Note added: In my talk on 10 June, the information on our past and forthcoming papers was not presented. They are listed on the last page of this pdf file. Accordingly, I could not acknowledge one of my essential collaborators, **Toshifumi Noumi (Kobe University)**.
I apologize that these important information was missing in my talk.

$1/K$ in Open String Field Theory

Toru Masuda
CEICO, Czech Academy of Sciences, Prague

Workshop on Fundamental Aspects of String Theory
ICTP-SAIFR/IFT-UNESP, 10th June 2020, Video Conference.

25min talk +5 min for questions

$1/K$ in Witten's Open String Field Theory

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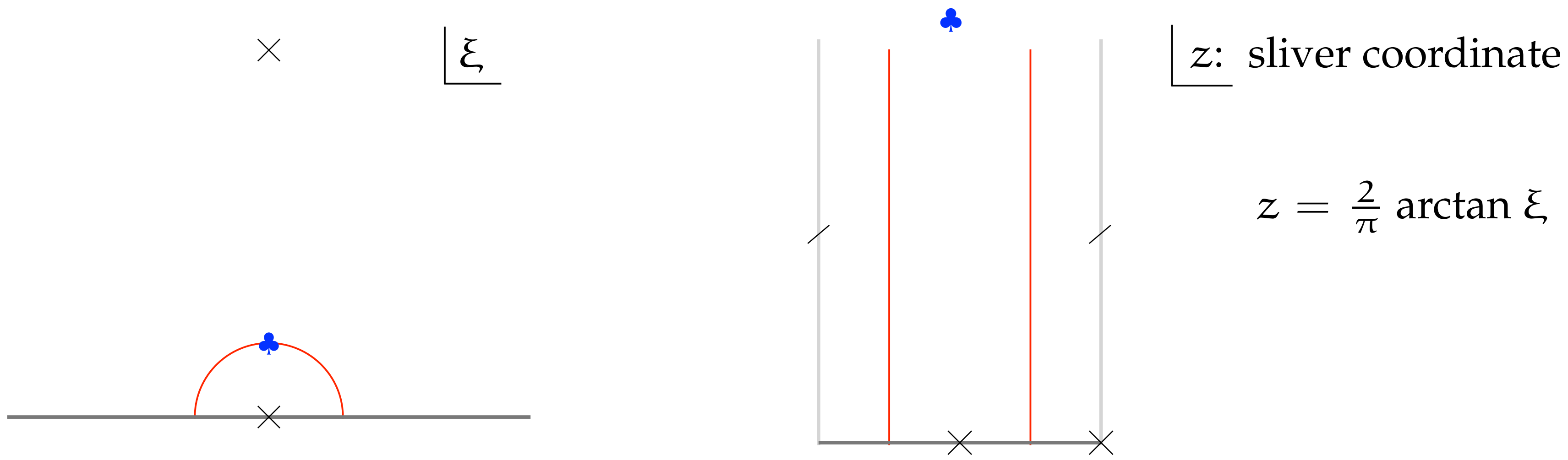
25min talk +5 min for questions

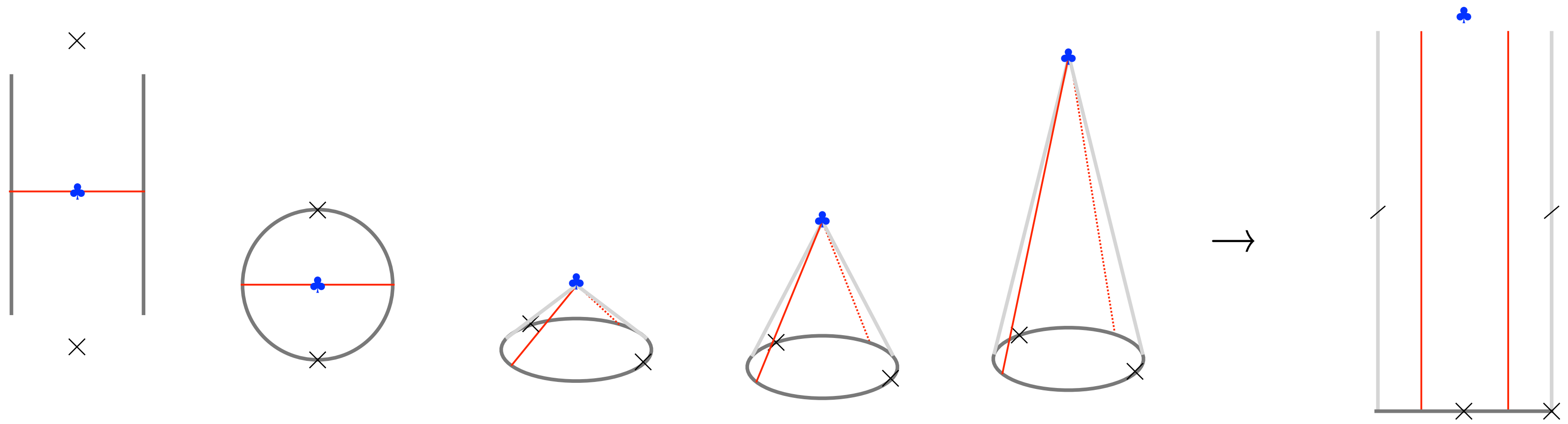
What is K ? : line integral of $\mathsf{T}(z)$ in the sliver coordinate



$$\mathsf{K} = \int_{+i\infty}^{-i\infty} \mathsf{T}(z) \frac{dz}{2\pi i}$$

Why interesting/important?

What is K :



a cone of infinite height

$$K = \int_{+i\infty}^{-i\infty} T(z) \frac{dz}{2\pi i}$$

Why interesting/important?

Because... 1/K may be related to

- Possible underlying geometry of (Witten's) Open SFT
- New Expression for Amplitudes

Outline

I. Overview (History)

II. S-matrix in tree level

~~III. 1/K and classical solution~~

I quit to take up this today.

