$ by $1/\beta$ for $\beta \neq 0$

– can be used to deal with power law divergences like $\int_0^1 \mathbf{dy}\,\mathbf{y}^{-2}$ in the earlier formulæ

For implementing this procedure, it is crucial that we transform the original integration variables (like y, v, x) to the q variables associated with the propagators in string field theory

The procedure for doing this will be explained in later lectures.

Comment 1: Making the correct change of variables is important for getting the correct result.

Replacement rule: $\int_0^1 \mathbf{dq\,q^{-2}} = -1$

Suppose we change variable to

$$q' = \frac{q}{(1 - c\,q)} \quad \Leftrightarrow \quad \frac{1}{q'} = \frac{1}{q} - c, \qquad c = \text{constant}$$

Then $dq\,q^{-2} = dq'\,q'^{-2}$

$$\Rightarrow \int_0^1 dq\,q^{-2} = \int_0^{1/(1-c)} dq' q'^{-2} = \int_0^1 dq' q'^{-2} + \int_1^{1/(1-c)} dq' q'^{-2}$$

If we now replace the first term on the rhs by -1 **using the replacement rule, we get**

$$-1 + 1 - (1 - c)^{-1} = -(1 - c)^{-1}$$

– a different answer!

Comment 2. The change of variables is determined by string field theory, but may not take a simple form.

e.g. consider the four point function of translational zero modes:

$$A = \int_0^1 dy \, \left[y^{-2} + (y-1)^{-2} + 1 \right]$$

The change of variable near y=0 takes the form:

$$y = 1 - \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1+\gamma)^2\,q} \, \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1-\gamma)^2\,q}$$

α, γ: parameters of string field theory

The final result is independent of α, γ.

This formula will be derived later.

$$(\mathbf{m^2})^{-1} = \int_0^1 \mathbf{dq}\, \mathbf{q^{m^2-1}},$$

For $m = 0$ both sides are divergent.

– associated with zero modes on the D-instanton

– produces logarithmic divergence in the world-sheet description

Strategy: Understand the physical origin of the zero modes and then find suitable remedy by drawing insights from QFT.

D-instantons have zero modes associated with translation of the instanton position along transverse directions

– known as collective coordinates ϕ

$\Rightarrow$ massless open string states

Treatment of these zero modes in QFT:

1. Carry out path integral over all modes of the instanton other than ϕ, in the background of ϕ

$\Rightarrow$ while evaluating Feynman diagrams we remove the ϕ contribution from the internal propagators but keep ϕ's as external states

After summing over Feynman diagrams we get a given closed string amplitude as a function F(ϕ).

2. Then we compute $\int$ dϕ F(ϕ)

Strategy: Follow the same procedure for D-instantons

In the world-sheet approach the first step demands:

a. Drop terms of the from $\int_0^1 dq\, q^{-1}$ coming from the collective coordinates

b. Allow external states to be both closed strings and the open string zero mode ϕ.

The second step is a finite dimensional integral over ϕ that needs to be performed.

After field redefinition, ϕ dependence is of the form $e^{ip.\phi}$

p: total momenta carried by the external closed strings

ϕ integration will generate the $\delta(p)$ factor.

An important point:

In general the zero mode of the open string will be related to the collective mode ϕ after a field redefinition.

Only after this field redefinition we have simple dependence on ϕ of the form $e^{ip.\phi}$ and the ϕ integral is performed easily.

The field redefinition may induce a Jacobian in the path integral measure that needs to be taken into account in the analysis.

We shall see an example of this in later lectures

However, string theory has other zero modes besides the ones associated with the collective coordinates

– arise from the ghost sector

– gives additional logarithmic divergence in the world-sheet integrals that is not removed by removing the collective modes from the propagators.

These divergences are clearly visible in various world-sheet expressions, including the BRY formula:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1-2\omega_1^2 y)+\int_0^1 dv\int_0^{1/4} dx\left\{2\,\frac{v^{-2}\overset{\Downarrow}{-}v^{-1}}{\sin^2(2\pi x)}+4\,\omega_1^2\, v^{-1}\right\}$$

Understanding the origin of these divergences and their treatment will require some knowledge of world-sheet string theory and string field theory.

World-sheet theory

Bosonic string world-sheet theory

– a c=26 matter CFT

– ghosts b, c, $\bar{b}$, $\bar{c}$ of conformal weights (2,0), (-1,0), (0,2), (0, -1)

Ghost number: 1 for c, $\bar{c}$, -1 for b, $\bar{b}$

Assume matter CFT has (euclidean) time coordinate X and a c=25 CFT

Examples of c=25 CFT:

– 25 free scalars $\Rightarrow$ D=26 bosonic string

– c=25 Liouville $\Rightarrow$ D=2 bosonic string

State - operator correspondence in CFT $\Rightarrow$ there must be a vertex operator for every string state

Closed string state $\Leftrightarrow$ vertex operator in the bulk

Open string state $\Leftrightarrow$ vertex operator on the boundary

We shall focus on open string theory on a D-instanton since the problem arises there.

Ghost boundary conditions: $\mathbf{c} = \bar{\mathbf{c}}$, $\mathbf{b} = \bar{\mathbf{b}}$

Matter boundary conditions: Dirichlet along all non-compact directions

Expansions of fields in the upper half plane:

$$c = \sum_n c_n z^{-n+1}, \quad b(z) = \sum_n b_n z^{-n-2}, \quad i\partial X = \sum_{n\neq 0} \alpha_n z^{-n-1}, \ \cdots$$

Note: Due to boundary condition on the real axis, the expansion coefficients of $\bar{b}$, $\bar{c}$, $i\bar{\partial}X$ are given by b_n, c_n, α_n

SL(2,R) invariant vacuum $|0\rangle$:

$$c_n|0\rangle = 0 \text{ for } n \geq 2, \quad b_n|0\rangle = 0 \text{ for } n \geq -1, \quad \alpha_n|0\rangle \text{ for } n \geq 0$$

State-operator correspondence: $|V\rangle = V(0)|0\rangle$, e.g.

$$c \Leftrightarrow c(0)|0\rangle = c_1|0\rangle, \quad b \Leftrightarrow b(0)|0\rangle = b_{-2}|0\rangle, \quad i\partial X \Leftrightarrow i\partial X(0)|0\rangle = \alpha_{-1}|0\rangle$$

Singular OPE:

$$\mathbf{b(z)c(w)} = \frac{\mathbf{1}}{\mathbf{z-w}}, \quad \partial\mathbf{X(z)}\partial\mathbf{X(w)} = \frac{\mathbf{1}}{\mathbf{(z-w)^2}} \quad \textbf{etc}$$

$$\Rightarrow \quad \{\mathbf{b_n}, \mathbf{c_m}\} = \delta_{\mathbf{m+n,0}}, \quad [\alpha_{\mathbf{m}}, \alpha_{\mathbf{n}}] = -\mathbf{m}\,\delta_{\mathbf{m+n,0}} \quad \textbf{etc}$$

States in Hilbert space H: Created by action on $|0\rangle$ of

c_{-n} for $n \geq -1$, b_{-n} for $n \geq 2$, α_{-n} for $n \geq 1$, etc.

Physical open string states have ghost number 1

– described by vertex operators with ghost number 1.

Time translation of D-instanton $\Rightarrow$ zero mode ϕ

$\Rightarrow$ described by the open string state $c_1 \alpha_{-1}|0\rangle$

– corresponding vertex operator: $i\, c\, \partial X$ (unintegrated)

Generic physical open string states have vertex operator cW, with W a dimension one primary in the matter sector.

Associated 'integrated' vertex operator is W

String amplitudes from a given Riemann surface:

1. Take a Riemann surface, possibly with boundaries

2. Choose some marked points (punctures) on the Riemann surface

– one bulk puncture for each external closed string

– one boundary puncture for each external open string

3. If the Riemann surface has conformal Killing vectors then use them to fix the locations of some punctures.

4. Insert unintegrated vertex operators at fixed punctures, integrated vertex operators at variable punctures.

5. Insert additional b-ghosts, one for each modulus of the Riemann surface.

6. Evaluate the correlation function, and integrate this over the locations of the punctures and moduli of the Riemann surface.

Example: Amplitudes of four zero mode fields ϕ from the upper half plane (UHP)

UHP has three conformal Killing vectors generating SL(2,R) isometry

$$z \to (az+b)/(cz+d)$$

– can be used to fix three of the punctures at 0, 1, ∞.

Put the fourth puncture at y.

