Algebraic structures of effective string field theory

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Workshop on Fundamental Aspects of String Theory
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In collaboration with: Carlo Maccaferri (Torino), Martin Schnabl, Jakub Vošmera (Prague) [1912.05463 + to appear]
Outline

1. Motivations
2. Perturbative description
3. Coalgebra description
4. Outlook
Effective actions

Goals:

- compute effective actions from string field theory
- study general structure (using $A_\infty$ and $L_\infty$ algebra)
- keep finite-momentum massless physical fields
- compute higher-derivative corrections
- include Ellwood invariant
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Two approaches:
- perturbative: intuitive, cumbersome beyond lowest orders
- coalgebra: all-order statements, natural field basis, simple deformations (homological perturbation)
Selected references

- **algebraic aspects** [hep-th/0107162, Lazaroiu; hep-th/0112228, Kajiura; math/0306332, Kajiura; 1609.00459, Sen; 1610.03251, Erler; 1901.08555, Matsunaga; 2003.05021, Masuda-Matsunaga]

- **explicit computations** [hep-th/0307019, Berkovits-Schnabl; 1801.07607, Maccaferri-Merlano; 1905.04958, Maccaferri-Merlano; 1912.05463, HE-Maccaferri-Vošmera]

- **level-truncation** [hep-th/0001201, Taylor; hep-th/0005085, David; hep-th/0306041, Coletti-Sigalov-Taylor; hep-th/0404102, Taylor; 1712.05935, Asada-Kishimoto]

- **related topics** [1902.10955, Mattiello-Sachs; 1910.00538, Vošmera; 2002.04043, Sen]

See also: Hiroaki’s, Jakub’s, Yuji’s talks
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4. Outlook
Classical string field theory

- $\mathcal{H}$ Hilbert space of string states, $\Psi \in \mathcal{H}$ string field
- action and equation of motion ($A_\infty$ and $L_\infty$)

$$S = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \langle \Psi, V(\Psi) \rangle$$

$$\mathcal{E}(\Psi) := Q\Psi + V'(\Psi), \quad V(\Psi) := \int_0^1 dt \ V'(t\Psi)$$

- $\langle \cdot, \cdot \rangle$ inner-product on $\mathcal{H}$, $Q := \ell_1$ BRST charge
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- $\langle \cdot, \cdot \rangle$ inner-product on $\mathcal{H}$, $Q := \ell_1$ BRST charge
- $L_\infty$ potential = interactions

$$V(\Psi) = \sum_{n \geq 2} \frac{1}{(n+1)!} \ell_n(\Psi^n), \quad \ell_n : \mathcal{H}^\otimes n \to \mathcal{H}$$

- $L_\infty$ relations $\Rightarrow$ gauge invariance

$$\delta_\Lambda \Psi = \sum_{n \geq 1} \frac{1}{n!} \ell_{n+1}(\Psi^n, \Lambda), \quad 0 = \sum_{k+\ell=n} \ell_{k+1}(\cdots, \ell_\ell(\cdots))$$
Definitions

- projector such that [1609.00459, Sen]

\[ [L_0, P] = [Q, P] = [b_0, P] = 0 \]
\[ P^\dagger = P, \quad \ker L_0 \in \text{Im } P \]

- light states \( P\mathcal{H} \)
- heavy states \( \bar{P}\mathcal{H} := (1 - P)\mathcal{H} \)
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- example: projector on massless states (open string)

\[
P_0 := e^{-\infty \hat{L}_0}, \quad L_0 = \alpha' k^2 + \hat{L}_0, \quad \hat{L}_0 =: \alpha' m^2
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- procedure \(\rightarrow\) effective action of light fields:

  1. Siegel gauge fixing heavy fields
  2. integrate out heavy fields
  3. check out-of-Siegel gauge constraints
  4. integrate out light auxiliary fields
Gauge fixing heavy fields (1)

- gauge fixing needed to invert kinetic term
- Siegel gauge projector

\[ \Pi_s := b_0 c_0, \quad \bar{\Pi}_s := c_0 b_0 \]

- field decomposition

\[ \Psi = \varphi + R_\downarrow + R_\uparrow \]

\[ \varphi := P\Psi, \quad R_\downarrow := \Pi_s \bar{P}\Psi, \quad R_\uparrow := \bar{\Pi}_s \bar{P}\Psi \]
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- BRST charge decomposition

\[ Q = c_0 L_0 - b_0 M_+ + \hat{Q}, \]