Emergence of $(1/K)$

1. Classical solutions formally being pure-gauge
   (Okawa 2006; Ellwood 2009; Erler-Maccaferri 2012 ...)

2. Murata-Schnabl solution (Murata-Schnabl 2011);
   Winding number (Hata-Kojita 2012, ...): **Geometric interpretation**

3. Tree level S-matrix (2019);
   Unconventional propagator (2020)

(1) Classical solutions are pure gauge (formally)

If we use $1/K$, all the classical solutions can be written as

$$\Psi = U^{-1} Q U$$

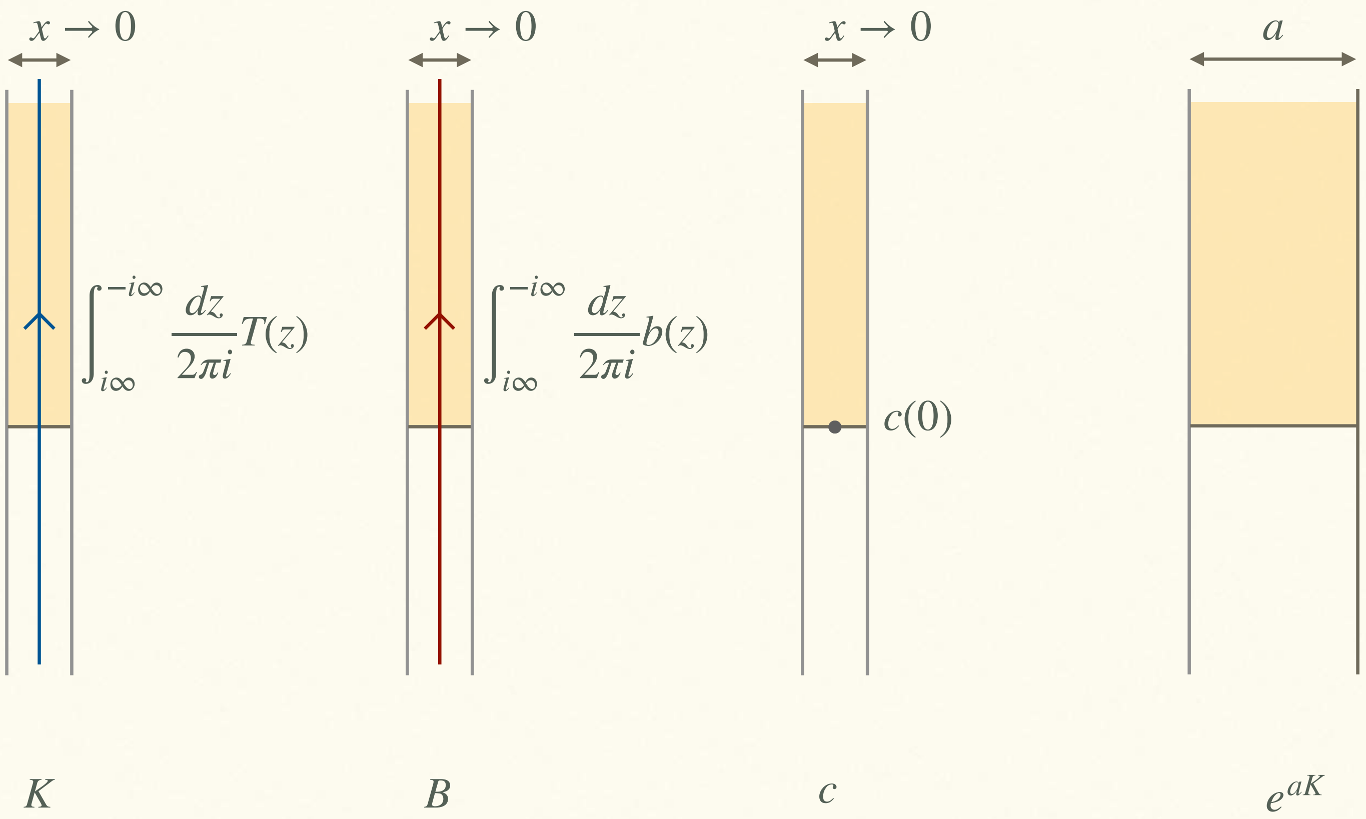
$1/K$ appears in U or U^{-1} unless Ψ is a pure-gauge solution.

In other words, $1/K$ is regarded as a singular object. This singularity symbolizes the non-triviality of Ψ.

Q. Why is $1/K$ considered to be singular ?

Because $1/K$ cannot be expressed as a superposition of wedge states e^{xK} $(0 < x < \infty)$. Usually $f(K)$ is defined by using it.

- K, B, c



$x \to 0$
$\int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} T(z)$
$x \to 0$
$\int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z)$
$x \to 0$
$c(0)$
a
K
B
c
e^{aK}

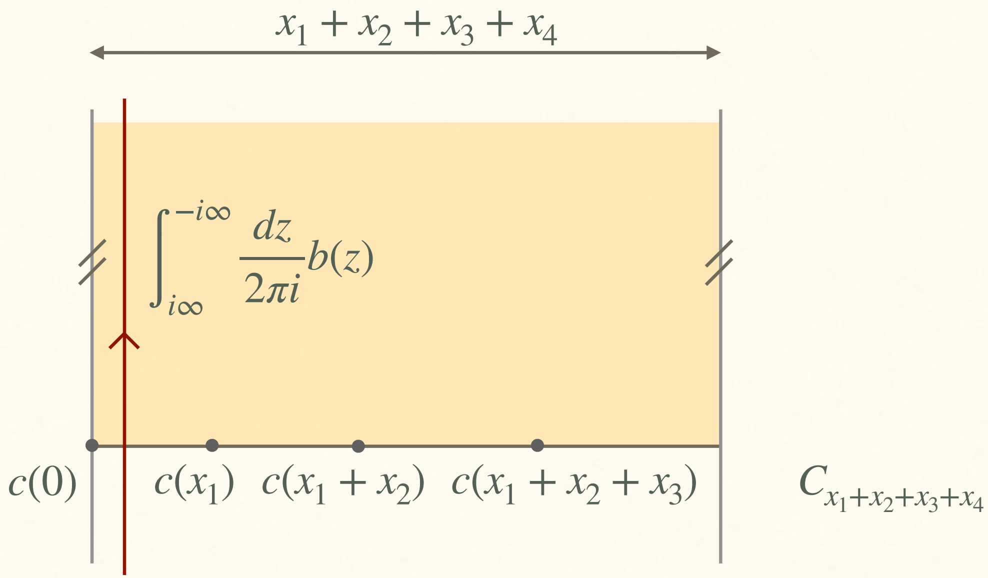
- KBc algebra

$$[K, B] = 0, \quad \{B, c\} = 1,$$

$$QB = K, \quad Qc = cKc$$

Relation with correlation functions on semi-infinite cylinders: an example

$$\int cBe^{x_1K}ce^{x_2K}ce^{x_3K}ce^{x_4K} = \int \frac{dz}{2\pi i} \left\langle c(0)b(z)c(x_1)c(x_1+x_2)c(x_1+x_2+x_3) \right\rangle_{C_{x_1+x_2+x_3+x_4}}$$



Note that: many of expressions with 1/K in this talk are **formal.**

(For example, "formally written as pure gauge" does not mean pure gauge.)

I hope that after a good understanding, we will be able to distinguish between correct and inappropriate expressions with 1/K. That is also the goal I would like to achieve.

