Dissection Quivers, Polytopes and Scattering Amplitudes

ϕ^4 Amplitudes from Positive geometries

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Based on earlier work with Aneesh, Pinaki, Mrunmay, Renjan and Sujoy

Ongoing work with Mrunmay Jagadale

Motivation

- In last several years, the "Amplituhedron program" for Scattering Amplitudes has seen a striking leap into the world of non-super symmetric QFTs.
- Arkani-Hamed, Bai, He and Yun (ABHY) showed that in the kinematic space formed by momenta, a Polytope known as Associahedron can be realised such that the *unique* canonical form associated to this polytope is the scattering Amplitude for bi-adjoint scalar ϕ^3 theories.

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- In last several years, the "Amplituhedron program" for Scattering Amplitudes has seen a striking leap into the world of non-super symmetric QFTs.
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- In this program, Amplitudes are thought of as differential forms as opposed to functions of momenta.
- The Locality and Unitarity of the S-matrix is a consequence of geometry of the Associahedron.
- This work also shed light on the "CHY formulae" for scattering Amplitudes as the CHY moduli space has a natural compactification as an Associahedron. (And many other beautiful results including a geometric understanding of color-kinematics duality)

Motivation

- In a Recent work, Arkani-Hamed, He, Salvatori and Thomas extended this program to loop integrands for ϕ^3 amplitudes.
- The ABHY Associahedron which contains information about tree-level amplitudes, are replaced with certain Cluster polytopes.
- Several ideas beautifully synthesize as both the Associahedra and cluster polytopes are geometric realisations of type-A and type-D cluster algebras respectively.

Our Questions

- Q1 : **Can scattering Amplitudes for massless scalar field theories with generic ϕ^p interactions understood via geometry of polytopes?**
- Q2 : **As the set of poles of an n-point (tree) amplitude for ϕ^p interaction is always a subset of poles that occur for ϕ^3 interaction, does geometric structures on Associahedra contain information about all tree-level Amplitudes?**

The Story so far

- We show that for all the interactions, a natural perspective is provided by considering a class of Quivers which are obtained from dissections of Polygons. (Pilaud, Maneville, Padrol, Palu,Plamondon and many others)
- These *dissection quivers* lead us to answers for both of the above questions which are intricately related.

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- We show that for all the interactions, a natural perspective is provided by considering a class of Quivers which are obtained from dissections of Polygons. (Pilaud, Maneville, Padrol, Palu,Plamondon and many others)
- These *dissection quivers* lead us to answers for both of the above questions which are intricately related.

- **A1** : There exists convex realisations of a large class of polytopes in the kinematic space whose canonical forms correspond to Amplitudes for arbitrary scalar interactions.
- In fact all these forms can be associated to the Associahedra as lower forms.
- **A2** : These ideas are fairly robust and naturally extend to 1-loop.
- In extending these ideas to 1-loop ϕ^4 integrand, we discover a new class of polytopes !

Disclaimer

- We will work with color ordered (planar) Amplitudes. So we implicitly assume that all our scalars have color degrees of freedom in the adjoint (or bi-adjoint) representation. Although we will never display the color indices explicitly.
- This is because the Amplituhedron ideas are perhaps best developed for planar Amplitudes.
- Although there has been progress in analysing non-planar channels .(Bern et al for Amplituhedron , and ABHY , Nick Early for ϕ^3 amplitudes.)

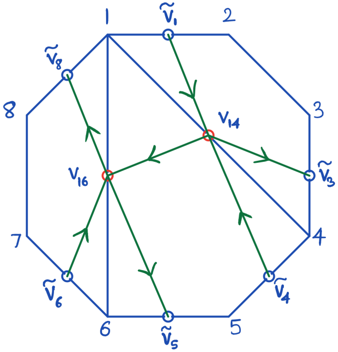
Scattering forms in Kinematic space

- We start with the planar Kinematic space for tree-level Amplitudes. defined by *planar* variables X_{ij}, $i < j$.

$$X_{ij} = (p_i + \ldots + p_n + \ldots + p_{j-1})^2$$

- Consider Polygon formed by n vertices where the vertices are labelled in clock-wise direction.
- Consider, the following quiver formed by the edges of the polygon and a dissection Q which break the polygon in cells (triangles, quadrilaterals, $\ldots$.)