Amplitude:

$$A = \int_0^1 dy \langle c\partial X(0)\, \partial X(y)\, c\partial X(1)\, c\partial X(\infty)\rangle$$

Use $\langle c(x_1)c(x_2)c(x_3)\rangle = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)$ to get:

$$A = \int_0^1 dy \left[y^{-2} + (1-y)^{-2} + 1\right]$$

Note, however, that there are plenty of states with $L_0 = 0$ other than the translational zero mode $c_1 \alpha_{-1}|0\rangle$

e.g. $|0\rangle$, $c_1 c_{-1}|0\rangle$ etc.

– will give additional logarithmic divergence in the world-sheet formulation.

To understand how to treat them, we need to turn to string field theory.

Lecture 2

String field theory

We need to study the field theory of open and closed strings, but in this discussion we shall focus on the open string sector where most of the subtleties arise.

H: Full vector space of open string states in matter + ghost sector

An off-shell open string field $|\psi\rangle$ is an arbitrary element of H.

Let $|\phi_r\rangle$ be a set of basis states in H.

Then $|\psi\rangle = \sum_r \chi_r |\phi_r\rangle$

χ_r are the dynamical variables over which we do (path) integration.

χ_r is grassmann even (odd) if $|\phi_r\rangle$ is grassmann odd (even).

Action:

$$S = \frac{1}{2}\langle\psi|Q_B|\psi\rangle + \text{interaction terms}$$

$$Q_B = \oint_0 dz\ [c(z)\,T_m(z) + b(z)c(z)\partial c(z)]$$

$T_m(z)$: matter stress tensor

$Q_B^2 = 0$

The action S is the master action in the BV formalism.

Recall ghost number assignment:

c: 1, b: -1, matter: 0

We choose a basis of states $|\varphi_{\mathbf{r}}\rangle$ of ghost number $\leq$ 1 and another basis of states $|\eta^{\mathbf{r}}\rangle$ of ghost number $\geq$ 2 such that

$$\langle\eta^{\mathbf{r}}|\varphi_{\mathbf{s}}\rangle = \delta^{\mathbf{r}}_{\mathbf{s}}, \quad \langle\eta^{\mathbf{r}}|\eta^{\mathbf{s}}\rangle = \mathbf{0} = \langle\varphi_{\mathbf{r}}|\varphi_{\mathbf{s}}\rangle$$

Expand the string fields as:

$$|\psi\rangle = \sum_{\mathbf{r}}(\psi^{\mathbf{r}}|\varphi_{\mathbf{r}}\rangle + \xi_{\mathbf{r}}|\eta^{\mathbf{r}}\rangle)$$

We declare $\psi^{\mathbf{r}}$ as fields and $\xi_{\mathbf{r}}$ as the conjugate anti-fields (up to sign).

Among ψ^{r}, those multiplying ghost number 1 states are called classical fields.

Those with ghost number $\leq$ 0 correspond to ghosts associated with gauge transformation parameters.

Note however, that in BV we already have the ghosts before gauge fixing.

S satisfies BV master equation

$$\frac{\partial_{\mathsf{R}} \mathsf{S}}{\partial \psi^{\mathsf{r}}} \; \frac{\partial_{\mathsf{L}} \mathsf{S}}{\partial \xi_{\mathsf{r}}} - \frac{\partial_{\mathsf{R}} \mathsf{S}}{\partial \xi_{\mathsf{r}}} \; \frac{\partial_{\mathsf{L}} \mathsf{S}}{\partial \psi^{\mathsf{r}}} + \frac{\partial_{\mathsf{R}}}{\partial \psi^{\mathsf{r}}} \frac{\partial_{\mathsf{L}}}{\partial \xi_{\mathsf{r}}} \mathsf{S} = \mathbf{0}$$

R,L: left, right derivatives

Rules for path integral:

Integrate over a 'Lagrangian submanifold' weighted by e^S.

Example of Lagrangian submanifold: For each pair (ψ^r, ξ_r) either set $\psi^r = 0$ or set $\xi_r = 0$

BV $\Rightarrow$ physical quantities are independent of the choice of Lagrangian submanifold.

Let us first take the Lagrangian submanifold as $\xi_r = 0$.

Left-over fields ψ^r: classical fields and ghost fields associated with gauge transformation parameters

The action $\left[\frac{1}{2}\langle\psi|Q_B|\psi\rangle + \text{interactions}\right]$ depends only on the classical fields (by ghost no. conservation)

The path integral reduces to the usual path integral over all classical fields.

Integration over the ghost fields $\Rightarrow$ division by the volume of the gauge group

This is formally the correct path integral but somewhat singular since we have not gauge fixed.

Siegel gauge fixing:

$$|\psi\rangle = \sum_{r} (\psi^{r}|\varphi_{r}\rangle + \xi_{r}|\eta^{r}\rangle), \qquad \langle\eta^{r}|\varphi_{s}\rangle = \delta^{r}_{s}$$

We can choose the basis states such that, either

$$b_0|\varphi_r\rangle = 0, \quad c_0|\eta^r\rangle = 0, \qquad e.g.\ |\varphi_r\rangle = c_1|0\rangle,\ |\eta^r\rangle = c_0c_1|0\rangle$$

or

$$c_0|\varphi_r\rangle = 0, \quad b_0|\eta^r\rangle = 0 \qquad e.g.\ |\varphi_r\rangle = c_0|0\rangle,\ |\eta^r\rangle = c_{-1}c_1|0\rangle$$

In the first case set $\xi_r = 0$ and in the second case set $\psi^r = 0$.

This gives

$$b_0|\psi\rangle = 0$$

$$S = \frac{1}{2}\langle\psi|Q_B|\psi\rangle + \cdots = \frac{1}{2}\langle\psi|c_0L_0|\psi\rangle + \cdots$$

$$T_m + T_{ghost} = \sum_n L_n z^{-n-2}$$

$$S = \frac{1}{2}\langle\psi|Q_B|\psi\rangle + \cdots = \frac{1}{2}\langle\psi|c_0 L_0|\psi\rangle + \cdots$$

This gives the propagator = $b_0 L_0^{-1} = b_0 \int_0^1 dq\, q^{L_0 - 1}$

From this we recover the usual world-sheet description of the amplitudes, with q's related to the integration variables of the world-sheet description

(to be seen in detail later)

Divergences arise from states with $L_0 \leq 0$

propagator = $b_0 L_0^{-1} = b_0 \int_0^1 dq\, q^{L_0 - 1}$

Tachyon: $L_0 = -1$ state $c_1|0\rangle \quad \Rightarrow \quad \int_0^1 dq\, q^{-2} = -1$

$L_0 = 0$ state $c_1 \alpha_{-1}|0\rangle = ic\partial X(0)|0\rangle$ is the zero mode associated with collective coordinate associated with X-translation

– can be dealt with by using the procedure described earlier

– remove q^{-1} factors from the world-sheet integrand and integrate over these zero modes at the end

Open strings have other $L_0 = 0$ states in the Siegel gauge from the ghost sector leading to additional zero modes.

$$|0\rangle, \qquad c_1 c_{-1}|0\rangle$$

These do not correspond to collective coordinates and cannot be treated using the procedure described earlier

– leads to logarithmic divergence in the loop amplitudes

We need to find new approach.

Solution: Consider the full expansion in the $L_0 = 0$ sector:

$$\psi^1 c_0|0\rangle + \psi^2|0\rangle + \xi_1 c_1 c_{-1}|0\rangle + \xi_2 c_1 c_0 c_{-1}|0\rangle$$

Siegel gauge choice: $\psi^1 = 0$, $\xi_2 = 0$: problematic

Go back to the original choice of Lagrangian subspace: $\xi_1 = 0$, $\xi_2 = 0$

$$|\chi\rangle = \psi^1 c_0|0\rangle + \psi^2|0\rangle + \cdots$$

$$|\chi\rangle = \mathbf{t}\,\mathbf{c_1}|\mathbf{0}\rangle + \psi^1\,\mathbf{c_0}|\mathbf{0}\rangle + \psi^2|\mathbf{0}\rangle + \cdots$$

$$\mathbf{S} = \frac{1}{2}\langle\chi|\mathbf{Q_B}|\chi\rangle = \frac{1}{2}\mathbf{t}^2 + (\psi^1)^2 + \cdots$$

ψ^1 has a non-vanishing kinetic term and therefore a finite propagator $-1/2$ in the convention in which tachyon propagator is -1

ψ^2 has no kinetic term $\Rightarrow$ a zero mode

However it decouples from the action completely by ghost number conservation

Physically, integration over ψ^2 corresponds to division by the volume of the rigid U(1) gauge group (after a field redefinition).