1/K also appears in (formal) homotopy states, which represents (non-)emptiness of the physical excitations around Ψ

$$"Q_\Psi A_\Psi = 1"$$

$$(Q_\Psi \phi = Q\phi + \Psi * \phi + (-)^{|\phi|} \phi * \Psi)$$

- if such a A_Ψ exists, there are no open string excitation around Ψ
- if A_Ψ exists only formally, there are phisical open string excitation around Ψ

Example: for $\Psi = 0$,

$$"Q \frac{B}{K} = 1"$$

(2) Murata-Schnabl solution

Ansatz for classical solution for n-multiple of the initial D-brane. For $n = 2$,

$$\Psi_{\text{double}} = \frac{1}{K} c \frac{K^2 B}{K-1} c.$$

They are almost solution but not rigorously realized so far:

- desirable energy and the Ellwood invariant are obtained by using some regularization
- EOM is broken. (difficulty: the double limit from $1/K$s)
- $n > 2$?   (c.f. Hata 2019, Kojita 2019)

(3-1) A new formula for S matrix

On another note..

On the basis of gauge invariance etc., we (I and H.Matsunaga) find a formula which expresses S-matrix around Ψ by using Ψ, Ψ_T.

$$I_{\Psi}^{(N)} \equiv I_{\Psi}^{(N)}(\mathcal{O}_1, ..., \mathcal{O}_N; \Psi_T)$$

Typically, this expresses S-matrix by using $1/K$.

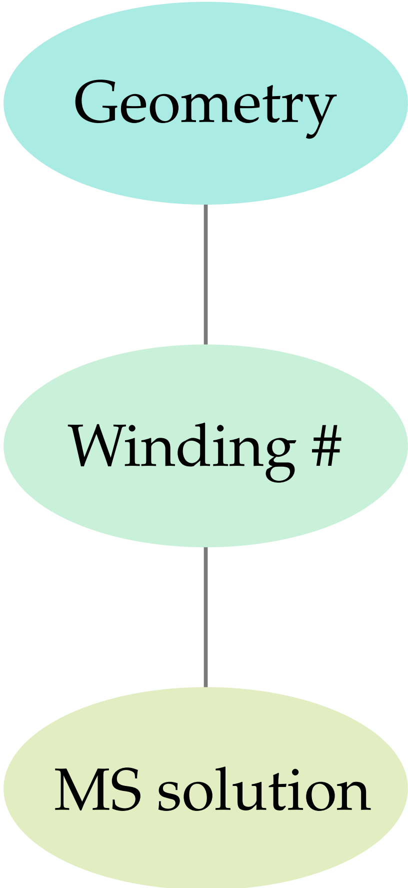
In the previous paper, we showed the calculation of $N = 4$ and proved that it agrees with the S matrix. (However, there was no proof for general N.) In this calculation, we seem to be dealing with $1/K$ well.