Dissection Quiver



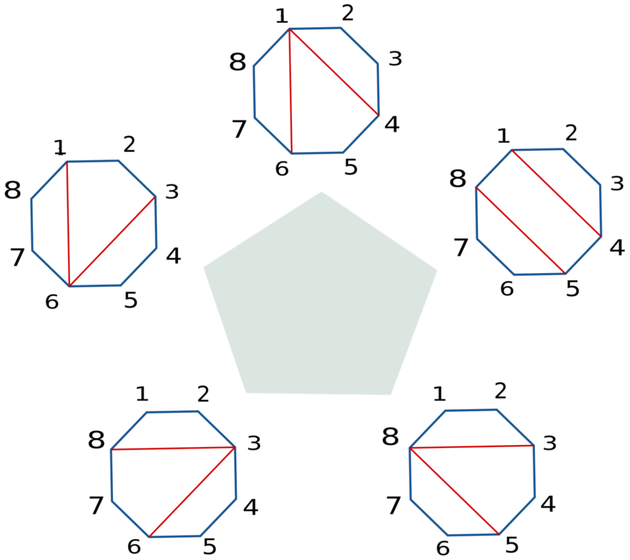
- Consider any path along the quiver that (1) begins and ends on the edges of the polygon, (2) that does not cross adjacent edges in any cell and (3) that does not begin and end on adjacent edges of the polygon. ($\tilde{v}_1\tilde{v}_6$, $\tilde{v}_3\tilde{v}_8$)

Polytopes from Dissection Quiver

- Draw a chord between hollow vertices $v_{\tilde{i}}$ and $v_{\tilde{j}}$ that we call $\mathcal{D}_{i_0 j_0}$.
- All such chords labelled by the hollow vertices form a polytope where

- Each vertex of the polytope is labelled by the maximal set of chords that do not intersect except in end points.
- Two vertices are adjacent if they differ in one and only one chord. We call such an operation a flip.
- The edge connecting such vertices is labelled by all the chords common to both of them.

An example of Accordiohedra with quadrangulation



Accordiohedra Polytopes

Some Properties

- For any dissection Q, we get one (combinatorial) polytope.
- If Q were a triangulation, the corresponding polytope is unique and is independent of the choice of Q. This polytope is the Associahedron.
- If Q is a quadrangulation, then dimension of the poytope is $\frac{n-4}{2}$. (Same as the number of propagators in any diagram wit n particles and involving only quartic vertices)
- For generic Q, these polytopes are called Accordiohedra : $\mathcal{AC}_Q$.
- **Their combinatorics and geometric realisations encode (planar/color-ordered) tree-level scattering Amplitudes of massless scalar field theories.**

From Combinatorial Polytope to Scattering forms

- Assign a sign to each vertex of the polytope $\mathcal{AC}_Q$, starting with any given vertex such that two adjacent vertices have opposite signs.
- Define a $\dim(\mathcal{AC}_Q)$ form on the planar kinematic space as,

$$\Omega_Q = \sum_{I=1}^{|\text{vertices}|} (-1)^v \wedge d \ln X_{ij} \tag{1}$$

- For any Q, the scattering form is projective : Invariant under $X_{ij} \rightarrow f(\{X\})\, X_{ij}$.
- If Q is a triangulation, there is a unique (projective) scattering form, as there is a unique polytope (Associahedron) (ABHY)

Planar Scattering forms from Accordiohedra

- Consider quadrangulation of a hexagon (6 pt. Scattering with ϕ^4 interactions.)
- Choose Reference as either Q as (14), 25 or 36

$$\begin{aligned} \Omega_{Q=(14)} &= d\ln X_{14} - d\ln X_{36} \\ \Omega_{Q=(25)} &= d\ln X_{25} - d\ln X_{14} \\ \Omega_{Q=(36)} &= d\ln X_{36} - d\ln X_{25} \end{aligned} \quad (2)$$