- equations of motion

\[ E_\varphi(\Psi) := P E(\Psi) = Q\varphi + PV'(\Psi) \]

\[ E_{R_\downarrow}(\Psi) := \bar{P}\bar{\Pi}_s E(\Psi) = c_0 L_0 R_\downarrow + \hat{Q} R_\uparrow + \bar{P}\bar{\Pi}_s V'(\Psi) \]

\[ E_{R_\uparrow}(\Psi) := \bar{P}\Pi_s E(\Psi) = \hat{Q} R_\downarrow - b_0 M_+ R_\uparrow + \bar{P}\Pi_s V'(\Psi) \]
Gauge fixing heavy fields (2)

- Siegel gauge for heavy field
  \[ b_0(\bar{P}\Psi) = 0 \implies R_\uparrow = 0 \]

- Gauge fixed equations of motion
  \[
  \mathcal{E}_{gf,\varphi}(\psi_{gf}) = Q\varphi + PV'(\psi_{gf}), \\
  \mathcal{E}_{gf,R_\downarrow}(\psi_{gf}) = c_0 L_0 R_\downarrow + \bar{P}\bar{\Pi}_s V'(\psi_{gf})
  \]
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- Siegel gauge propagator (contracting homotopy operator)
  \[ \Delta := \frac{b_0}{L_0} \bar{P}_0, \quad \{Q, \Delta\} = \bar{P}_0 \]

- Eom for \( R_\downarrow \)
  \[ \mathcal{E}_{gf,R_\downarrow}(\psi_{gf}) = 0 \implies R_\downarrow = -\frac{b_0}{L_0} \bar{P}V'(\varphi + R_\downarrow) \]

\( L_0 \) can be singular in \( \bar{P}\mathcal{H} \)
Gauge fixing heavy fields (2)

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  \]

\( L_0 \) can be singular in \( \bar{P}\mathcal{H} \)

- Effective theory \( \Rightarrow \) momentum cut-off \( \alpha' k^2 \ll \min \hat{L}_0 \) in \( \bar{P}\mathcal{H} \)
Out-of-Siegel gauge constraints

- out-of-Siegel gauge constraints (∼ Gauss constraints)

\[ \mathcal{E}_{gf,R^\uparrow}(\Psi_{\text{eff}}) = \hat{Q}R^\downarrow + \bar{P}\Pi_s V'(\Psi_{\text{eff}}) \]

\[ \Psi_{\text{eff}} := \varphi + R^\downarrow(\varphi) \]

- cannot be derived from effective action

\[ S_{\text{eff}} = \frac{1}{2} \langle \varphi, Q\varphi \rangle + \langle \varphi, PV(\Psi_{\text{eff}}) \rangle \]

\[ + \left\langle \Pi_s V'(\Psi_{\text{eff}}), \frac{b_0}{L_0} \bar{P} \left( \frac{V'(\Psi_{\text{eff}})}{2} - V(\Psi_{\text{eff}}) \right) \right\rangle \]

→ must impose \( \mathcal{E}_{gf,R^\uparrow} = 0 \) on the side
Out-of-Siegel gauge constraints

- out-of-Siegel gauge constraints (\(\sim\) Gauss constraints)

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\( \rightarrow \) must impose \( E_{gf,R\uparrow} = 0 \) on the side

- result: automatic if light eom holds

\[ E_{gf,\varphi} = 0 \implies E_{gf,R\uparrow} = 0 \]
Perturbative solution

expand all fields and potential with $\mu \ll 1$

$$\varphi = \sum_{n \geq 1} \mu^n \varphi_n, \quad R_{\downarrow} = \sum_{n \geq 1} \mu^n R_n, \quad V' = \sum_{n \geq 2} \mu^n V'_n$$

solve order by order in $\mu$ and resum

$$R_{\downarrow} = -\frac{1}{2} \frac{b_0}{L_0} P \ell_2 (\varphi^2) + \frac{1}{2} \frac{b_0}{L_0} P \ell_2 \left( \varphi, \frac{b_0}{L_0} P \ell_2 (\varphi^2) \right)$$