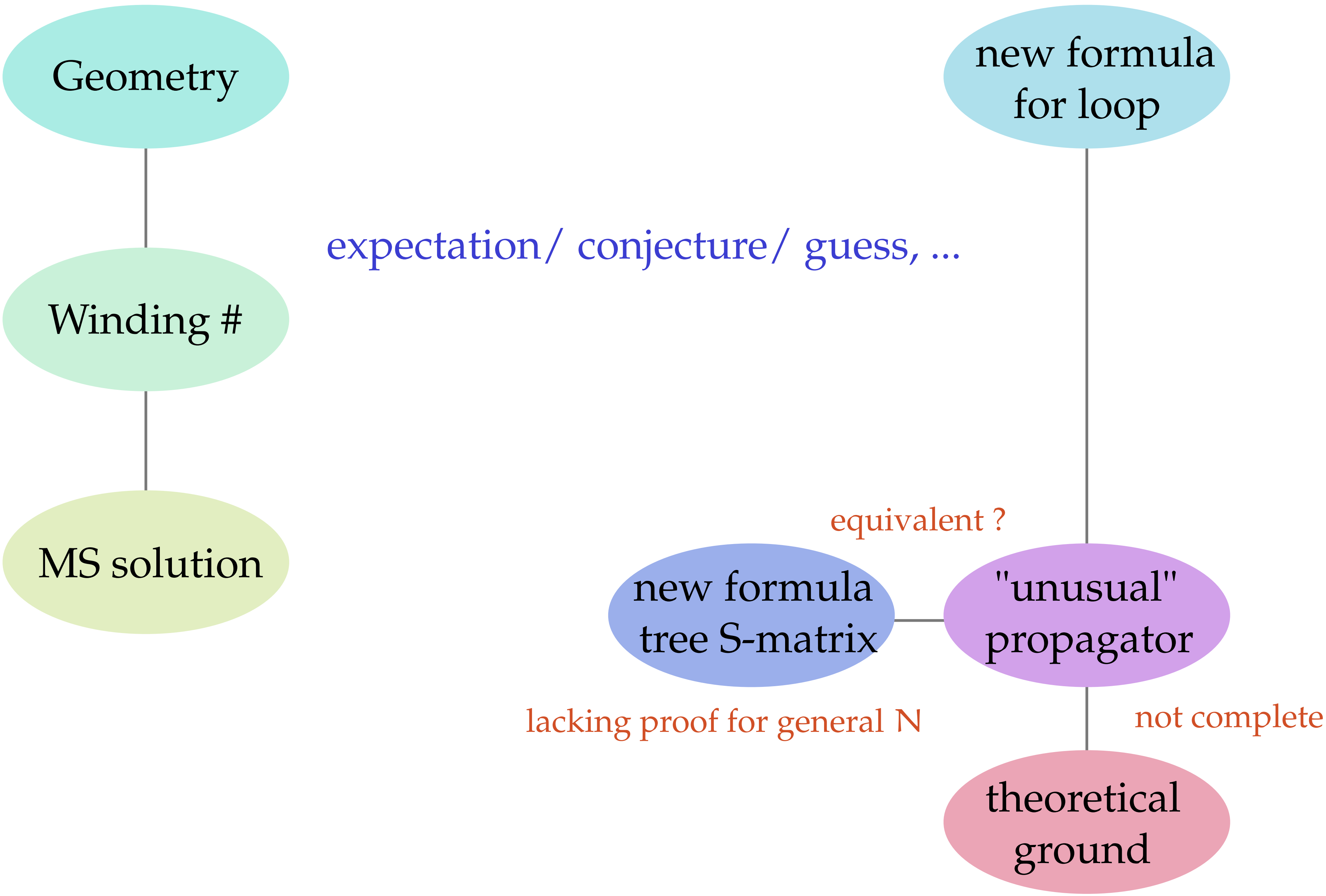
(3-2) Unconventional propagator

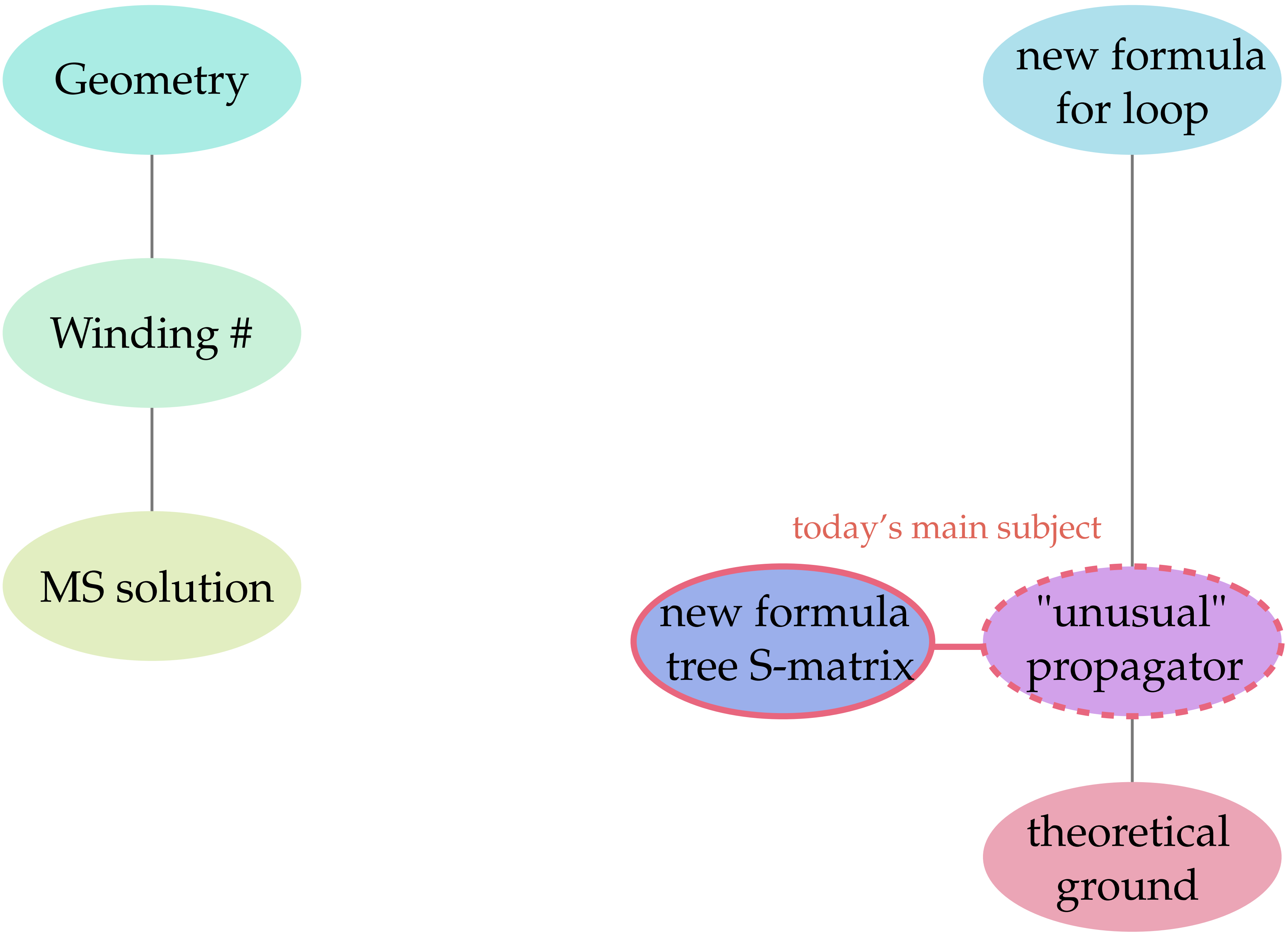
Later on, we proposed a gauge-fixing condition for a Feynman rule with a very "unconventional" propagator $\mathcal{P}_M$, which utilizes $1/K$.

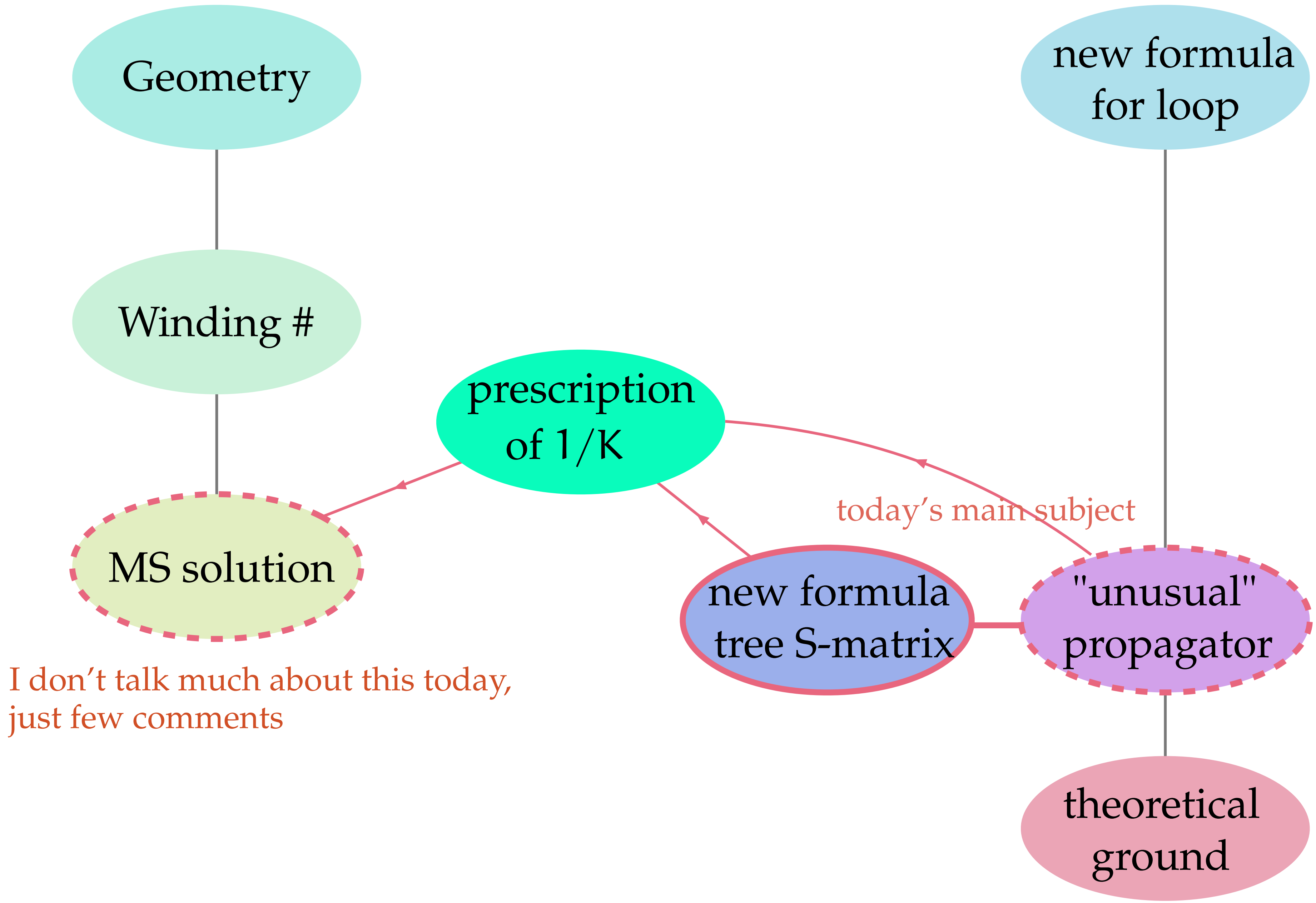
This $\mathcal{P}_M$ is derived by using Homological Perturbation Lemma, though a part of assumption ($\ker \mathcal{P}_M = \mathcal{H}_{\text{phys}}$) is not justified. (so its theoretical ground is not complete at present)