Therefore integration over ψ^2 factorizes and can be dropped (possibly after a field redefinition).

Algorithm on the world-sheet:

1. Remove all logarithmically divergent integrals of the form $\int_0^1 dq/q$, including those that come from the Siegel gauge ghost zero mode pairs ξ_1, ψ^2.

2. Explicitly add the contribution from ψ^1 propagators

3. Account for required field redefinition of the zero modes and the contribution of the measure generated during the field redefinition to the amplitude

Example 1

Four point function of translational zero modes

Recall the tree level four point amplitude of the zero mode ϕ associated to time translation:

$$\mathbf{A} = \int_0^1 \mathbf{dy} \, \langle \mathbf{c}\partial\mathbf{X}(0) \, \partial\mathbf{X}(\mathbf{y}) \, \mathbf{c}\partial\mathbf{X}(1) \, \mathbf{c}\partial\mathbf{X}(\infty)\rangle$$
$$= \int_0^1 \mathbf{dy} \left\{\mathbf{y}^{-2} + (1-\mathbf{y})^{-2} + 1\right\}$$

This diverges at $y = 0$ and $y = 1$.

We shall show how to get a finite, unambiguous value for this integral.

Strategy: Change variable from y to q near the singular points.

Replace the divergent integrals $\int_0^1 dq q^{-1+h}$ by

1/h for $h < 0$

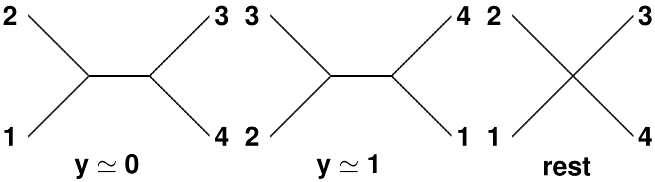
0 for h=0

Expectation:

Let us denote the amplitude as

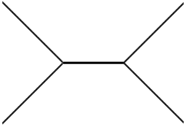
$$A = \int_0^1 dy \, \langle cW_1(0) \, W_2(y) \, cW_3(1) \, cW_4(\infty) \rangle$$

We expect the $y \simeq 0$ region to be covered by the s-channel diagram, the $y \simeq 1$ region to be covered by the u-channel diagram and the rest of the region by contact term.

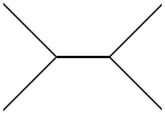


We need to find how much region each diagram covers and what is the change of variable near y=0 and y=1.

Consider a Feynman diagram with an internal propagator.



Question: How to translate the Schwinger parameter q associated with the propagator to y?



To calculate this, we need a choice of the off-shell 3-point **vertex** of three open string states with unintegrated vertex operator V_1, V_2, V_3.

Should be related to $\langle V_1(z_1)V_2(z_2)V_3(z_3)\rangle$

However the result depends on z_1, z_2, z_3 unless each V_i is a dimension zero primary, i.e. an on-shell state.

We need systematic characterization of the choice of vertex.

Choose 'local coordinate' w_1, w_2, w_3 near z_1, z_2, z_3, related to the global UHP coordinate z as

$$z = f_i(w_i) \text{ near } z = z_i, \quad f_i(0) = z_i$$

Insert the vertex operator V_i in the w_i coordinate system

Off-shell 3-point function

$$S[\langle f_1 \circ V_1(0)\, f_2 \circ V_2(0)\, f_3 \circ V_3(0)\rangle]$$

S: symmetrization under permutations of V_1, V_2, V_3

$f \circ V(0)$: conformal transform of V under f at $w = 0$.

If V is a primary of weight h, then

$$f \circ V(0) = f'(0)^h\, V(f(0))$$

Different choice of f_i's lead to different string field theories, but they are related by field redefinition.

Some more examples:

$$\mathbf{f} \circ \mathbf{c}(\mathbf{w}) = (\mathbf{f}'(\mathbf{w}))^{-1}\mathbf{c}(\mathbf{f}(\mathbf{w}))$$

$$\mathbf{f} \circ \partial\mathbf{c}(\mathbf{w}) = \partial_{\mathbf{w}}\left[(\mathbf{f}'(\mathbf{w}))^{-1}\mathbf{c}(\mathbf{f}(\mathbf{w}))\right] = \partial\mathbf{c}(\mathbf{f}(\mathbf{w})) - \frac{\mathbf{f}''(\mathbf{w})}{(\mathbf{f}'(\mathbf{w}))^{2}}\mathbf{c}(\mathbf{f}(\mathbf{w}))$$

etc.

We shall make a particular choice and do the calculations.

$$f_1(w_1) = \frac{2w_1}{2\alpha + w_1(1-\gamma)}, \quad f_2(w_2) = \frac{2\alpha + w_2(1-\gamma)}{2\alpha - w_2(1+\gamma)},$$
$$f_3(w_3) = -\frac{2\alpha - w_3(1+\gamma)}{2w_3}.$$

α, γ: constants. α large

Note: In the z plane the three punctures are at:

$$z_1 = f_1(0) = 0, \quad z_2 = f_2(0) = 1, \quad z_3 = f_3(0) = \infty$$

$z \to 1/(1-z)$ cyclically permutes f_1, f_2, f_3 and so also z_1, z_2, z_3.

$\Rightarrow$ interaction vertex is already invariant under cyclic permutation of V_1, V_2, V_3.

Since the 3-point vertex we shall use always has two identical states $c\partial X$, we do not need to symmetrize.

Given the f_i's, the relation between y and q are found as follows:

1. Take two copies of UHP (one for each interaction vertex) with global coordinates z and z' and related local coordinates w_i, w'_i.

2. Glue the two surfaces via the identification:

$$w_2 w'_2 = -q$$

(average over $w_2 \to w_1, w_3$ and $w'_2 \to w'_1, w'_3$ if needed)

3. This relates z to z' as $z = F(z'; q)$

4. In the new UHP, the external vertex operators are at z_1, z_3, $F(z_1; q)$ and $F(z_3; q)$.

5. Bring three of them to 0, 1, ∞ and identify the fourth one as y

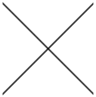
– gives the relation between y and q.

This prescription follows from properties of CFT and the form of Siegel gauge propagator b_0/L_0.

Once we have found the relation between y and q, the range $0 \leq q \leq 1$ corresponds to certain range of y

– covered by Feynman diagram with an internal propagator.

The rest of the region in y space is covered by the four point contact interaction.



We shall now do this for the choice:

$$f_1(w_1) = \frac{2w_1}{2\alpha + w_1(1-\gamma)}, \quad f_2(w_2) = \frac{2\alpha + w_2(1-\gamma)}{2\alpha - w_2(1+\gamma)},$$
$$f_3(w_3) = -\frac{2\alpha - w_3(1+\gamma)}{2w_3}.$$

Inverse relations:

$$\mathbf{w_1} = \alpha\, \frac{\mathbf{2\,z}}{\mathbf{2 - z + \gamma\, z}}, \quad \mathbf{w_2} = \mathbf{2}\,\alpha\, \frac{\mathbf{z - 1}}{\mathbf{z + 1 + \gamma(z - 1)}},$$

$$\mathbf{w_3} = \mathbf{2}\,\alpha\, \frac{\mathbf{1}}{\mathbf{1 + \gamma - 2\,z}}\,.$$

$$\mathbf{w_2 w_2'} = -\mathbf{q}$$

$$\Rightarrow \mathbf{4}\,\alpha^{\mathbf{2}}\, \frac{\mathbf{z - 1}}{\mathbf{z + 1 + \gamma(z - 1)}}\; \frac{\mathbf{z' - 1}}{\mathbf{z' + 1 + \gamma(z' - 1)}} = -\mathbf{q}$$

External vertex operators are inserted at $\mathbf{z} = \infty$, 0, $\mathbf{z'} = \infty, \mathbf{0}$

$$\Rightarrow \qquad \mathbf{z} = \infty,\ \mathbf{0},\ \frac{\mathbf{4}\alpha^{\mathbf{2}} + (\gamma^{\mathbf{2}} - \mathbf{1})\mathbf{q}}{\mathbf{4}\alpha^{\mathbf{2}} + (\mathbf{1} + \gamma)^{\mathbf{2}}\mathbf{q}},\ \frac{\mathbf{4}\,\alpha^{\mathbf{2}} + (\mathbf{1} - \gamma)^{\mathbf{2}}\,\mathbf{q}}{\mathbf{4}\alpha^{\mathbf{2}} + (\gamma^{\mathbf{2}} - \mathbf{1})\mathbf{q}}$$

$$z = \infty,\ 0,\ \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1+\gamma)^2 q},\ \frac{4\,\alpha^2 + (1-\gamma)^2\, q}{4\alpha^2 + (\gamma^2 - 1)q}$$

Large α / small q $\Rightarrow$ last two punctures are close.

