D-instanton Perturbation Theory

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Plan:

1. Overview of the problem and the solution

2. Review of basic aspects of world-sheet string theory and string field theory

3. Some explicit computations

A.S., arXiv:1908.02782, 2002.04043, work in progress

The problem

String theory began with Veneziano amplitude

– tree level scattering amplitude of four tachyons in open string theory

World-sheet expression for the amplitude (in $\alpha' = 1$ unit)

$$\int_0^1 dy\, y^{2p_1.p_2}(1-y)^{2p_2.p_3}$$

– diverges for $2p_1.p_2 \le -1$ or $2p_2.p_3 \le -1$.

Our convention: $a.b \equiv -a^0 b^0 + \vec{a}.\vec{b}$

Conventional viewpoint: Define the amplitude for $2p_1.p_2 > -1$, $2p_2.p_3 > -1$ and then analytically continue to the other kinematic regions.

However, analytic continuation does not always work

It may not be possible to move away from the singularity by changing the external momenta

Examples: Mass renormalization, Vacuum shift

– discussed earlier

In these lectures we shall discuss another situation where analytic continuation fails

– D-instanton contribution to string amplitudes

D-instanton: A D-brane with Dirichlet boundary condition on all non-compact directions including (euclidean) time.

D-instantons give non-perturbative contribution to string amplitudes that are important in many situations

Example: KKLT moduli stabilization uses non-perturbative contribution from D-instanton (euclidean D3-brane)

Systematic computation of string amplitudes in such backgrounds will require us to compute amplitudes in the presence of D-instantons

Problem: Open strings living on the D-instanton do not carry any continuous momenta

$\Rightarrow$ we cannot move away from the singularities by varying the external momenta

Some examples:

Let X be the (euclidean) time direction

Since the D-instanton is localized at some given euclidean time, it has a zero mode that translates it along time direction

4-point function of these zero modes:

$$A = \int_0^1 dy \, \left[y^{-2} + (y-1)^{-2} + 1 \right]$$

Derivation of this expression will be discussed later.

$$\mathbf{A} = \int_0^1 \mathbf{dy}\ \left[\mathbf{y}^{-2} + (\mathbf{y}-1)^{-2} + 1\right]$$

– diverges from near y=0 and y=1.

In this case no analytic continuation is possible since open strings on D-instantons do not carry momentum.

On physical grounds, we expect this amplitude to vanish since translation along X is an exactly marginal deformation of the world-sheet theory.

In the first example we shall study, we shall see how to get 0 from this divergent integral.

Related work:

Berkovits, Schnabl: hep-th/0307019

Maccaferri, Merlano, arXiv:1801.07607,1905.04958

Erbin, Maccaferri, Vosmera, arXiv:1912.05463, $\cdots$

Another example: Bosonic string theory in two dimensions

World-sheet theory: A free scalar X describing time coordinate and a Liouville field theory with central charge 25

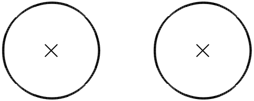
Total central charge adds up to 26, cancelling anomalies on the world-sheet

In this case the closed string ‘tachyon’ is actually a massless state of the theory

In arXiv:1907.07688 Balthazar, Rodriguez and Yin (BRY) computed the D-instanton contribution to the two point amplitude of closed string tachyons

This model is interesting because there is a dual matrix model description that gives the exact results.

The leading contribution comes from the product of two disk one point functions.



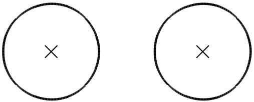
Result:

$$8\,\pi\,\mathbf{N}\,\mathbf{e}^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\mathbf{sinh}(\pi|\omega_1|)\,\mathbf{sinh}(\pi|\omega_2|)$$

N: An overall normalization constant

g_s: string coupling constant

$-\omega_1, \omega_2$: energies of incoming / outgoing ‘tachyons’



Naively one might have expected this to be proportional to $\delta(\omega_1)\delta(\omega_2)$

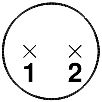
However, for D-instanton boundary conditions, individual disk amplitudes do not conserve energy, since time translation invariance is broken

The energy conservation is restored at the end after integration over the collective coordinates

– will be discussed later.