From this Feynman rule, we obtain an expression for S-matrices with B/K. Apparently, it looks different from $I_\Psi^{(N)}$ for $N \geq 5$, though the two are very similar. (Only the weight for each term is different.)


Geometry
Winding #
MS solution


Geometry
Winding #
MS solution
expectation/ conjecture/ guess, ...
new formula
for loop
equivalent ?
new formula
tree S-matrix
"unusual"
propagator
lacking proof for general N
not complete
theoretical
ground


Geometry
Winding #
MS solution
new formula
for loop
today's main subject
new formula
tree S-matrix
"unusual"
propagator
theoretical
ground


Geometry
new formula
for loop
Winding #
prescription
of 1/K
today’s main subject
MS solution
new formula
tree S-matrix
"unusual"
propagator
I don’t talk much about this today,
just few comments
theoretical
ground

2.1. S-matrix from clsscl solutions

We had obtained the following formula for S-matrix from gauge invariance etc.:

$$I_{\Psi}^{(N)} = -\frac{N}{N-3}\sum{}' \int \prod_{j=1}^{N} (A + W_{\Psi})\mathcal{O}_j,$$

where

$$W_{\Psi} = Q_{\Psi}A_T - 1 \quad (\to -e^K),$$

$$A = A_T - A_{\Psi} \quad (\to e^K \frac{B}{K}),$$

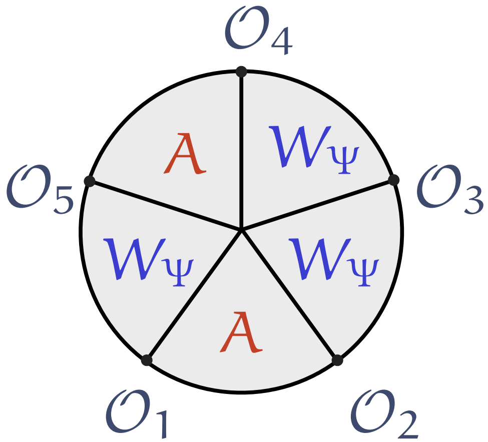
$$\mathcal{O}_j\text{: external states satisfying } Q_{\Psi}O_j = 0,$$

$$\sum{}' = \text{symmetrization over } \{j\}$$

Example: for $N = 5$,

$$
\begin{aligned}
I_{\Psi}^{(5)} = -\frac{5}{2}\Bigg(& \int W_{\Psi}\mathcal{O}_1 W_{\Psi}\mathcal{O}_2 W_{\Psi}\mathcal{O}_3 A\mathcal{O}_4 A\mathcal{O}_5 \\
& + \int W_{\Psi}\mathcal{O}_1 W_{\Psi}\mathcal{O}_2 A\mathcal{O}_3 W_{\Psi}\mathcal{O}_4 A\mathcal{O}_5 \\
& + \ldots \\
& + \int W_{\Psi}\mathcal{O}_2 W_{\Psi}\mathcal{O}_1 W_{\Psi}\mathcal{O}_3 A\mathcal{O}_4 A\mathcal{O}_5 \\
& + \ldots \Bigg)
\end{aligned}
$$

We call these terms using W_Ψ, A, and O_j "urchins"



From PIXABAY

$$\int W_\Psi \mathcal{O}_1 A \mathcal{O}_2 W_\Psi \mathcal{O}_3 W_\Psi \mathcal{O}_4 A \mathcal{O}_5$$

We proved

$$I_{\Psi}^{(N)} = \text{S-matrix}$$

by relating $I_{\Psi}^{(N)}$ with **the Feynman rules in the dressed $\mathcal{B}_0$ gauge.**

The dressed B_0 gauge is... (We will review from the next slide)

- a "singular" gauge condition in the sense that the propagator does not generate propagation of the open string midpoint.
- However, we can use it for the tree-level calculation.
- For a proper treatment of loop amplitudes, we need to consider it as a limit of a class of regular gauges, as discussed in Kiermaier-Sen-Zwiebach [arXiv:0712.0627]

Reminder: the dressed $\mathcal{B}_0$ gauge

Reference: Appendix C of Erler-Schnabl [arXiv: 0906.0979].
The gauge fixing condition is

$$\mathcal{B}_{F,G}\Psi = 0.$$

Here $\mathcal{B}_{F,G}$ is defined by

$$\mathcal{B}_{F,G} \bullet = \frac{1}{2} F(K) \mathcal{B}_0^- \left[F(K)^{-1} \bullet G(K)^{-1} \right] G(K)$$

with $\mathcal{B}_0^- = \mathcal{B}_0 - \mathcal{B}_0^\star$. Let us also introduce

$$\mathcal{L}_{F,G} \bullet = \frac{1}{2} F(K) \mathcal{L}_0^- \left[F(K)^{-1} \bullet G(K)^{-1} \right] G(K)$$

with $\mathcal{L}_0^- = \mathcal{L}_0 - \mathcal{L}_0^\star$.

Notably, $\mathcal{B}_0^-$ and $\mathcal{L}_0^-$ are derivatives (under star products) and mutually commuting. Their action on $\{K, B, c\}$ reads

$$\frac{1}{2}\mathcal{B}_0^- K = B\,, \quad \frac{1}{2}\mathcal{L}_0^- K = K\,,$$
$$\frac{1}{2}\mathcal{B}_0^- B = 0\,, \quad \frac{1}{2}\mathcal{L}_0^- B = B\,,$$
$$\frac{1}{2}\mathcal{B}_0^- c = 0\,, \quad \frac{1}{2}\mathcal{L}_0^- c = -c\,.$$

$\frac{1}{2}\mathcal{B}_0^-$ is trivial in the matter sector and $\frac{1}{2}\mathcal{L}_0^-$ simply counts the scaling dimension of the operator inserted to the wedge state of zero width.