For finding the relation between q and y near $y = 1$, we need to make SL(2,R) transformation so that the vertex operators are at 0,1, ∞ and $y < 1,\ y \simeq 1$ (u-channel diagram)

Similarly for finding the relation between q and y near y=0, we need to make SL(2,R) transformation so that the vertex operators are at 0,1, ∞ and $y > 0,\ y \simeq 0$ (s-channel diagram)

$$\hat{z} = z\,\frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1-\gamma)^2\,q}$$

brings the punctures at

$$\hat{z} = \infty, 0, y, 1$$

$$y \equiv \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1-\gamma)^2\,q}\,\frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1+\gamma)^2 q} = 1 - \frac{q}{\alpha^2} + \frac{(1+\gamma^2)q^2}{2\alpha^4} + \mathcal{O}(\frac{q^3}{\alpha^6})$$

$$\tilde{z} = 1 - \frac{1}{z}\,\frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1+\gamma)^2\,q}\,,$$

brings the punctures at

$$\tilde{z} = 1, \infty, 0, y$$

$$y \equiv 1 - \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1+\gamma)^2\,q}\,\frac{4\alpha^2 + (\gamma^2 - 1)q}{4\,\alpha^2 + (1-\gamma)^2\,q} = \frac{q}{\alpha^2} - \frac{(1+\gamma^2)q^2}{2\alpha^4} + \mathcal{O}(\frac{q^3}{\alpha^6})$$

These give contributions from u and s channel diagrams.

Lecture 3

We have been studying the four point function of the collective modes:

$$\mathbf{A} = \int_0^1 \mathbf{dy}\,\left\{\mathbf{y}^{-2} + (1-\mathbf{y})^{-2} + 1\right\}$$

This diverges near y=0 and y=1.

Using string field theory, we found the change of variables from y to q near y=0 and y=1

$$\mathbf{y} \equiv \frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\,\alpha^2 + (1-\gamma)^2\,\mathbf{q}}\,\frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\alpha^2 + (1+\gamma)^2\mathbf{q}} = 1 - \frac{\mathbf{q}}{\alpha^2} + \frac{(1+\gamma^2)\mathbf{q}^2}{2\alpha^4} + \mathcal{O}(\frac{\mathbf{q}^3}{\alpha^6})$$

near y=1, and

$$\mathbf{y} \equiv 1 - \frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\,\alpha^2 + (1+\gamma)^2\,\mathbf{q}}\,\frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\,\alpha^2 + (1-\gamma)^2\,\mathbf{q}} = \frac{\mathbf{q}}{\alpha^2} - \frac{(1+\gamma^2)\mathbf{q}^2}{2\alpha^4} + \mathcal{O}(\frac{\mathbf{q}^3}{\alpha^6})$$

near y=0

α, γ arbitrary, but we take α to be large.

Near y=1, $y = 1 - \frac{q}{\alpha^2} + \frac{(1+\gamma^2)q^2}{2\alpha^4} + \mathcal{O}(\frac{q^3}{\alpha^6})$

The range $0 \le q \le 1$ corresponds to:

$$1 - \frac{1}{\alpha^2} + \frac{(1+\gamma^2)}{2\alpha^4} + \mathcal{O}(\alpha^{-6}) \le y \le 1\,.$$

$$(1-y)^{-2}dy = -\alpha^2\, q^{-2}\, dq + \mathcal{O}(\alpha^{-2})$$

$$\int_0^1 dy\,(1-y)^{-2} = \int_0^{1-\frac{1}{\alpha^2}+\frac{(1+\gamma^2)}{2\alpha^4}+\mathcal{O}(\alpha^{-6})} dy\,(1-y)^{-2} + \alpha^2 \int_0^1 \frac{dq}{q^2} + \mathcal{O}(\alpha^{-2})$$

$$= \left\{\frac{1}{\alpha^2} - \frac{(1+\gamma^2)}{2\alpha^4}\right\}^{-1} - 1 - \alpha^2 + \mathcal{O}(\alpha^{-2}) = \frac{\gamma^2-1}{2} + \mathcal{O}(\alpha^{-2})$$

Note: In principle we should also change variable near y=0, but since the integrand is non-singular there, it makes no difference.

Similarly

$$\int_0^1 dy\, y^{-2} = \frac{\gamma^2 - 1}{2}$$

Also

$$\int_0^1 dy = 1$$

Total:

$$A = \frac{\gamma^2 - 1}{2} + \frac{\gamma^2 - 1}{2} + 1 = \gamma^2$$

The result depends on γ!

What is missing?

1. No logarithmic divergence $\Rightarrow$ no zero mode in the internal propagator to be subtracted.

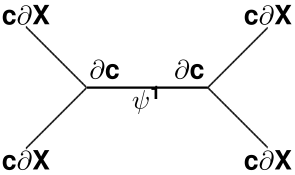
2. The change in integration measure due to redefinition of zero mode variables affect the result at loop level, but not at the tree level.

3. The only missing ingredient is possible contribution from internal ψ^1 modes.

Recall that ψ^1 exchange is not captured by world-sheet expressions since it is not present in the Siegel gauge

– needs to be included separately.

We need to add contribution from:



Quadratic term in action = $(\psi^1)^2$

$\Rightarrow \psi^1$ **propagator is** $-1/2$

The three point functions are:

$$\langle \mathbf{f_1} \circ \mathbf{c}\partial \mathbf{X}(\mathbf{0})\ \mathbf{f_2} \circ \partial \mathbf{c}(\mathbf{0})\ \mathbf{f_3} \circ \mathbf{c}\partial \mathbf{X}(\mathbf{0})\rangle$$

$$= \left\langle \mathbf{c}\partial\mathbf{X}(\mathbf{f_1}(\mathbf{0})) \left\{ \partial\mathbf{c}(\mathbf{f_2}(\mathbf{0})) - \frac{\mathbf{f_2''}(\mathbf{0})}{\mathbf{f_2'}(\mathbf{0})^2}\mathbf{c}(\mathbf{f_2}(\mathbf{0})) \right\} \mathbf{c}\partial\mathbf{X}(\mathbf{f_3}(\mathbf{0})) \right\rangle$$

$$\mathbf{f_1}(\mathbf{w_1}) = \frac{\mathbf{2w_1}}{\mathbf{2}\alpha + \mathbf{w_1}(\mathbf{1}-\gamma)}, \quad \mathbf{f_2}(\mathbf{w_2}) = \frac{\mathbf{2}\alpha + \mathbf{w_2}(\mathbf{1}-\gamma)}{\mathbf{2}\alpha - \mathbf{w_2}(\mathbf{1}+\gamma)},$$

$$\mathbf{f_3}(\mathbf{w_3}) = -\frac{\mathbf{2}\alpha - \mathbf{w_3}(\mathbf{1}+\gamma)}{\mathbf{2w_3}}\,.$$

$$\langle \mathbf{c}(\mathbf{z_1})\mathbf{c}(\mathbf{z_2})\mathbf{c}(\mathbf{z_3})\rangle = (\mathbf{z_3}-\mathbf{z_1})(\mathbf{z_3}-\mathbf{z_2})(\mathbf{z_2}-\mathbf{z_1})$$

$$\langle \mathbf{c}(\mathbf{z_1})\partial\mathbf{c}(\mathbf{z_2})\mathbf{c}(\mathbf{z_3})\rangle = \frac{\partial}{\partial \mathbf{z_2}}\{(\mathbf{z_3}-\mathbf{z_1})(\mathbf{z_3}-\mathbf{z_2})(\mathbf{z_2}-\mathbf{z_1})\}$$

– can be used to calculate the 3-point function $\mathbf{C}_{\phi\phi\psi^1}$

The normalization can be fixed by comparing with the tachyon exchange contribution.