- So we have three distinct projective forms , one for each Q

From Combinatorial to convex Polytope

- We can use the path algebra generated by the dissection quiver to give convex realisations of any Accordiohedron in Kinematic space (Padrol, Palu, Pilaud, Plamondon)
- Quotient the path algebra by an ideal generated by paths of length two which belong to the same cell : The resulting algebra is known as a Gentle algebra.
- For any Q, this algebra can be used to define a set of constraints which locate the polytope $\mathcal{AC}_Q$ in the positive region of Kinematic space. (Padrol, Palu, Pilaud, Plamondon)

$$s_{ij} = (p_i + p_j)^2 = X_{ij} + X_{i+1,j+1} - X_{ij+1} - X_{i+1j} \tag{3}$$

- If T is a triangulation and if T^c is a triangulation obtained by rotating T counter-clockwise, then the constraints are simply

$$s_{ij} = -c_{ij} \, \forall \, (ij) \notin T^c, \; |i-j| \geq 2 \tag{4}$$

- Intuitively, these constraints freeze all the Mandelstam variables which will never occur as poles in a ϕ^3 amplitude.
- If Q is a quandrangulation (or any other dissection) contained in T then the convex polytope is realised in the (positive) region of Kinematic space simply by adjoining to 4, a following set of constraints

$$X_{ij} = d_{ij} \, \forall \, (ij) \, \mathrm{in} \, T/Q. \tag{5}$$

- Intuitively, the extra constraints freeze all the propagators in the cubic channels which will not occur ϕ^4 channels.

Some nice properties

- For every triangulation of an n-gon, we get distinct convex realisations of the Associahedra which contain the ABHY Associahedra (A family of realisations discovered by Arkani-Hamed, Bai, He, Yan , as well as Bazier-Matte et al. These realisations come from type-A cluster algebra as opposed to gentle algebras.)
- If Q is a quadrangulation, then the convex realisation is a polytope which (modulo some technical conditions) can always be embedded inside one of the Gentle Associahedra $\mathcal{A}_T$ as long as $Q \subset T$. (Aneesh et al)
- The Last property shows why Associahedra are the "master polytopes" in kinematic space.

- we can now try to answer both the questions asked at the beginning

From Scattering forms to Scattering Amplitudes

- Consider a specific example of 6 point scattering with ϕ^4 interactions. There are three possible quadrangulations of a hexagon labelled by $Q = \{(14)\,(25)\,(36)\}$.
- For each choice of Q, the gentle algebra generated by the dissection quiver, gives one constraint

$$
\begin{aligned}
C(14) &: X_{14} + X_{36} = c_1 \\
C(25) &: X_{25} + X_{14} = c_2 \\
C(36) &: X_{36} + X_{25} = c_3
\end{aligned}
$$

(All c_is are positive)

- Each Constraint gives a realisation of $\mathcal{AC}_Q$ in the positive region of Kinematic space (in fact inside an Associahedron)
- $\Omega_Q|_{\mathcal{AC}(Q)}$ is the canonical top form on $\mathcal{AC}_Q$ which has simple poles on all the boundaries.

From Scattering forms to Amplitudes

- But we can even forget about the polytope $\mathcal{AC}_Q$ and work only with Associahedra. Consider a (convexly realised) Associahedron $\mathcal{A}_T$ obtained from $T = (13, 14, 15)$. If we restrict Ω_{14} onto this Associahedron then the resulting form is,

$$\Omega_{(14)}|_{\mathcal{A}_T} = (\frac{1}{X_{14}} + \frac{1}{X_{36}})dX_{14} =: m_{(14)}dX_{14} = \Omega_{(14)}|_{\mathcal{AC}_{(14)}} \tag{6}$$

- We then immediately see that a weighted sum over such lower forms produce the complete (planar) amplitude.

$$\sum_Q \alpha_Q \, m_Q = \mathcal{M}_6$$

where $\alpha_Q = \frac{1}{2} \, \forall \, Q$.