At the next order, there are two contributions.

1. Two point function on the disk.



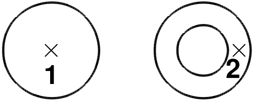
Result:

$$8\,\pi\,N\,e^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$
$$\times\frac{1}{2}\,g_s\int_0^1 dy\,y^{-2}(1+2\omega_1\omega_2 y)+\text{ finite terms}$$

Note the divergences from the $y\to 0$ limit

– cannot be tamed by deforming the ω_i's.

2. Product of disk one point function and annulus one point function.



$+ (1 \Leftrightarrow 2)$

Result:

$$8\,\pi\,N\,e^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$

$$\times\ g_s\int_0^1 dv\int_0^{1/4}dx\left\{2\,\frac{v^{-2}-v^{-1}}{\sin^2(2\pi x)}+2(\omega_1^2+\omega_2^2)\,v^{-1}\right\}$$

$$+\ \text{Finite terms}$$

Note the divergences from $x\to 0$ and $v\to 0$ that cannot be tamed by adjusting the ω_i's.

Finite terms include divergences that can be tamed by analytic continuation in ω_1,ω_2.

After setting $\omega_2 = -\omega_1$, the total divergent factor is:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1-2\omega_1^2 y) + \int_0^1 dv \int_0^{1/4} dx \left\{ 2\,\frac{v^{-2}-v^{-1}}{\sin^2(2\pi x)} + 4\,\omega_1^2\, v^{-1} \right\}$$

BRY replaced this by

$$A + B\,\omega_1^2$$

with unknown constants A and B.

They then numerically compared the result with matrix model results as function of ω_1.

The best fit results:

$$A = -0.496, \qquad B = -1.399$$

Question: Can we get these results from string theory without invoking the matrix model?

Answer: $B = -\log 4 \simeq -1.386...$, $\qquad A = ?$

The solution

We shall use string field theory (SFT) to deal with the divergences arising in the world-sheet theory.

SFT is a regular quantum field theory (QFT) with infinite number of fields

Perturbative amplitudes: sum of Feynman diagrams

Each diagram covers part of the integration region over the world-sheet variables (moduli space of Riemann surfaces and locations of vertex operators)

Sum of the diagrams covers the full integration region.

How do we get integral over world-sheet variables from a Feynman diagram?

Express internal propagator as

$$(k^2+m^2)^{-1} = \int_0^\infty ds\, e^{-s(k^2+m^2)} = \int_0^1 dq\, q^{k^2+m^2-1}, \quad q \equiv e^{-s}$$

The integration over q gives integration over world-sheet variables after a change of variable.

Divergences come from the $q \to 0$ region for $k^2+m^2 \le 0$.

All divergences in string theory are of this kind.

For D-instantons k=0, and we cannot analytically continue in momenta to make $k^2+m^2 > 0$.

$$(m^2)^{-1} = \int_0^1 dq\, q^{m^2-1}$$

This equation is:

1. An identity for $m^2 > 0$.

2. For $m^2 < 0$ the lhs is finite but the rhs is divergent

$\Rightarrow$ use lhs to define the integral.

– Change variables from the moduli of Riemann surfaces to the variables $q_1, q_2, \cdots$ associated with the propagators

– Replace $\int_0^1 dq\, q^{\beta-1}$ by $1/\beta$ for $\beta \neq 0$

– can be used to deal with power law divergences like $\int_0^1 dy\, y^{-2}$ in the earlier formulæ

For implementing this procedure, it is crucial that we transform the original integration variables (like y, v, x) to the q variables associated with the propagators in string field theory

The procedure for doing this will be explained in later lectures.

Comment 1: Making the correct change of variables is important for getting the correct result.

Replacement rule: $\int_0^1 dq\, q^{-2} = -1$

Suppose we change variable to

$$q' = \frac{q}{(1 - c\, q)} \quad \Leftrightarrow \quad \frac{1}{q'} = \frac{1}{q} - c, \qquad c = \text{constant}$$

Then $dq\, q^{-2} = dq'\, q'^{-2}$

$$\Rightarrow \int_0^1 dq\, q^{-2} = \int_0^{1/(1-c)} dq' q'^{-2} = \int_0^1 dq' q'^{-2} + \int_1^{1/(1-c)} dq' q'^{-2}$$

If we now replace the first term on the rhs by -1 using the replacement rule, we get

$$-1 + 1 - (1 - c)^{-1} = -(1 - c)^{-1}$$

– a different answer!