Two differences:

The three point functions for tachyon exchange are:

$$\langle \mathbf{f_1} \circ \mathbf{c}\partial\mathbf{X}(0)\mathbf{f_2} \circ \mathbf{c}(0)\mathbf{f_3} \circ \mathbf{c}\partial\mathbf{X}(0)\rangle = \left\langle \mathbf{c}\partial\mathbf{X}(\mathbf{f_1}(0))(\mathbf{f_2'}(0))^{-1}\mathbf{c}(\mathbf{f_2}(0))\mathbf{c}\partial\mathbf{X}(\mathbf{f_3}(0))\right\rangle$$

Tachyon propagator $=-1$

Therefore the ψ^1 exchange contribution is:

$$\frac{1}{2}\left(\frac{C_{\phi\phi\psi^1}}{C_{\phi\phi t}}\right)^2 \times \text{tachyon exchange contribution}$$

Tachyon exchange contribution is the term containing $\int_0^1 dq q^{-2}$ in the s and u-channel diagrams $\Rightarrow -\alpha^2 - \alpha^2$

Result for ψ^1 exchange contribution: $-\gamma^2$

Recall the previous result from world-sheet computation: γ^2

Total: $\gamma^2 - \gamma^2 = 0$

– independent of γ as expected

Also consistent with the interpretation of $c\partial X$ as translational zero modes for which there should be no potential.

Note: If we had chosen $\gamma = 0$ then $C_{\phi\phi\psi^1}$ would have vanished and we could get the result without the ψ^1 exchange diagram

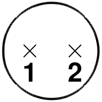
– related to the existence of a Z_2 symmetry called twist symmetry

– cannot be used to eliminate ψ^1 exchange contributions in loop diagrams

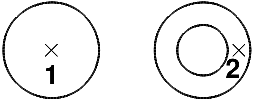
Example 2

Closed string tachyon two point function in two dimensional string theory

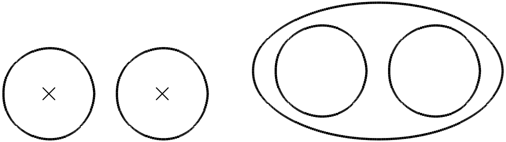
We need to compute:



$$= \mathbf{f}(\omega_1, \omega_2)$$



$$= \mathbf{g}(\omega_2)$$



$$= \mathbf{C}$$

We have factored out the overall factor:

$$8\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$

$\mathcal{O}(g_s)$ contribution to tachyon 2-point function:

$$8\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)\{f(\omega_1,\omega_2)+g(\omega_1)+g(\omega_2)+C\}$$

Using the same functions we can also get the expressions for tachyon n-point function at order g_s:

$$2^{n+1}\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\cdots\omega_n)\,\prod_{i=1}^{n}\sinh(\pi|\omega_i|)\,\left[\sum_{j<k}f(\omega_j,\omega_k)+\sum_j g(\omega_j)+C\right]$$

– receives contribution from 3 types of diagrams:

1. $(n-2)$ disk one point functions and one disk 2-point function

2. $(n-1)$ disk one point functions and one annulus 1-point function

3. n disk one point functions and a disk with 2 holes / torus with one hole

f, g and C have divergences.

Divergent parts (BRY):

$$f_{div}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y) = A_f + B_f\, \omega_1\omega_2$$

$$g_{div}(\omega) = \int_0^1 dv \int_0^{1/4} dx \left\{ \frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1} \right\} = A_g + B_g \omega^2$$

Tachyon n-point function at order g_s:

$$2^{n+1}\, \pi\, N\, e^{-1/g_s}\, g_s\, \delta(\omega_1 + \cdots \omega_n) \prod_{i=1}^{n} \sinh(\pi|\omega_i|) \left[\sum_{j<k} f(\omega_j, \omega_k) + \sum_j g(\omega_j) + C \right]$$

$$= 2^{n+1}\, \pi\, N\, e^{-1/g_s}\, g_s\, \delta(\omega_1 + \cdots \omega_n) \prod_{i=1}^{n} \sinh(\pi|\omega_i|)$$

$$\times \left[\frac{n(n-1)}{2} A_f + n\, A_g + C + \left\{ B_g - \frac{B_f}{2} \right\} \sum_j \omega_j^2 + \text{finite} \right]$$

Tachyon n-point function at order g_s:

$$= 2^{n+1}\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1 + \cdots \omega_n)\prod_{i=1}^{n}\sinh(\pi|\omega_i|)$$

$$\times\left[\frac{n(n-1)}{2}A_f + n\,A_g + C + \left\{B_g - \frac{B_f}{2}\right\}\sum_j \omega_j^2 + \text{finite}\right]$$

Comparing this with the known matrix model results we can get information on A_f, A_g, C and $2B_g - B_f$ separately.

BRY comparison for n=2 gives

$$A_f + 2A_g + C \simeq -.496, \quad 2B_g - B_f \simeq -1.399$$

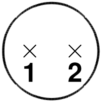
Our results so far (λ is an SFT parameter):

$$A_f = -\frac{1}{2}, \quad B_f = \ln\lambda^2, \quad B_g = \frac{1}{2}\ln\frac{\lambda^2}{4} \;\Rightarrow 2B_g - B_f \simeq -\ln 4 \simeq -1.386$$

A_g in progress, C not in progress (=0?)

We shall first describe the analysis of $f_{div}(\omega_1, \omega_2)$.

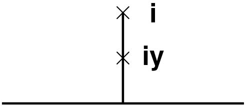
– related to the divergent part of disk / UHP two point function:



$$f_{div}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 dy\, y^{-2} (1 + 2\omega_1\omega_2 y)$$

Balthazar, Rodriguez, Yin

On the UHP, closed string vertex operators are located at i and iy



$$f_{div}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

We need to find the relation between y and q to deal with the divergence at y=0.

Feynman diagrams:

Thick lines: Closed strings　　Thin lines: open strings

For computing (a), we need the off-shell two point vertex of an open string and a closed string.

Since the closed strings are always on-shell, we need a choice of local coordinates at the open string puncture.

C-O interaction vertex

Put C at i, O at 0

Choose local coordinate at O to be

$$\mathbf{w} = \lambda\, \mathbf{z}$$

λ: an arbitrary constant, taken to be large for convenience

(a) (b)

For diagram (a), the sewing relation is

$$\mathbf{w}\mathbf{w}' = -\mathbf{q} \quad \Rightarrow \quad \lambda^2 \mathbf{z}\mathbf{z}' = -\mathbf{q}$$

The punctures are located at

$$\mathbf{z} = \mathbf{i}, \qquad \mathbf{z}' = \mathbf{i} \quad \Rightarrow \quad \mathbf{z} = \mathbf{i}\,\mathbf{q}/\lambda^2 \equiv \mathbf{i}\,\mathbf{y}$$

This gives $\mathbf{y} = \mathbf{q}/\lambda^2$.

$$\mathbf{y} = \mathbf{q}/\lambda^2$$

$$0 \le \mathbf{q} \le 1 \qquad \Rightarrow \quad 0 \le \mathbf{y} \le 1/\lambda^2$$

Analyze f_{div} using this:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1+2\omega_1\omega_2 y) = \frac{1}{2}\left\{\int_0^{1/\lambda^2} + \int_{1/\lambda^2}^1\right\} dy\, y^{-2}(1+2\omega_1\omega_2 y)$$

$$= \frac{1}{2}\int_0^1 dq\{\lambda^2 q^{-2} + 2\omega_1\omega_2 q^{-1}\} + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1+2\omega_1\omega_2 y)$$

$$\Rightarrow -\frac{1}{2}\lambda^2 + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1+2\omega_1\omega_2 y) = -\frac{1}{2} + 2\,\omega_1\omega_2 \ln\lambda$$

$-\omega_1, \omega_2$ **are energies of incoming and outgoing C.**

For the choice of local coordinates we have made, the C-ψ^1 vertex vanishes.

$\Rightarrow$ no need to include ψ^1 exchange contribution.

Final result:

$$f_{div}(\omega_1, \omega_2) = -\frac{1}{2} + 2\omega_1\omega_2 \ln\lambda \equiv A_f + B_f\omega_1\omega_2$$

This gives

$$A_f = -\frac{1}{2}, \qquad B_f = \ln\lambda^2$$

Note: If we had chosen a different local coordinate for the C-O vertex, the result will be different

– compensated by ψ^1 exchange diagram for A_f.

For B_f some part may also cancel against contribution to $2\,B_g$.

We now turn to the divergent part of the annulus one point function:

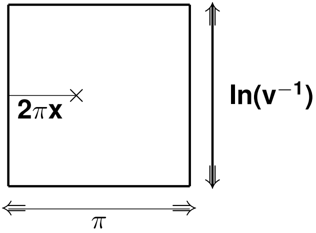
$$g_{div}(\omega) = \int_0^1 dv \int_0^{1/4} dx \left\{ \frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1} \right\}$$

Interpretation of v, x:

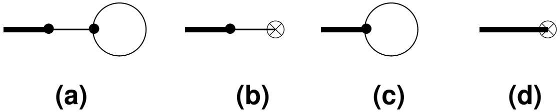
Describe the annulus by

$$w \equiv w + i\ln(v^{-1}), \qquad 0 \le \mathrm{Re}(w) \le \pi$$

$2\pi x$: Value of Re(w) of the closed string puncture



$g_{div}(\omega)$ from C-amplitude on the annulus



$\otimes$ denotes one point vertex on the annulus

Therefore we need O-O-O interaction vertex on the disk, C-O-O interaction vertex on the disk and O interaction vertex on the annulus.