$$- \frac{1}{3!} \frac{b_0}{L_0} P \ell_3 (\varphi^3) + O(\varphi^4)$$

$$\varphi = \mu \varphi_1 + \mu^2 \varphi_2 + O(\mu^3)$$

effective action

$$S_{\text{eff}} = \frac{1}{2} \langle \varphi, Q \varphi \rangle + \frac{1}{3!} \langle \varphi, P \ell_2 (\varphi^2) \rangle + \frac{1}{4!} \langle \varphi, P \ell_3 (\varphi^3) \rangle$$

$$- \frac{1}{8} \left< \Pi_{s} \ell_2 (\varphi^2), \frac{b_0}{L_0} P \ell_2 (\varphi^2) \right> + O(\varphi^5)$$

many interesting cases: compute with localization [Jakub’s talk]
Effective gauge invariance

- equation of motion

\[ \mathcal{E}_{gf,\varphi} = Q\varphi + \sum_{n \geq 2} \frac{1}{n!} \tilde{\ell}_n(\varphi^n) \]

- effective \( L_\infty \) structure

\[ \tilde{\ell}_1(A_1) = PQA_1, \quad \tilde{\ell}_2(A_1, A_2) = P\ell_2(A_1, A_2), \]
\[ \tilde{\ell}_3(A_1, A_2, A_3) = P\ell_3(A_1, A_2, A_3) - P\ell_2 \left( A_1, \frac{b_0}{L_0} \bar{P}\ell_2(A_2, A_3) \right) \]
\[ - (\bar{P})^{A_1(A_2+A_3)} P\ell_2 \left( A_2, \frac{b_0}{L_0} \bar{P}\ell_2(A_3, A_1) \right) \]
\[ - (\bar{P})^{A_3(A_1+A_2)} P\ell_2 \left( A_3, \frac{b_0}{L_0} \bar{P}\ell_2(A_1, A_2) \right) \]

- effective (non-canonical) gauge invariance

\[ \delta_\lambda \varphi = Q\lambda + \tilde{\ell}_2(\varphi, \lambda) + \frac{1}{2} \tilde{\ell}_3(\varphi^2, \lambda) + O(\varphi^3), \quad \bar{P}\lambda = 0 \]
Integrating out auxiliary massless fields (1)

- consider massless gauge field (open bosonic string)
- $\hat{L}_0 = 0$ field at $N_{gh} = 1$

$$\varphi_A := \frac{\sqrt{2}}{\alpha'} A_\mu(k) c i \partial X^\mu e^{ik \cdot X}, \quad \varphi_B := \frac{B(k)}{\sqrt{2}} \partial c e^{ik \cdot X}$$

- $A_\mu(k)$ gauge field: $\varphi_A$ primary if $k \cdot A = 0$, on-shell for $k^2 = 0$

- $B(k)$ Nakanishi–Lautrup (NL) auxiliary field, $\varphi_B$ not primary
  note: massless ghost zero-mode from [2002.04043, Sen]
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- Siegel gauge condition + constraint:

\[
\begin{align*}
& b_0 \varphi_B = 0 \\
& \hat{Q} \varphi_A = 0 \\
\end{align*} \quad \Rightarrow \quad \begin{align*}
& B(k) = 0 \\
& k \cdot A(k) = 0 \\
\end{align*}
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► Siegel gauge condition + constraint:

$$\begin{cases} b_0 \varphi_B = 0 \\ \hat{Q} \varphi_A = 0 \end{cases} \implies \begin{cases} B(k) = 0 \\ k \cdot A(k) = 0 \end{cases}$$

► keep gauge invariance $\to$ integrate out NL field
Integrating out auxiliary massless fields (2)

Integrate out $B(k)$ field:

- solve equation

\[ \Pi_s \mathcal{E}_{gh,\varphi} = 0 \quad \Rightarrow \quad \varphi_B = c_0 M_-(\hat{Q}\varphi_A - PV'(\varphi_A + \varphi_B)) \]

- use $SU(1,1)$ algebra

\[
[M_+, M_-] = \hat{N}_{gh}, \quad \quad \quad \quad \quad \quad \quad \quad [\hat{N}_{gh}, M_\pm] = \pm 2M_\pm
\]

$\hat{N}_{gh}$ ghost number without zero-mode
Integrating out auxiliary massless fields (2)

Integrate out $B(k)$ field:

- solve equation

\[ \Pi_s \mathcal{E}_{gh,\varphi} = 0 \quad \implies \quad \varphi_B = c_0 \mathcal{M}_- \left( \hat{Q} \varphi_A - PV' \left( \varphi_A + \varphi_B \right) \right) \]

- use $SU(1,1)$ algebra

\[ [\mathcal{M}_+, \mathcal{M}_-] = \hat{\mathcal{N}}_{gh}, \quad [\hat{\mathcal{N}}_{gh}, \mathcal{M}_\pm] = \pm 2 \mathcal{M}_\pm \]

\( \hat{\mathcal{N}}_{gh} \) ghost number without zero-mode

- \( \varphi_B = O(\varphi_A) \)

- free action before integrating out

\[ S = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \left[ A_\mu(k) k^2 A_\mu(-k) - B(k) B(-k) + 2k \cdot A(k) B(-k) \right] \]
Integrating out auxiliary massless fields (3)

Better approach:

- field redefinition to make state with $A_\mu$ primary

\[ \tilde{\varphi}_A := \frac{A_\mu(k)}{\sqrt{2}} \left( \frac{2}{\alpha'} c i \partial X^\mu + k^\mu \partial c \right) e^{ik \cdot X} \]

\[ \varphi_\beta := \frac{\beta(k)}{\sqrt{2}} \partial c e^{ik \cdot X}, \quad \beta(k) := B(k) - k \cdot A(k) \]
Integrating out auxiliary massless fields (3)

Better approach:

- **field redefinition** to make state with $A_\mu$ primary

  \[ \tilde{\varphi}_A := \frac{A_\mu(k)}{\sqrt{2}} \left( \frac{2}{\alpha'} c i \partial X^\mu + k^\mu \partial c \right) e^{i k \cdot X} \]

  \[ \varphi_\beta := \frac{\beta(k)}{\sqrt{2}} \partial c e^{i k \cdot X}, \quad \beta(k) := B(k) - k \cdot A(k) \]

- can be implemented with **modified projector** $\Pi$

  \[ \tilde{\varphi}_A := \Pi(\varphi_A + \varphi_B), \quad \varphi_\beta := \bar{\Pi}(\varphi_A + \varphi_B) \]

- **free action**

  \[ S = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \left[ A_\mu(k)(k^2 \eta^{\mu\nu} - k^\mu k^\nu)A_\nu(-k) - \beta(k)\beta(-k) \right] \]

- $\varphi_\beta = O(\tilde{\varphi}_A^2)$
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Motivations

Advantages of coalgebra description:

▶ no need for explicit field decomposition
▶ optimal projector clearly characterized
▶ package perturbative expansion and effective interactions
▶ read directly effective $L_\infty$ structure
▶ perform both projections at the same time (“horizontal composition”)
▶ deformation (e.g. Ellwood invariant) combined directly with projection (“vertical composition”)

Applications ($A_\infty$ SFT, but works also for $L_\infty$):

▶ effective action for gauge and NL fields
▶ effective action for gauge field only
▶ effective action with Ellwood invariant
Coalgebra description (1)

- tensor product Hilbert space
  \[ T\mathcal{H} := \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \otimes^2 + \cdots, \quad \pi_k : T\mathcal{H} \to \mathcal{H} \otimes^k \]

- coderivation \rightarrow embed \ A_\infty \ products
  \[ m_n : T\mathcal{H} \to T\mathcal{H}, \quad m := \sum_{n \geq 1} m_n \]
  \[ m_n \pi_N = \sum_{k=0}^{N-n} 1_{\mathcal{H}} \otimes (N-n-k) \otimes m_n \otimes 1_{\mathcal{H}} \]

- \( A_\infty \) relation
  \[ [m, m] = 0 \]

- group-like element
  \[ \frac{1}{1 - A} = 1_{T\mathcal{H}} + A + A \otimes^2 + \cdots \]

- symplectic form
  \[ \omega : \mathcal{H} \otimes^2 \to \mathbb{C} \]
Coalgebra description

- action

\[ S = \int_0^1 dt \omega \left( \pi_1 \partial_t \frac{1}{1 - \psi(t)} \otimes \pi_1 m \frac{1}{1 - \psi(t)} \right) \]