Comment 2. The change of variables is determined by string field theory, but may not take a simple form.

e.g. consider the four point function of translational zero modes:

$$\mathbf{A} = \int_0^1 \mathbf{dy}\ \left[\mathbf{y}^{-2} + (\mathbf{y}-1)^{-2} + 1\right]$$

The change of variable near y=0 takes the form:

$$\mathbf{y} = 1 - \frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\,\alpha^2 + (1+\gamma)^2\,\mathbf{q}}\ \frac{4\alpha^2 + (\gamma^2 - 1)\mathbf{q}}{4\,\alpha^2 + (1-\gamma)^2\,\mathbf{q}}$$

α, γ: parameters of string field theory

The final result is independent of α, γ.

This formula will be derived later.

$$(m^2)^{-1} = \int_0^1 dq\, q^{m^2-1},$$

For $m = 0$ both sides are divergent.

– associated with zero modes on the D-instanton

– produces logarithmic divergence in the world-sheet description

Strategy: Understand the physical origin of the zero modes and then find suitable remedy by drawing insights from QFT.

D-instantons have zero modes associated with translation of the instanton position along transverse directions

– known as collective coordinates ϕ

$\Rightarrow$ massless open string states

Treatment of these zero modes in QFT:

1. Carry out path integral over all modes of the instanton other than ϕ, in the background of ϕ

$\Rightarrow$ while evaluating Feynman diagrams we remove the ϕ contribution from the internal propagators but keep ϕ's as external states

After summing over Feynman diagrams we get a given closed string amplitude as a function F(ϕ).

2. Then we compute $\int$ dϕ F(ϕ)

Strategy: Follow the same procedure for D-instantons

In the world-sheet approach the first step demands:

a. Drop terms of the from $\int_0^1 dq\, q^{-1}$ coming from the collective coordinates

b. Allow external states to be both closed strings and the open string zero mode ϕ.

The second step is a finite dimensional integral over ϕ that needs to be performed.

After field redefinition, ϕ dependence is of the form $e^{ip.\phi}$

p: total momenta carried by the external closed strings

ϕ integration will generate the $\delta(p)$ factor.

An important point:

In general the zero mode of the open string will be related to the collective mode ϕ <u>after a field redefinition</u>.

Only after this field redefinition we have simple dependence on ϕ of the form $e^{ip.\phi}$ and the ϕ integral is performed easily.

The field redefinition may induce a Jacobian in the path integral measure that needs to be taken into account in the analysis.

We shall see an example of this in later lectures

However, string theory has other zero modes besides the ones associated with the collective coordinates

– arise from the ghost sector

– gives additional logarithmic divergence in the world-sheet integrals that is not removed by removing the collective modes from the propagators.

These divergences are clearly visible in various world-sheet expressions, including the BRY formula:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1-2\omega_1^2 y) + \int_0^1 dv \int_0^{1/4} dx \left\{ 2\, \frac{v^{-2} \overset{\Downarrow}{-} v^{-1}}{\sin^2(2\pi x)} + 4\,\omega_1^2\, v^{-1} \right\}$$

Understanding the origin of these divergences and their treatment will require some knowledge of world-sheet string theory and string field theory.

World-sheet theory

Bosonic string world-sheet theory

– a c=26 matter CFT

– ghosts b, c, $\bar{b}$, $\bar{c}$ of conformal weights (2,0), (-1,0), (0,2), (0, -1)

Ghost number: 1 for c, $\bar{c}$, -1 for b, $\bar{b}$

Assume matter CFT has (euclidean) time coordinate X and a c=25 CFT

Examples of c=25 CFT:

– 25 free scalars $\Rightarrow$ D=26 bosonic string

– c=25 Liouville $\Rightarrow$ D=2 bosonic string

State - operator correspondence in CFT $\Rightarrow$ there must be a vertex operator for every string state

Closed string state $\Leftrightarrow$ vertex operator in the bulk

Open string state $\Leftrightarrow$ vertex operator on the boundary

We shall focus on open string theory on a D-instanton since the problem arises there.