O-O-O vertex:

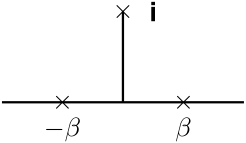
Three punctures are taken to be at 0, 1, ∞ with local coordinates:

$$w_1 = \alpha \frac{2z}{2-z}, \quad w_2 = -2\,\alpha\,\frac{1-z}{1+z}, \quad w_3 = \alpha\,\frac{2}{1-2z}$$

Note: This is $\gamma = 0$ version of the vertex we described earlier.

C-O-O vertex:

Take C at i, and the pair of O's at $\pm\beta$.



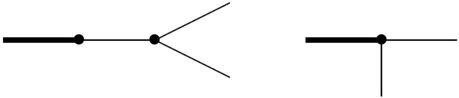
We need to specify the local coordinates at the O's and the range of β.

To determine the C-O-O vertex we need to first examine the C-O-O amplitude

1. Determine the range of β covered by Feynman diagrams with known vertices

2. The C-O-O vertex should cover the missing range

C-O-O amplitude has two Feynman diagrams.



(a) **(b)**

The region in β space covered by Fig.(a) can be evaluated by sewing the C-O vertex to the O-O-O vertex.

This also gives the local coordinates at the O's inherited from the O-O-O vertex.

$$z_1 = -\frac{q}{2\widetilde{\lambda}}, \qquad z_2 = \frac{q}{2\widetilde{\lambda}}, \qquad \widetilde{\lambda} \equiv \lambda\alpha, \qquad 0 \le q \le 1$$

$$w_1 = -\,-\,2\,\alpha\,\frac{z - z_1}{z + 3\,z_1}, \qquad w_2 = 2\,\alpha\,\frac{z - z_2}{z + 3z_2}$$

– covers the range $|\beta| < (2\widetilde{\lambda})^{-1}$.

Therefore the C-O-O vertex must cover the region $|\beta| > (2\widetilde{\lambda})^{-1}$ and the local coordinates at the O's should match the ones above at $\beta = (2\widetilde{\lambda})^{-1}$.

Our choice of C-O-O vertex:

$$1/(2\widetilde{\lambda}) \leq \beta \leq 1$$

Local coordinates at the pair of O's:

$$w_a = \alpha\widetilde{\lambda}\,\frac{4\widetilde{\lambda}^2+1}{4\widetilde{\lambda}^2}\,\frac{z-z_a}{(1+z_a\,z)+\widetilde{\lambda}\,f(z_a)(z-z_a)},$$
$$a = 1,2, \qquad z_1 = -\beta, \quad z_2 = \beta, \quad \widetilde{\lambda} \equiv \alpha\,\lambda\,.$$

f is an arbitrary function satisfying:

$$f(-\beta) = -f(\beta), \quad f(1/(2\widetilde{\lambda})) = 1/2, \quad f(1) = 0$$

–needed for smooth matching with the Feynman diagram induced from C-O and O-O-O joining.

Note: $|\beta| > 1$ is related to $|\beta| < 1$ by $z \to -1/z$ transformation that leaves the closed string puncture fixed at i.

O-interaction vertex on the annulus:

Describe the annulus as $z \equiv v^{-1} z$ identification in UHP.

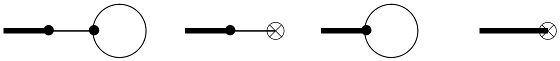
$$v \geq \left\{ \alpha^2 - \frac{1}{2} \right\}^{-1}$$

Local coordinate at O:

$$w = 2\,\alpha \frac{(4 + 3\,\alpha^{-2}) z - 4 + 3\,\alpha^{-2}}{(4 - \alpha^{-2}) z + 4 - 7\,\alpha^{-2}}$$

This smoothly matches the local coordinates induced from the diagram obtained by joining two O's of O-O-O

With this, we can evaluate the 'world-sheet contribution' to g_{div}:



(a) (b) (c) (d)

Results:

$$(\mathbf{a}):\ \frac{\widetilde{\lambda}\,\alpha^2}{2\pi},\qquad (\mathbf{b}):\ -\frac{1}{2\pi}\widetilde{\lambda}\,\alpha^2+\frac{\widetilde{\lambda}}{4\pi}+\frac{\widetilde{\lambda}}{\pi}\ln\alpha,\qquad (\mathbf{c}):\ -\frac{1}{4}\alpha^2\widetilde{\lambda}^2-\frac{\alpha^2}{8}+\frac{\alpha^2\,\widetilde{\lambda}}{2\pi}$$

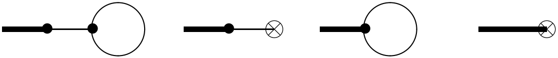
$$(\mathbf{d}):\ \frac{1}{4}\alpha^2\widetilde{\lambda}^2-\frac{\alpha^2\widetilde{\lambda}}{2\pi}+\frac{\alpha^2}{8}-\frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1}\mathrm{d}\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2 f(\beta)^2-\frac{\widetilde{\lambda}}{\pi}\ln\alpha+\frac{1}{2}\omega^2\ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

$$\widetilde{g}_{div}(\omega)=-\frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1}\mathrm{d}\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\,f(\beta)^2+\frac{\widetilde{\lambda}}{4\pi}+\frac{1}{2}\omega^2\ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

Lecture 4

We have been computing the 'divergent' part of the annulus one point function $\mathbf{g}(\omega)$ of the closed string tachyon

'World-sheet contribution' to $\mathbf{g}_{\mathbf{div}}$:



Results:

$$(\mathbf{a}) : \frac{\widetilde{\lambda}\,\alpha^2}{2\pi}, \qquad (\mathbf{b}) : -\frac{1}{2\pi}\widetilde{\lambda}\,\alpha^2 + \frac{\widetilde{\lambda}}{4\pi} + \frac{\widetilde{\lambda}}{\pi}\ln\alpha, \qquad (\mathbf{c}) : -\frac{1}{4}\alpha^2\widetilde{\lambda}^2 - \frac{\alpha^2}{8} + \frac{\alpha^2\,\widetilde{\lambda}}{2\pi}$$

$$(\mathbf{d}) : \frac{1}{4}\alpha^2\widetilde{\lambda}^2 - \frac{\alpha^2\widetilde{\lambda}}{2\pi} + \frac{\alpha^2}{8} - \frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2 f(\beta)^2 - \frac{\widetilde{\lambda}}{\pi}\ln\alpha + \frac{1}{2}\omega^2\ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

$$\widetilde{g}_{div}(\omega) = -\frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\,f(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi} + \frac{1}{2}\omega^2\ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

ψ^1 exchange contributions comes from two diagrams:

(a) **(b)**

Vanishing of C-ψ^1 amplitude eliminates the other two diagrams.

Recall that the state multiplying ψ^1 is $c_0|0\rangle$

$\Leftrightarrow$ vertex operator ∂c.

This has two complications:

1. ∂c is not primary and so $f \circ \partial c$ has extra terms

2. Construction of integrated vertex operator is more complicated.

Nevertheless there is well defined algorithm and this can be computed.

We shall discuss this procedure at the end

Result from ψ^1 exchange terms:

$$\mathbf{g}_{\psi^1}(\omega) = \frac{2}{\pi} \int_{(2\widetilde{\lambda})^{-1}}^{1} \mathbf{d}\beta \, (1 + \beta^2)^{-1} \, \widetilde{\lambda}^2 \, \mathbf{f}(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi}$$

Field redefinition of the translational zero mode:

ϕ: zero mode of open string field theory associated with the state $c_1\alpha_{-1}|0\rangle$

We expect ϕ to be related by field redefinition to the collective coordinate $\widetilde{\phi}$ that translates the D-instanton along the time direction.

On the other hand, a given amplitude in background $\widetilde{\phi}$ field is expected to have $\widetilde{\phi}$ dependence of the form

$$e^{-i\omega\widetilde{\phi}} = 1 - i\omega\widetilde{\phi} + \frac{1}{2}(i\omega\widetilde{\phi})^2 + \cdots$$

ω: total energy of all the vertex operators in the amplitude

This result follows from the fact that $\widetilde{\phi}$ is the position of the D-instanton along the time direction

This gives simple result for the $\widetilde{\phi}$ integral:

$$\int \mathbf{d}\widetilde{\phi}\, \mathbf{e}^{-\mathbf{i}\omega\widetilde{\phi}} = \mathbf{2}\pi\delta(\omega)$$

– gives us the energy conserving delta function.