For $F = G = e^{\frac{K}{2}}$,

$$\mathcal{B}_{F,G} = \mathcal{B}_0, \quad \mathcal{L}_{F,G} = \mathcal{L}_0$$

and this gauge condition reduces to the Schnabl gauge.

Propagator

We will use the following “simplified" propagator

$$\mathcal{P}_D \equiv \frac{\mathcal{B}_{F,G}}{\mathcal{L}_{F,G}} = \int_0^\infty ds e^{-s\mathcal{L}_{F,G}} \mathcal{B}_{F,G} \,.$$

Actually, this $\mathcal{P}$ is incomplete, because it violates the BPZ property ($\mathcal{P}^\star \neq \mathcal{P}$). The genuine propagator is given by

$$\mathcal{P}'_D = \mathcal{P}^\star_D Q \mathcal{P}_D \,.$$

Nevertheless, we cen use the simplified propagator to calculate on-shell, tree-level amplitudes,

$$A_N(\varphi_1, ..., \varphi_N) = A_N(\varphi_1, ..., \varphi_N)\big|_{\mathcal{P}'_D \to \mathcal{P}_D} \,.$$

(We confirmed this proposition.)

Physical states

Let us assume $F = G$. The physical states in this gauge is

$$\varphi_i = F(K)\mathcal{O}_i F(K)$$

where $\mathcal{O}_i = cV_i$ is an identity-based state at the ghost number 1, satisfying $Q\mathcal{O}_i = 0$.

In terms of W_Ψ, A, $\mathcal{O}_i$,

$$\varphi_i = \sqrt{-W_\Psi}\,\mathcal{O}_i\sqrt{-W_\Psi}$$

Notation for Feynman rules

Feynman diagrams are expressed by using the following three maps:

i. star product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

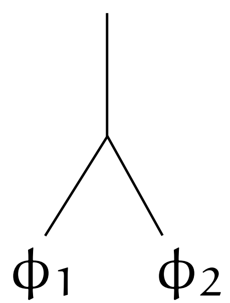
$$m(\phi_i, \phi_j) = \phi_i * \phi_j,$$

ii. propagator $\mathcal{P}_D$,

iii. inner product $I : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$

$$I(\phi_i, \phi_j) = \int \phi_i * \phi_j.$$

We also define $Y(\phi_i, \phi_j) = \mathcal{P}_D[m(\phi_i, \phi_j)]$ for notational simplicity.



Now, the formula to convert Feynman graphs and urchins is given by

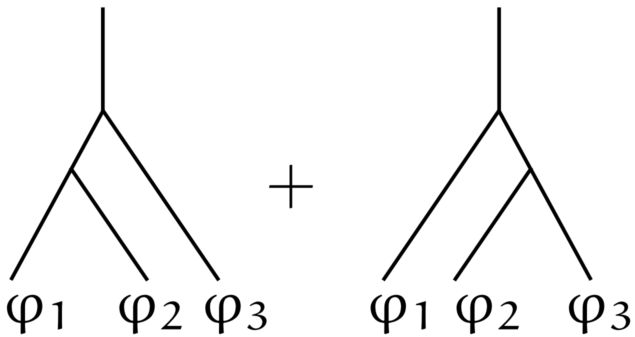
$$T_n(\varphi_1, ..., \varphi_n) = \sqrt{-W}\mathcal{O}_1 A ... \mathcal{O}_{n-1} A \mathcal{O}_n \sqrt{-W}.$$

where $T_n(\varphi_1, ..., \varphi_n)$ is recursively defined by

$$T_1(\varphi_1) = \varphi_1$$

$$T_n(\varphi_1, ..., \varphi_n) = \sum_{i=1}^{n-1} Y(T_i(\varphi_1, ..., \varphi_i), T_{n-i}(\varphi_{i+1}, ..., \varphi_n)).$$

For $n = 3$,



$$= \sqrt{W_\Psi}\mathcal{O}_1 A \mathcal{O}_2 A \mathcal{O}_3 \sqrt{W_\Psi}.$$

Derivation of this formula for $n = 2$,

$$\begin{aligned}
&\frac{\mathcal{B}_{F,F}}{\mathcal{L}_{F,F}} \left[(F(K)\mathcal{O}_k F(K))(F(K)\mathcal{O}_l F(K))\right] \\
&= \int_0^\infty ds F(K) e^{-s\frac{1}{2}\mathcal{L}_0^-} \frac{1}{2}\mathcal{B}_0^- \left[\mathcal{O}_k F(K)^2 \mathcal{O}_l\right] F(K) \\
&= -\int_0^\infty ds F(K) e^{-s\frac{1}{2}\mathcal{L}_0^-} \left[\mathcal{O}_k H'(K) B \mathcal{O}_l\right] F(K) \\
&= -\int_0^\infty ds F(K) \left[\mathcal{O}_k H'(K e^{-s}) B e^{-s} \mathcal{O}_l\right] F(K) \\
&= F(K) \left[\mathcal{O}_k H(e^{-s} K) \frac{B}{K} \mathcal{O}_l\right]_{s=0}^{s=\infty} F(K) \\
&= F(K)\mathcal{O}_k (1 - F(K)^2) \frac{B}{K} \mathcal{O}_l F(K)
\end{aligned}$$

where $H(K) = F(K)^2$. This equals $\sqrt{-W_\Psi}\mathcal{O}_k A_T \mathcal{O}_l \sqrt{-W_\Psi}$.

Closer look: To obtain the A_Ψ term, we need to use the regularized propagator. There are several options. Let us take..

$$\bar{\mathcal{P}}_D \equiv \left(\int_0^\Lambda e^{-s\mathcal{L}_{F,F}} + \int_\Lambda^{\Lambda+i\infty} e^{-s(\mathcal{L}_{F,F}-i\epsilon)} \right) \mathcal{B}_{F,F} \quad (\Lambda\epsilon \ll 1).$$

$$\begin{aligned}
&\int_\Lambda^{\Lambda+i\infty} e^{-s(\mathcal{L}_{F,F}-i\epsilon)} \mathcal{B}_{F,F} \left[(F\mathcal{O}_1 F)(F\mathcal{O}_2 F) \right] \\
&= F \left[\int_\Lambda^{\Lambda+i\infty} ds e^{-s(\frac{1}{2}\mathcal{L}_0^- - i\epsilon)} \frac{1}{2} \mathcal{B}_0^- (\mathcal{O}_1 H \mathcal{O}_2) \right] F \\
&= F \left[\int_\Lambda^{\Lambda+i\infty} ds e^{i\epsilon s} e^{-s} \mathcal{O}_1 B H'(e^{-s}K) \mathcal{O}_2 \right] F \\
&= F\mathcal{O}_1 \left[\frac{1}{K} H(e^{-\Lambda}K) \right] B\mathcal{O}_2 F + O(\epsilon) \,,
\end{aligned}$$