\( \psi(1) = \psi, \quad \psi(0) = 0, \quad \pi_1 \partial_t \pi_1 = \partial_t \)

- equation of motion

\[ \pi_1 m \frac{1}{1 - \psi(t)} = 0 \]

- gauge transformation

\[ \delta \Lambda \frac{1}{1 - \psi} = [m, \Lambda] \frac{1}{1 - \psi} \]

\[ \pi_1 \Lambda \pi_0 = \Lambda \in \mathcal{H} \]
Homological perturbation lemma (1)

- encode free SFT as strong deformation retract:
  - vector space $\mathcal{T}\mathcal{H}$
  - differential = BRST charge $Q = m_1$
  - contracting operator = free propagator $\Delta$
  - projector $P$

- describe interactions as perturbation
  - perturbation $\delta m = \text{interactions } m_n \text{ for } n \geq 2$
  - full differential $m = m_1 + \delta m$
  - full contracting operator $\eta$
  - full projector $\Pi$
**Homological perturbation lemma (1)**

- encode free SFT as strong deformation retract:
  - vector space $T\mathcal{H}$
  - differential = BRST charge $Q = m_1$
  - contracting operator = free propagator $\Delta$
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- describe interactions as perturbation
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  - full differential $m = m_1 + \delta m$
  - full contracting operator $\eta$
  - full projector $\Pi$

- conditions on $Q$ and $m$

\[
[P, Q] = 0, \quad Q^2 = 0, \quad [\Pi, m] = 0, \quad m^2 = 0
\]

- gauge-fixing and Hodge–Kodaira decomposition

\[
\Delta \psi = 0, \quad [Q, \Delta] = 1 - P, \quad P\Delta = \Delta P = \Delta^2 = 0
\]
\[
\eta \psi = 0, \quad [m, \eta] = 1 - \Pi, \quad \Pi \eta = \eta \Pi = \eta^2 = 0
\]
Homological perturbation lemma (2)

- Theory diagram: projector $P$ and perturbation $\delta m$

\[
\Delta \otimes (\mathcal{T}\mathcal{H}, Q) \xrightarrow{\delta m} (TP\mathcal{H}, Q) \xrightarrow{\delta \tilde{m}} (TP\tilde{m}) \\
\eta \otimes (\mathcal{T}\mathcal{H}, m) \xrightarrow{\eta} (T\Pi\mathcal{H}, \tilde{m})
\]

- Homological perturbation lemma

\[
\delta \tilde{m} = P \delta m \frac{1}{1 + \Delta \delta m}, \quad \eta = \Delta - \Delta \delta \tilde{m} \Delta, \\
\Pi = \frac{1}{1 + \Delta \delta m} P \frac{1}{1 + \delta m \Delta}
\]
Application: integrate out heavy fields

- \( P \) integrates out heavy fields, \( \delta m \) adds interactions

**Theory diagram**

\[
\text{free theory} \xrightarrow{\delta m} \text{interacting theory} \xrightarrow{\Pi} \text{interacting effective theory} \]

\[
\text{free effective theory} \xrightarrow{\delta \tilde{m}} \text{interacting effective theory}
\]

- read effective interactions by expanding

\[
\tilde{m}_2(\varphi^2) = Pm_2(\varphi^2),
\]

\[
\tilde{m}_3(\varphi^3) = Pm_3(\varphi^3) - 2 \ Pm_2(\Delta m_2(\varphi^2), \varphi)
\]
Horizontal composition

- theory diagram: two successive projections $P_1$ and $P_2$

$$
\begin{align*}
\Delta_1 & \xrightarrow{P_1} \Delta_2 \\
\eta_1 & \xrightarrow{P_1} \eta_2
\end{align*}
$$

$$
\begin{align*}
\delta m & \xrightarrow{\delta \tilde{m}} \\
\delta \tilde{m} & \xrightarrow{\delta m'}
\end{align*}
$$

$$
\begin{align*}
\Pi_1 & \xrightarrow{P_2} \Pi_2 \\
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$$
\begin{align*}
\Delta_1 \xrightarrow{(T \mathcal{H}, Q)} & \xrightarrow{P_1} \Delta_2 \xrightarrow{(TP_1 \mathcal{H}, Q)} \\
\eta_1 \xrightarrow{(T \mathcal{H}, m)} & \xrightarrow{\Pi_1} \eta_2 \xrightarrow{(T \Pi_1 \mathcal{H}, \tilde{m})} \\
\delta m & \xrightarrow{\delta \tilde{m}} \\
\delta \tilde{m} & \xrightarrow{\delta m'}
\end{align*}
$$