General Result for n point Amplitude

- Take the form Ω_Q^n and project it onto any Associahedron $\mathcal{A}_T$ such that $T \subset Q$.

$$\Omega_Q|_{\mathcal{A}_T} = m_Q^n \wedge_{(ij)\in Q} dX_{ij}$$

- We thus see that a weighted sum over all such lower forms on Associahedra produce the complete Amplitude.

$$\mathcal{M}_n = \sum_Q \alpha_Q \, m_Q^n$$

n-point Amplitudes

- We thus see that the complete Amplitude is not determined by a single Polytope but by summing over all the polytopes of a given dimensions.
- But the sum depends on pesky weight factors.
- The Weights are determined from the combinatorics of the Polytope and use a Striking property of all such Accordiohedras, namely Facet of any Accordiohedra is a product of lower dimensional Accordiohedra of the same type. (Prashanth Raman, Ryota Kojima)

Main reults

- For any ϕ^p interaction, there is a set of projective planar scattering forms in the Kinematic space.
- for $p = 3$, this form is unique but for all $p > 3$ there is a whole set of them parametrized by dissections Q.
- These forms when restricted to the Associahedra produce lower forms which have singularities associated to poles in the Feynman diagrams with ϕ^p vertices.
- Each such form is a canonical top form on a convex polytope called Accordiohedra which can be realised inside a Associahedron in Kinematic space.
- Locality and Unitarity emerge due to the remarkable property of all Accordiohedra that all their faces belong to the same family of polytopes.

Towards 1-loop integrand for quartic interactions

Work in Progress with Mrunmay Jagadale : 2007.xxxxx

Cluster Polytopes for ϕ^3 integrand.

- In a recent work, Arkani-Hamed, He, Salvatori and Thomas extended the "Associahedron program" to include 1-loop integrands for cubic interactions.
- The striking aspect of their work is not only a discovery of polytopes in a Kinematic space whose canonical form determines ϕ^3 integrand, but the fact that these polytopes are geometric realisations of type-D cluster algebras just as ABHY Associahedra are geomtric realisations of type-A Cluster algebras.

Questions

- Is there a generalisation of Accordiohedra which determine ϕ^4 1-loop integrands?
- Can one also understand these integrands as lower forms on type-D Cluster polytopes?

Dissection Quivers for type-D Polytopes

- Recall : Combinatorially, Associahedron is determined by triangulation of a polygon.
- It turns out that One "dissection model" that determines the (combinatorial) type-D polytope is ". centrally symmetric Pseudo-triangulations" of a polygon with Annulus. (Ceballos and Pilaud)
- A Pseudo-triangle is any (curved) polygon which has precisely three convex coerners

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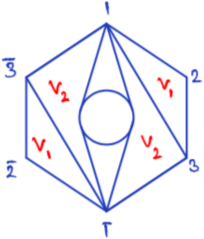


Kinematic space for 1-loop integrands

- All the dissections of a centrally symmetric Pseudo-triangulation form the kinematic space for 1-loop amplitude.
- For $n = 3$, Kinematic Space $= \{X_{13},\, X_{2\bar{1}},\, X_{3\bar{2}},\, Y_1,\, \tilde{Y}_1,\, \ldots\}$.
- Y_1 is an arc which emerges from 1 and touches the Annulus on the left and $\tilde{Y}_1$ is the one that touches on the right.

From X_{ij} to s_{ij}

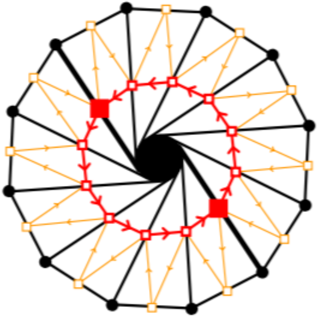
- For any two vertices i, j on the (doubled) polygon, we can define s_{ij} in terms of X_{ij} as in the case of Planar Kinematics space.

$$s_{i\overline{i+1}} = X_{i\overline{i+1}} + X_{i+1\overline{i+2}} - X_{i\overline{i+2}} - Y_i - \tilde{Y}_{i+1} \tag{7}$$

From Pseudo-triangulation to Convex Polytope

Various Realisations of the type-D polytope

- Consider a reference Pseudo triangulation and the corresponding dissection Quiver drawn below.