However in open string field theory the path integral is to be performed over ϕ, not over $\widetilde{\phi}$.

Strategy:

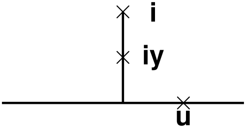
1. Make the change of variable from ϕ to $\widetilde{\phi}$, keeping track of the measure.

2. Then carry out $\widetilde{\phi}$ integral.

For this we have to find the change of variable from ϕ to $\widetilde{\phi}$.

Compute C-C-O amplitude with ϕ as external O state.

Naive result:



If the O is inserted at u, and the C-'s are at i and iy then the integrand has the structure:

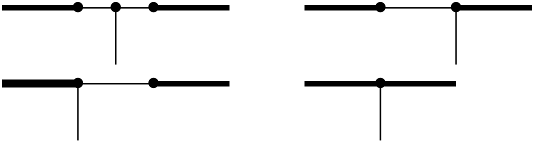
$$-\frac{1}{2\pi}\left\{\omega_1\left(\frac{1}{u-i}-\frac{1}{u+i}\right)+\omega_2\left(\frac{1}{u-iy}-\frac{1}{u+iy}\right)\right\} I_{CC}$$

I_{CC} is the integrand for C-C amplitude.

Closing the u contour in UHP and picking up resides at i, iy we get the integrand to be $-i(\omega_1+\omega_2)\, I_{CC}$

– gives the expected result from a collective coordinate.

However there are divergences as $y \to 0$ and we need to deal with them using string field theory.



Result:

$$-i(\omega_1+\omega_2)\left\{-\frac{1}{2}+\frac{\tilde{\lambda}}{\pi}+2\omega_1\omega_2 \ln\tilde{\lambda}\right\} 4\,N\,e^{-1/g_s} g_s \sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)$$

Expected result for $\tilde{\phi}$ coupling is $-i(\omega_1+\omega_2)$ times the C-C amplitude:

$$-i(\omega_1+\omega_2)\left\{-\frac{1}{2}+2\omega_1\omega_2 \ln\lambda\right\} 4\,N\,e^{-1/g_s} g_s \sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)$$

The extra term

$$-i(\omega_1+\omega_2)\left\{\frac{\tilde{\lambda}}{\pi}+2\omega_1\omega_2 \ln\frac{\tilde{\lambda}}{\lambda}\right\} 4\,N\,e^{-1/g_s} g_s \sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)$$

has to be accounted for by postulating appropriate relation between ϕ and $\tilde{\phi}$.

If $\mathbf{C}(\omega)$ denotes closed string tachyon field of energy ω, then the extra term may be regarded as an 'unwanted term' in the effective action:

$$-\frac{\mathbf{i}}{\mathbf{2}}\,\mathbf{g_s}\,\phi\int\frac{\mathbf{d}\omega_\mathbf{1}}{\mathbf{2}\pi}\frac{\mathbf{d}\omega_\mathbf{2}}{\mathbf{2}\pi}\,(\omega_\mathbf{1}+\omega_\mathbf{2})\,\mathbf{4\,sinh}(\pi|\omega_\mathbf{1}|)\,\mathbf{sinh}(\pi|\omega_\mathbf{2}|)$$
$$\left\{\frac{\mathbf{1}}{\pi}\,\widetilde{\lambda}+\mathbf{2}\,\omega_\mathbf{1}\,\omega_\mathbf{2}\,\mathbf{ln}\,\frac{\widetilde{\lambda}}{\lambda}\right\}\mathbf{C}(\omega_\mathbf{1})\,\mathbf{C}(\omega_\mathbf{2})$$

There is also an expected ϕ-C interaction term:

$$-\mathbf{i}\,\phi\int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\,\omega\,\mathbf{2\,sinh}(\pi|\omega|)\,\mathbf{C}(\omega)$$

A field redefinition of the form:

$$\phi=\widetilde{\phi}+\mathbf{g_s}\,\widetilde{\phi}\int\frac{\mathbf{d}\omega'}{\mathbf{2}\pi}\,\mathbf{2\,sinh}(\pi|\omega'|)\,\mathbf{a}(\omega')\,\mathbf{C}(\omega')$$

$$\mathbf{a}(\omega)=-\left[\frac{\mathbf{1}}{\pi}\,\widetilde{\lambda}+\omega^\mathbf{2}\mathbf{ln}\,\frac{\widetilde{\lambda}^\mathbf{2}}{\lambda^\mathbf{2}}\right]$$

will get rid of the extra term to this order.

$$\phi = \widetilde{\phi} + \mathbf{g_s}\, \widetilde{\phi} \int \frac{\mathbf{d}\omega'}{\mathbf{2}\pi}\, \mathbf{2\,sinh}(\pi|\omega'|)\, \mathbf{a}(\omega')\, \mathbf{C}(\omega')$$

$$\mathbf{a}(\omega) = -\left[\frac{\mathbf{1}}{\pi}\,\widetilde{\lambda} + \omega^{\mathbf{2}}\mathbf{ln}\,\frac{\widetilde{\lambda}^{\mathbf{2}}}{\lambda^{\mathbf{2}}}\right]$$

Change of variable from $\phi \to \widetilde{\phi} \Rightarrow$ a Jacobian

$$\mathbf{1} + \mathbf{g_s} \int \frac{\mathbf{d}\omega'}{\mathbf{2}\pi}\, \mathbf{2\,sinh}(\pi|\omega'|)\, \mathbf{a}(\omega')\, \mathbf{C}(\omega') = \mathbf{exp}\left[\mathbf{g_s} \int \frac{\mathbf{d}\omega'}{\mathbf{2}\pi}\, \mathbf{2\,sinh}(\pi|\omega'|)\, \mathbf{a}(\omega')\, \mathbf{C}(\omega')\right]$$

$\Rightarrow$ a new term in the effective action:

$$\mathbf{g_s} \int \frac{\mathbf{d}\omega'}{\mathbf{2}\pi}\, \mathbf{2}\ \mathrm{sinh}(\pi|\omega'|)\, \mathbf{a}(\omega')\, \mathbf{C}(\omega')$$

This gives a new contribution to one point function of C:

$$\Rightarrow \quad \mathbf{g_{jacobian}}(\omega) = \mathbf{a}(\omega) = -\frac{\widetilde{\lambda}}{\pi} - \omega^{\mathbf{2}}\mathbf{ln}\frac{\widetilde{\lambda}^{\mathbf{2}}}{\lambda^{\mathbf{2}}}$$

We are not done yet, but let us add up the results for $g_{div}(\omega) = A_g + B_g\,\omega^2$:

$$\textbf{Annulus:} -\frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\, f(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi} + \frac{1}{2}\omega^2 \ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

$$\psi^1 \textbf{ exchange: } \frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\, f(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi}$$

$$\phi \textbf{ redefinition: } -\frac{\widetilde{\lambda}}{\pi} - \omega^2 \ln\frac{\widetilde{\lambda}^2}{\lambda^2}$$

Adding these up and using $\widetilde{\lambda} = \alpha\lambda$, we get:

$$-\frac{\widetilde{\lambda}}{2\pi} + \frac{1}{2}\omega^2 \ln\frac{\lambda^2}{4}$$

Contribution to $g_{div} = A_g + B_g\,\omega^2$ so far:

$$-\frac{\widetilde{\lambda}}{2\pi} + \frac{1}{2}\omega^2 \ln\frac{\lambda^2}{4}$$

What remains is to take into account the redefinition of the ghost field and the associated Jacobian

Since ghost field coupling is not sensitive to ω, on general grounds one can argue that the ghost field redefinition will not affect the coefficient of the term proportional to ω^2.

B_g receives no further corrections.

$$B_g = \frac{1}{2}\omega^2 \ln\frac{\lambda^2}{4}$$

Ghost (ψ^2) field redefinition:

ψ^2 multipies $|0\rangle$ in the string field expansion

Ghosts $\Leftrightarrow$ gauge transformation parameters

There is a string field theory gauge transformation parameter associated with the state $|0\rangle$

Call that θ

Integration over the ghost field $\psi^2 \Rightarrow$ division by $\int d\theta$

Strategy: Compare the transformation generated by θ to the transformation generated by the rigid U(1) gauge transformation parameter $\widetilde{\theta}$

– will give the relation between θ and $\widetilde{\theta}$

Express dθ as J d$\widetilde{\theta}$ then then drop division by $\int$ d$\widetilde{\theta}$

Then $1/\mathrm{J} = \exp[-\ln \mathrm{J}]$ will give correction to the effective action / amplitudes.