Ghost boundary conditions: $c = \bar{c}$, $b = \bar{b}$

Matter boundary conditions: Dirichlet along all non-compact directions

Expansions of fields in the upper half plane:

$$c = \sum_n c_n z^{-n+1}, \quad b(z) = \sum_n b_n z^{-n-2}, \quad i\partial X = \sum_{n\neq 0} \alpha_n z^{-n-1}, \ \cdots$$

Note: Due to boundary condition on the real axis, the expansion coefficients of $\bar{b}$, $\bar{c}$, $i\bar{\partial}X$ are given by b_n, c_n, α_n

SL(2,R) invariant vacuum $|0\rangle$:

$$c_n|0\rangle = 0 \text{ for } n \geq 2, \quad b_n|0\rangle = 0 \text{ for } n \geq -1, \quad \alpha_n|0\rangle \text{ for } n \geq 0$$

State-operator correspondence: $|V\rangle = V(0)|0\rangle$, e.g.

$$c \Leftrightarrow c(0)|0\rangle = c_1|0\rangle, \quad b \Leftrightarrow b(0)|0\rangle = b_{-2}|0\rangle, \quad i\partial X \Leftrightarrow i\partial X(0)|0\rangle = \alpha_{-1}|0\rangle$$

Singular OPE:

$$\mathbf{b(z)c(w)} = \frac{\mathbf{1}}{\mathbf{z-w}}, \quad \partial\mathbf{X(z)}\partial\mathbf{X(w)} = \frac{\mathbf{1}}{\mathbf{(z-w)^2}} \quad \textbf{etc}$$

$$\Rightarrow \quad \{\mathbf{b_n}, \mathbf{c_m}\} = \delta_{\mathbf{m+n,0}}, \quad [\alpha_{\mathbf{m}}, \alpha_{\mathbf{n}}] = -\mathbf{m}\,\delta_{\mathbf{m+n,0}} \quad \textbf{etc}$$

States in Hilbert space H: Created by action on $|0\rangle$ of

c_{-n} for $n \geq -1$, b_{-n} for $n \geq 2$, α_{-n} for $n \geq 1$, etc.

Physical open string states have ghost number 1

– described by vertex operators with ghost number 1.

Time translation of D-instanton $\Rightarrow$ zero mode ϕ

$\Rightarrow$ described by the open string state $c_1 \alpha_{-1}|0\rangle$

– corresponding vertex operator: $i\, c\, \partial X$ (unintegrated)

Generic physical open string states have vertex operator cW, with W a dimension one primary in the matter sector.

Associated ‘integrated’ vertex operator is W

String amplitudes from a given Riemann surface:

1. Take a Riemann surface, possibly with boundaries

2. Choose some marked points (punctures) on the Riemann surface

– one bulk puncture for each external closed string

– one boundary puncture for each external open string

3. If the Riemann surface has conformal Killing vectors then use them to fix the locations of some punctures.

4. Insert unintegrated vertex operators at fixed punctures, integrated vertex operators at variable punctures.

5. Insert additional b-ghosts, one for each modulus of the Riemann surface.

6. Evaluate the correlation function, and integrate this over the locations of the punctures and moduli of the Riemann surface.

Example: Amplitudes of four zero mode fields ϕ from the upper half plane (UHP)

UHP has three conformal Killing vectors generating SL(2,R) isometry

$$z \to (az+b)/(cz+d)$$

– can be used to fix three of the punctures at 0, 1, ∞.

Put the fourth puncture at y.

Amplitude:

$$A = \int_0^1 dy \langle c\partial X(0)\, \partial X(y)\, c\partial X(1)\, c\partial X(\infty)\rangle$$

Use $\langle c(x_1)c(x_2)c(x_3)\rangle = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)$ to get:

$$A = \int_0^1 dy\, \left[y^{-2} + (1-y)^{-2} + 1\right]$$