Result: equivalent to

$$
\begin{align*}
\Delta_1 \xrightarrow{(T \mathcal{H}, Q)} & \xrightarrow{P_1} \Delta_2 \xrightarrow{(TP_1 \mathcal{H}, Q)} \\
\eta_1 \xrightarrow{(T \mathcal{H}, m)} & \xrightarrow{\Pi_1} \eta_2 \xrightarrow{(T \Pi_1 \mathcal{H}, \tilde{m})} \\
\delta m & \xrightarrow{\delta \tilde{m}} \\
\delta \tilde{m} & \xrightarrow{\delta m'}
\end{align*}
$$

$$
\begin{align*}
P_1 \Pi_1 & = P_2 \\
\Pi_1 \Pi_2 & = \Pi_2 \\
\Delta_1 \Pi_1 & = \Delta_1 + \Delta_2 \\
\eta_1 \Pi_1 & = \eta_1 + \eta_2
\end{align*}
$$
Horizontal composition

- theory diagram: two successive projections $P_1$ and $P_2$

\[
\begin{align*}
\Delta_1 \circ (T\mathcal{H}, Q) \xrightarrow{P_1} \Delta_2 \circ (TP_1\mathcal{H}, Q) \xrightarrow{P_2} (TP_2P_1\mathcal{H}, Q) \\
\eta_1 \circ (T\mathcal{H}, m) \xrightarrow{\Pi_1} \eta_2 \circ (T\Pi_1\mathcal{H}, \tilde{m}) \xrightarrow{\Pi_2} (T\Pi_2\Pi_1\mathcal{H}, m')
\end{align*}
\]

- result: equivalent to

\[
\begin{align*}
\Delta_{12} \circ (T\mathcal{H}, Q) \xrightarrow{P_{12}} (TP_{12}\mathcal{H}, Q) & & P_{12} = P_2 P_1 \\
\eta_{12} \circ (T\mathcal{H}, m) \xrightarrow{\Pi_{12}} (T\Pi_{12}\mathcal{H}, \tilde{m}) & & \Pi_{12} = \Pi_2 \Pi_1 \\
\Delta_{12} = \Delta_1 + \Delta_2 P_1 & & \Delta_{12} = \Delta_1 + \Delta_2 P_1 \\
\eta_{12} = \eta_1 + \eta_2 \Pi_1 & & \eta_{12} = \eta_1 + \eta_2 \Pi_1
\end{align*}
\]
Application: integrate out NL field

- $P_1 = P_0$ (resp. $P_2 = \Pi$) integrates out heavy (resp. NL) fields
- Hodge–Kodaira decomposition for $P_0 Q$ fixes $\Pi, \Delta_2$

$$\Delta_2 = c_0 M^- P_0 = \frac{1}{2} c_0 b_{-1} b_1 P_0$$

$$\Pi = \Pi_s - c_0 W, \quad W := [M^-, \hat{Q}]$$

$\Pi$ corrected w.r.t. Siegel projector $\Pi_s$
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- horizontal composition $\rightarrow$ vertices for physical field $\tilde{\varphi}_A$

\[
\Delta_{\text{eff}} = \frac{b_0}{L_0} \bar{P}_0 + c_0 M^- P
\]

\[
m'_1(A_2) = \Pi P_0 QA_1, \quad m'_2(A_1, A_2) = \Pi P_0 m_2(A_1, A_2)
\]

\[
m'_3(A_1, A_2, A_3) = \Pi P_0 m_2 \left( \Delta_{\text{eff}} m_2(A_1, A_2), A_3 \right) + \cdots
\]

additional algebraic propagator from NL field [2002.04043, Sen]

- $k = 0 \Rightarrow m'_1 = 0$ ($\Pi P \rightarrow$ cohomology) $\rightarrow$ minimal model
Vertical composition