Also in the last equality we used

$$\begin{aligned}
&\int_{\Lambda}^{\Lambda+i\infty} ds e^{i\epsilon s} e^{-s} H'(e^{-s}K) \\
&= \int_{e^{-(\Lambda+i\infty)}}^{e^{-\Lambda}} dx x^{-i\epsilon} H'(xK) \\
&= \left[x^{-i\epsilon} \frac{1}{K} H(xK) \right]_{e^{-(\Lambda+i\infty)}}^{e^{-\Lambda}} + i\epsilon \int_{e^{-(\Lambda+i\infty)}}^{e^{-\Lambda}} dx x^{-1-i\epsilon} \frac{1}{K} H(xK) \\
&= \frac{1}{K} H(e^{-\Lambda}K) + O(\epsilon) \, ,
\end{aligned}$$

where we used $\lambda^{\epsilon} = e^{-\epsilon\Lambda} \simeq 1$ and assumed that the integral in the third line does not give any $1/\epsilon$ singularity.

Note: In our previous work, we observed that A_Ψ $(= B/K)$ works as "a boundary term" which removes world-sheet UV divergence (c.f. Sen [arXiv:1902.00263 [hep-th]]).

Now, we understand that A_Ψ corresponds to a regularization term for $\mathcal{P}_D$ in the dressed $\mathcal{B}_0$ gauge.

We are halfway down the road to the proof; by a combinatorial argument, we can confirm "$I_\Psi^{(N)} =$ S-matrix in tree level".

(But I would like to omit this part because it is too technical.)

Note: from relation to Feynman diagrams, we find that the urchins satisfy the following relation, which is very important in our proof:

$$\sum_{i=1}^{y-1}[x, i, y-i] = \sum_{j=1}^{x-1}[y, j, x-j].$$

Here $[l, m, n]$ is the partial sum of the urchins, given by

$$\sum \int (A\mathcal{O})_{1}^{l-1} W_{\Psi}\mathcal{O}_{l} \, (A\mathcal{O})_{l+1}^{l+m-1} W_{\Psi}\mathcal{O}_{l+m} \, (A\mathcal{O})_{l+m+1}^{l+m+n-1} W_{\Psi}\mathcal{O}_{l+m+n}$$

where

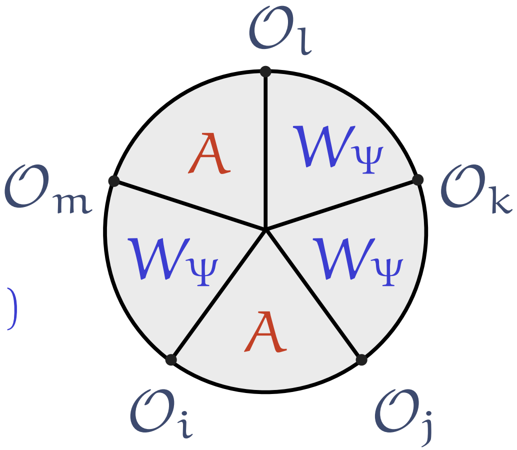
$$(A\mathcal{O})_{p}^{q} = A\mathcal{O}_{p} A\mathcal{O}_{p+1} A\mathcal{O}_{p+2} ... A\mathcal{O}_{q}.$$

This satisfies

$$[l, m, n] = [m, n, l]$$

Example of our notation [l, m, n]

$$\sum_{(i,j,k,l,m)}$$



From PIXABAY

$$= \sum_{(i,j,k,l,m)} \int W_\Psi \mathcal{O}_i A \mathcal{O}_j W_\Psi \mathcal{O}_k W_\Psi \mathcal{O}_l A \mathcal{O}_m = [2, 1, 2]$$

2.2. "Unconventional" propagator

There is another way to obtain a similar expression for the S-matrix; a Feynman rule with what we call "the tachyon vacuum's A_T gauge" and the following "propagator," $\mathcal{P}_M$[*1]

$$\mathcal{P}_M \phi = \frac{1}{2W_\Psi} A * \phi + \phi * A \frac{1}{2W_\Psi}.$$

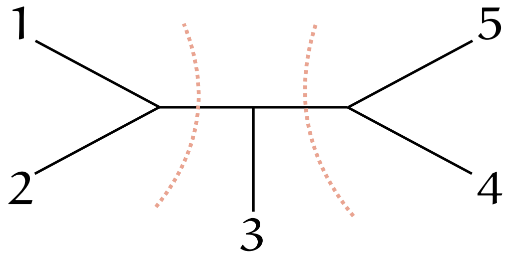
I was sceptical when my collaborator proposed this based on HPL, but..

This Feynman rule gives a correct result at least for tree S-matrices. The result is sum of urchins, but their weight is different from $I_\Psi^{(N)}$. By a combinatorial discussion, we proved agreement with $I_\Psi^{(N)}$.

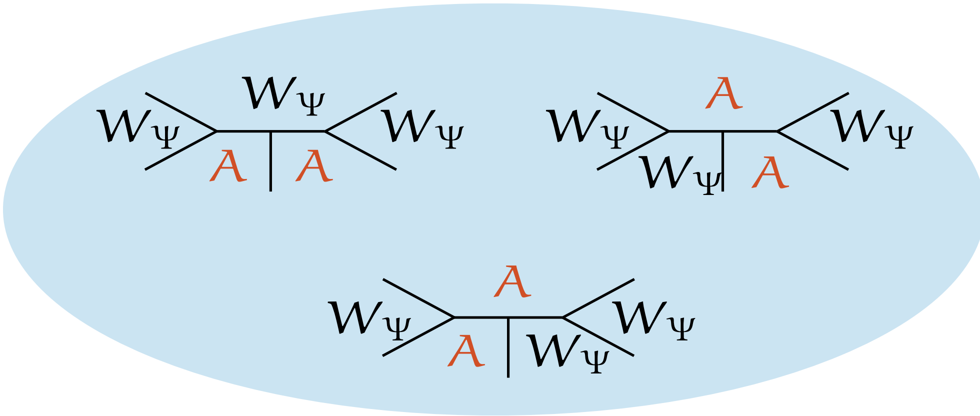
[*1] To be precise, we used the expression $A_T/(1+W_\Psi)$ instead of A_Ψ.