Under rigid U(1) all open string states on the D-instanton are neutral.

We need to bring another spectator D-instanton on top of the original D-instanton.

Now an open string stretched from the original D-instanton to the spectator is charged under rigid U(1).

Open strings stretched from the spectator to the original D-brane are oppositely charged under the rigid U(1).

χ: any string field component in the open string sector that stretches from the original D-instanton to the spectator D-instanton

cW: vertex operator associated with this field.

Under rigid U(1), $\delta\chi = \mathrm{i}\widetilde{\theta}\chi$ with no further corrections

Goal: compare this with $\delta\chi$ generated by the open string gauge transformation parameter θ.

Leading term in the transformation law comes from the disk three point function of $\mathrm{cW}, \mathrm{c}\partial\mathrm{cW}, \mathrm{I}$

$\Rightarrow \delta\chi = \mathrm{i}\theta\chi$

$\Rightarrow$ at the leading order $\widetilde{\theta} = \theta$.

However there are corrections.

With corrections, open string gauge transformation laws take the form:

$$\delta\chi = \mathbf{i}\theta\chi\left[\mathbf{1} + \mathbf{g_s}\int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right]$$

$\mathbf{C}(\omega)$: closed string tachyon field of energy ω

$\mathbf{h}(\omega)$ is a computable function

– given by C-O-O-O amplitude with the three O's as cW, c∂cW and I.

This will give:

$$\widetilde{\theta} = \theta\left[\mathbf{1} + \mathbf{g_s}\int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right]$$

$$\mathbf{d}\theta = \mathbf{d}\widetilde{\theta}\left[\mathbf{1} - \mathbf{g_s}\int\frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right]$$

$$\mathbf{d}\theta = \mathbf{d}\widetilde{\theta}\left[\mathbf{1} - \mathbf{g_s}\int \frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right]$$

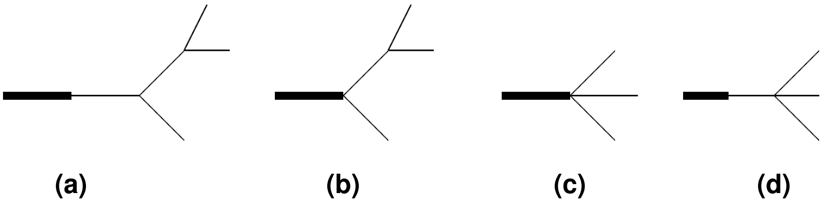
So division by $\int \mathbf{d}\theta$ may be regarded as division by $\int \mathbf{d}\widetilde{\theta}$ up to the extra factor of

$$\left[\mathbf{1} + \mathbf{g_s}\int \frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right] = \mathbf{exp}\left[\mathbf{g_s}\int \frac{\mathbf{d}\omega}{\mathbf{2}\pi}\mathbf{h}(\omega)\mathbf{C}(\omega) + \cdots\right]$$

This leads to new one point function of $\mathbf{C}(\omega)$ leading to additional contribution to $\mathbf{g}(\omega)$.

The analysis of the relevant C-O-O-O amplitude to determine $\mathbf{h}(\omega)$ is still in progress.

Feynman diagrams:



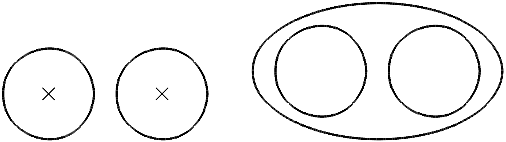
External open string states: cW, c∂cW, I

Chan-Paton factors:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

Only one cyclic ordering contributes.

Finally we shall discuss C



$= \mathbf{C}$

This multiplies the leading term by $\mathbf{C}g_s$ for some constant C.

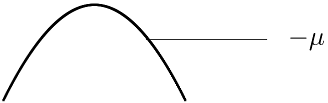
This term can be interpreted as a renormalization of the instanton action by $(1 - C\, g_s^2)$:

$$e^{-(1-Cg_s^2+\cdots)/g_s} = e^{-1/g_s}(1 + C\, g_s + \cdots)$$

We shall fix C by comparison to the instanton action in the matrix model.

Matrix model: Free fermion in inverted harmonic oscillator potential.

$$\mathbf{H} = \frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^2}{2}$$



Instanton causes tunneling from one side of the potential to another at fermi level $-\mu$.

In matrix model we have no quantum correction to instanton action beyond one loop since the potential is quadratic

$\Rightarrow$ **instanton action = $2\pi\mu$ + constant**

In order to compare the matrix model answer to the string theory answer we have to first fix the relation between the parameters of the two theories.

BRY fixed this by using $2\pi\mu = 1/g_s$ to all orders

Therefore the matrix model instanton action = $1/g_s$ + constant

Identifying the matrix model instanton with D-instanton we see that the D-instanton action should be $1/g_s$ + constant

No correction of the form $C\, g_s \Rightarrow C=0$

Caution: By strict rules, we are only allowed to compare physical amplitudes on both sides, not instanton action which is not an observable

$\Rightarrow$ the argument can have loopholes

Summary:

$$\mathbf{f_{div}}(\omega_1, \omega_2) = \mathbf{A_f} + \mathbf{B_f}\omega_1\,\omega_2, \quad \mathbf{g_{div}}(\omega) = \mathbf{A_g} + \mathbf{B_g}\omega^2$$

Results so far:

$$\mathbf{A_f} = -\frac{1}{2}, \quad \mathbf{2B_g} - \mathbf{B_f} = -\ln 4$$

$$\mathbf{A_g} = -\frac{\widetilde{\lambda}}{2\pi} + \text{ghost jacobian contribution}$$

$$\mathbf{C} = 0? \quad \text{(general arguments)}$$

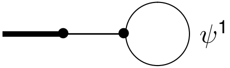
BRY result:

$$\mathbf{A_f} + 2\mathbf{A_g} + \mathbf{C} \simeq -.496, \qquad 2\mathbf{B_g} - \mathbf{B_f} \simeq -1.399$$

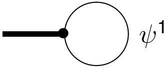
In principle A_f, A_g, C can be extracted individually by comparison with the matrix model results for higher point functions.

A technical point:

In our analysis we encounter amplitudes in which the external vertex operators involve ∂c.



(a)



(b)

Converting this to integrated vertex operator involves some extra effort.

We shall illustrate this with the C-O-O vertex with ψ^1 as the external O's.

In UHP, C is at i, O's are at $\pm\beta$.

We have to integrate over β.

Let $\mathbf{w_a = F_a(z, \beta)}$ be the local coordinates at the O's for a=1,2.

Then the integrand will involve a correlation function of all the unintegrated vertex operators $\mathbf{c\bar{c}W_C(i)}$, $\mathbf{\partial c(w_1 = 0)}$, $\mathbf{\partial c(w_2 = 0)}$ and

$$\mathbf{d\beta \sum_{a=1}^{2} \int_a \frac{\partial F_a}{\partial \beta} \, b(w_a) \, dw_a}$$

$\int_a$ anti-clockwise contour around $\mathbf{w_a = 0}$.

After evaluating the integral we need to conformally map this to the z-coordinate system.

Suppose

$$\frac{\partial F_a(z,\beta)}{\partial\beta} = A_a(\beta) + B_a(\beta) w_a + \mathcal{O}(w_a^2) \quad \text{near } z = z_a, \text{ i.e. } w_a = 0$$

$$a = 1, 2, \quad z_1 = -\beta, \quad z_2 = \beta$$

Then

$$\begin{aligned} & \sum_{a=1}^{2} \int_a \frac{\partial F_a}{\partial\beta}\, b(w_a)\, dw_a\, \partial c(w_1 = 0) \partial c(w_2 = 0) \\ = \quad & B_1(\beta) \partial c(w_2 = 0) - B_2(\beta) \partial c(w_1 = 0) \end{aligned}$$

Now convert to z coordinate:

$$B_1(\beta) f_2 \circ \partial c(0) - B_2(\beta) f_1 \circ \partial c(0)$$

f_1, f_2: **inverse functions of** F_1, F_2:

$$z = f_1(w_1), \quad z = f_2(w_2)$$

After this we have a correlation function of $c\bar{c}(i)$ **and another c or** ∂c **at** $\pm\beta$**, and we can proceed as usual.**