- theory diagram: two successive deformations $\delta m_1$ and $\delta m_2$

$$\Delta \circ (TH, Q) \xrightarrow{P} (TPH, Q)$$

$$\eta_1 \circ (TH, m) \quad \xrightarrow{\Pi_1} \quad (T\Pi_1H, \tilde{m})$$

$$\eta_2 \circ (TH, M) \quad \xrightarrow{\Pi_2} \quad (T\Pi_2H, \tilde{M})$$

$\delta m_1 \quad \delta \tilde{m}_1$

$\delta m_2 \quad \delta \tilde{m}_2$
Vertical composition

- theory diagram: two successive deformations $\delta m_1$ and $\delta m_2$

\[ \Delta \bigcirc (TH, Q) \xrightarrow{P} (TPH, Q) \]

\[ \delta m_1 \downarrow \delta \tilde{m}_1 \]

\[ \eta_1 \bigcirc (TH, m) \xrightarrow{\Pi_1} (T\Pi_1 H, \tilde{m}) \]

\[ \delta m_2 \downarrow \delta \tilde{m}_2 \]

\[ \eta_2 \bigcirc (TH, M) \xrightarrow{\Pi_2} (T\Pi_2 H, \tilde{M}) \]

- result: equivalent to

\[ \Delta \bigcirc (TH, Q) \xrightarrow{P} (TPH, Q) \]

\[ \delta m_{12} \downarrow \delta \tilde{m}_{12} \]

\[ \eta_2 \bigcirc (TH, M) \xrightarrow{\Pi_2} (T\Pi_2 H, \tilde{M}) \]

\[ \delta m_{12} = \delta m_1 + \delta m_2 \]

\[ \delta \tilde{m}_{12} = \delta \tilde{m}_1 + \delta \tilde{m}_2 \]
Application: Ellwood invariant (1)

Ellwood invariant: open string 1-point function [Yuji’s talk]

\[ E[\Psi] := \langle \mathcal{V}(i)f \circ \Psi \rangle_{UHP} := \langle \Psi, e_0 \rangle \]

\( \mathcal{V} \) on-shell closed string state at mid-point \( \rightarrow 0 \)-product \( e_0 \)

Result: effective invariant for massless \( k = 0 \) fields

\[ \tilde{E}[\varphi] = \sum_{n \geq 0} \frac{1}{n + 1} \omega(\varphi, \tilde{e}_n(\varphi^n)) = E[\Psi_{\text{eff}}(\varphi)] \]

\[ \tilde{e}_n(\varphi^n) = -\tilde{m}_{n+1}(-\Delta e_0, \varphi^n) + \text{perms} \]

Off-shell gauge invariant \( \rightarrow \) deform the action

\[ S_{\text{ell}}[\Psi] = S[\Psi] + \lambda E[\Psi], \quad S_{\text{eff,ell}}[\varphi] = S_{\text{eff}}[\varphi] + \lambda \tilde{E}[\varphi] + O(\lambda^2) \]

Result: 0-product = tadpole \( \rightarrow \) vacuum shift [1404.6254, Pius-Rudra-Sen]

Obstruction to vacuum shift of full SFT = massless eom of effective SFT (simple cases: reduce to tadpole in \( S_{\text{eff,ell}} \))
Application: Ellwood invariant (2)

- vertical composition → effective action [Jakub's, Yuji's talks]
  - $\delta m_1 = m_2 + \cdots$
  - $\delta m_2 = \lambda e$

- result: effective products

\[
\tilde{M} = P(m + \lambda e) \frac{1}{1 + \Delta(m + \lambda e - Q)} = \tilde{m} + \lambda \Pi_1 e \frac{1}{1 + \lambda \eta_1 e}
\]

→ implements automatically vacuum shift of products

note: $\tilde{m}$, $\Pi_1$, $\eta_1$ effective theory of light fields (with NL field)

- in components

\[
\tilde{M}_n(A_1, \ldots, A_n) = \sum_{k \geq 0} \tilde{m}_{n+k} \left( (-\Delta e_0)^k, A_1, \ldots, A_n \right) + \text{perms}
\]
Outline

1. Motivations
2. Perturbative description
3. Coalggebra description
4. Outlook
Conclusion and outlook

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- efficient way to combine multiple projections and perturbations
- understand better the role of the NL field
- effective action with Ellwood’s invariant

Outlook:

- compute quartic effective interaction with full $\alpha'$ corrections (Witten’s open SFT)
- generalize to open-closed SFT
- compute ghost-dilaton contributions
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