Example: A Feynman graph for a 5 point amplitude

External states: $\phi_j = \sqrt{W_\Psi}\mathcal{O}_j\sqrt{W_\Psi}$

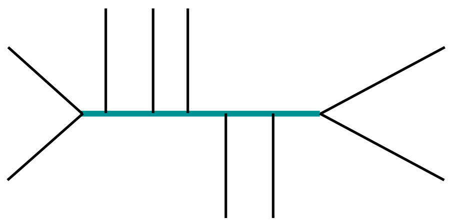


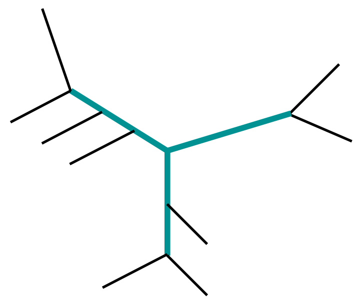
$\rightarrow$



$$
= \frac{1}{4}\Bigg(\int \mathcal{O}_1 W \mathcal{O}_2 A \mathcal{O}_3 A \mathcal{O}_4 W \mathcal{O}_5 W + \int \mathcal{O}_1 W \mathcal{O}_2 W \mathcal{O}_3 A \mathcal{O}_4 W \mathcal{O}_5 A \\
+ \int \mathcal{O}_1 W \mathcal{O}_2 A \mathcal{O}_3 W \mathcal{O}_4 W \mathcal{O}_5 A \Bigg)
$$

A notable feature of this Feynman rule is that most of the Feynman diagrams vanish except for those of the following two types:





I-type: After removing all the external lines, the resulting subgraph is "I-shape"

Y-type: After removing all the external lines, the resulting subgraph is "Y-shape"

The contribution from the Feynman diagrams of I-type is

$$\left(\frac{1}{2}\right)^{N-3} \sum_{p=0}^{N-4} \binom{N-4}{p} [N-p-2, p+2]$$

while the contribution from the Feynman diagrams of Y-type is

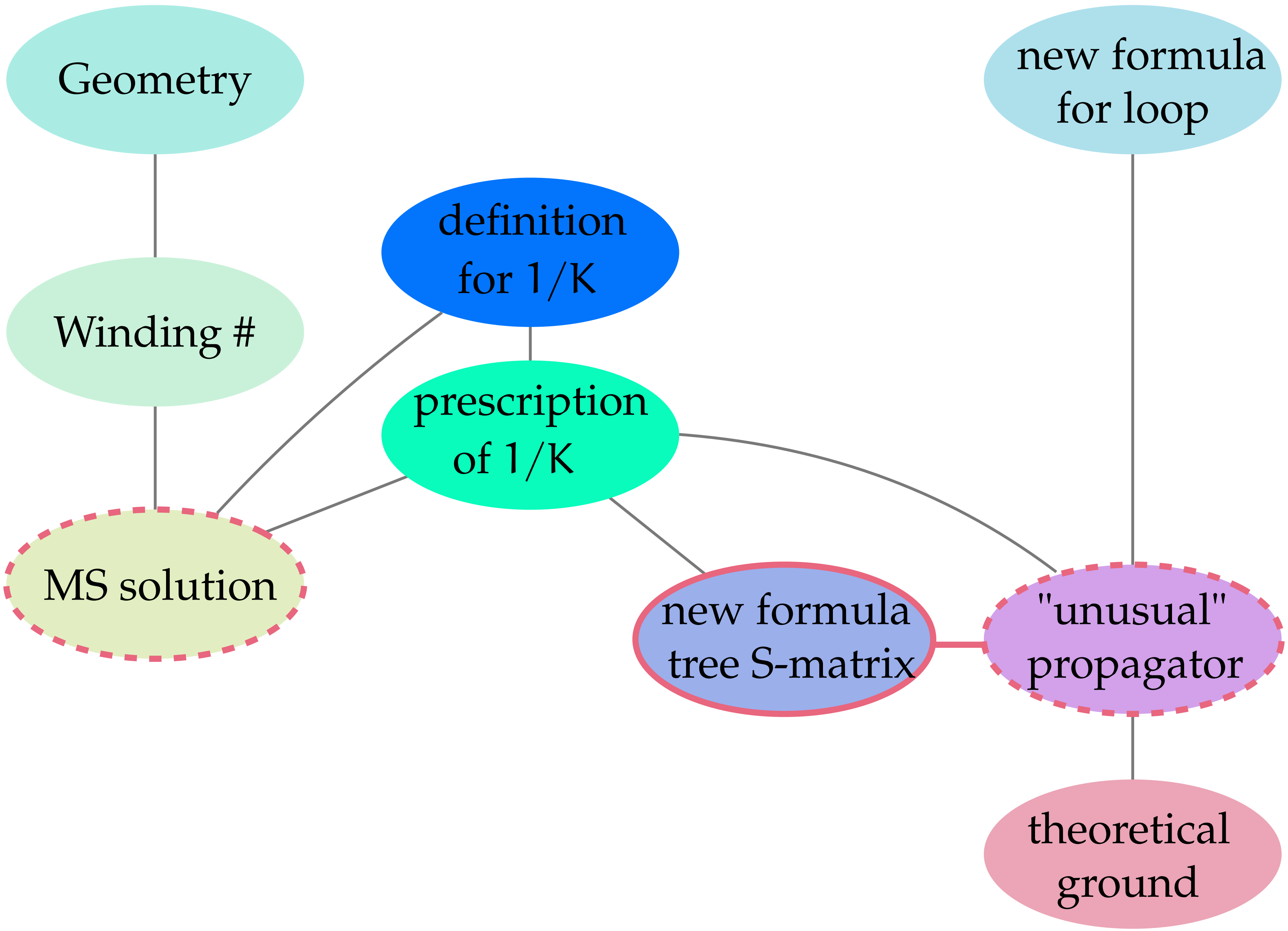
$$\frac{1}{3}\left(\frac{1}{2}\right)^{N-3} \sum_{p+q+r=N-6} 2f(p,q,r)[p+2, q+2, r+2]$$

where

$$f(p,q,r) = \sum_{p_1=0}^{p} \sum_{q_1=0}^{q} \sum_{r_1=0}^{r} \binom{p_1+r-r_1}{p_1} \binom{q_1+p-p_1}{q_1} \binom{r_1+q-q_1}{r_1}.$$

We can prove that the sum of these expressions equals $I_{\Psi}^{(N)}$.

Concluding remarks



Main references for this talk:

- 1908.09784 [hep-th], w. H. Matsunaga (Charles Univ., Czech A.S.)
  ... presentation of $I_{\Psi}^{(N)}$

- 2003.05021 [hep-th], w. H. Matsunaga:
  ... presentation of the "unconventional propagator"

- to appear soon, w. H. Matsunaga, and T. Noumi (Kobe Univ.):
  ... relation to Feynman diagrams in the dressed $\mathcal{B}_0$ gauge;
  combinatorial proof of
  - "$I_{\Psi}^{(N)} =$ S-matrix in tree leel,"
  - "S-matrix from the unconventional propagator $= I_{\Psi}^{(N)}$"