Convex realisations of type-D polytope from the Quiver

- We can again consider an algebra generated by the paths of the quiver which begin and end on an external edge.
- Precisely mirroring the Associahedron scenario, this algebra generates certain linear constraints.

$$s_{ij} = -c_{ij} \,\forall\, (ij) \notin PT^c$$

- **One caveat** : A single quiver leads to too many constraints ! But there is a canonical split into a optimum set of constraints (namely, $n^2 - n$ in number).
- Solving these constraints produce *various* realisations of the type-D polytope one of which is the realisation obtained by Arkani-Hamed, He, Salvatori and Thomas

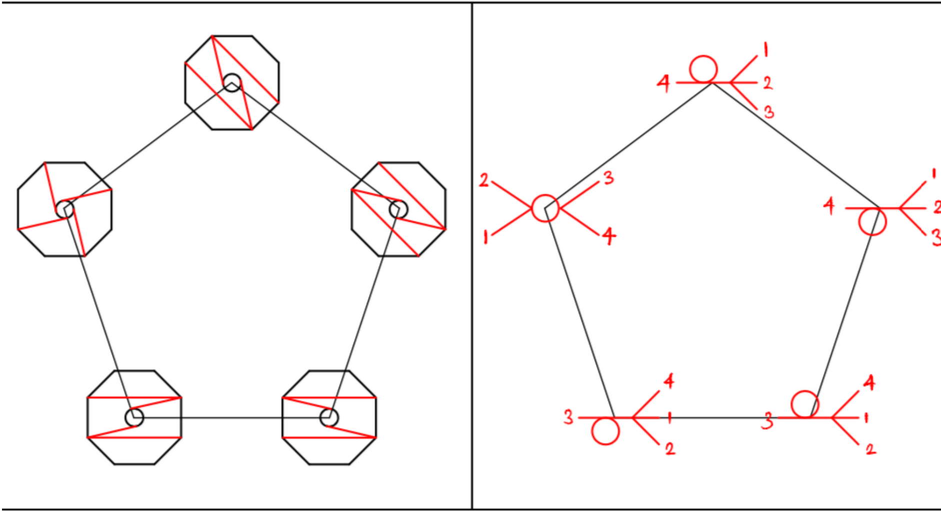
A Polytope for 1-loop ϕ^4 Amplitude

- We can now consider a centrally symmetric Pseudo-Quandrangulations.
- Proceeding exactly as in the case of Accordiohedra, we append to the constraints that produce convex realisation of type-D polytope,

$$X_{ij} = d_{ij} \, \forall \, (ij) \in \text{a PT}$$

- These constraints can be consistently solved and we obtain a convex polytope of dimension $\frac{n}{2}$.
- Vertices of this convex polytope are in 1-1 correspondence with a set of Feynman diagrams of ϕ^4 amplitude at 1-loop level.
- The polytope retains memory of reference Pseudo-Quadrangulation.

An example



A new class of polytopes?

- We directly spoke about the convexly realised polytope resulting from Pseudo-Quadrangulations as opposed to it's combinatorial source.
- The corresponding combinatorial polytope has not been investigated in literature and is an intriguing object for following reasons.

Associahedra : **type-A Cluster Algebra or Gentle Algebra of triangulation Quiver**

Accordiohedra : **Gentle algebra of other dissection quivers**

Type-D Cluster Polytope : **type-D Cluster algebra**

Polytopes associated to Pseudo-quadrangulations.

Are they realisations of any Algebra?

Conclusions

- Although we focused on quartic interactions at 1-loop, we believe that everything should generalise to other monomial (and perhaps even Polynomial) interactions.
- For ϕ^4 interactions, we have shown that the language of dissection quivers, Accordiohedra and Gentle Algebras offers a unifying framework to define scattering forms and analyse scattering amplitudes in terms of Polytopes.
- It will be interesting to understand the algebraic origins of the new polytopes we have found in the context of 1-loop quartic amplitudes.
- The tree level lower forms on Associahedra have interesting connection with lower forms on CHY Moduli space. (Can this connection be refined using Binary geometries for Accordiohedra? He, Li, Raman, and Zhang.)

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Thank you all for listening !