The pure spinor $b$ ghost in curved backgrounds

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Plan of the talk

• Review of the pure spinor string in a flat background
• Pure spinor string in a curved background
• Non-minimal variables in a curved background
• Construction of the $b$ ghost in the heterotic string in a curved background
• Final remarks
Review of pure spinor in a flat background

(Berkovits, hep-th/0001035)

Given the superspace coordinates in ten dimensions \((X^m, \theta^\alpha)\), their momenta and a pair of conjugate pure spinor variables.

\[
S = \int d^2z \frac{1}{2} \partial X^m \bar{\partial} X^m + p_\alpha \bar{\partial} \theta^\alpha + \omega_\alpha \bar{\partial} \lambda^\alpha,
\]

\[
Q = \oint \lambda^\alpha \left( p_\alpha - \frac{1}{2} (\gamma_m \theta)_\alpha \partial X^m - \frac{1}{8} (\gamma_m \theta)_\alpha (\theta \gamma^m \partial \theta) \right) \equiv \oint \lambda^\alpha d_\alpha.
\]

\(Q\) is nilpotent because the pure spinor constraint \(\lambda \gamma^m \lambda = 0\)

\[
Q^2 = \oint (\lambda \gamma_m \lambda) \Pi^m, \quad \left( \Pi^m = \partial X^m + \frac{1}{2} (\theta \gamma^m \partial \theta) \right)
\]

⇒ \(Q\) is declared as the BRST charge of the theory.

Note that \(\omega_\alpha\) is defined up to \((\lambda \gamma_m)_\alpha \Lambda^m\).
Review of pure spinor in a flat background

It reproduces the superstring physical spectrum through the cohomology of $Q$.

$$Q(\lambda^\alpha A_\alpha(X, \theta)) = 0, \quad \lambda^\alpha A_\alpha(X, \theta) \sim \lambda^\alpha A_\alpha(X, \theta) + \lambda^\alpha D_\alpha \Omega.$$  

For example,

$$A_\alpha(X, \theta) = (\gamma^m \theta)_\alpha a_m(X) + (\theta \gamma^{mnp} \theta)(\gamma_{mnp} \psi(X))_\alpha + \cdots ,$$

$a_m$ describes a photon and $\psi$ a photino.

This model is conformal, spacetime supersymmetric and BRST invariant.

This is the minimal pure spinor string in a flat background.
Review of pure spinor in a flat background

The stress-energy tensor of the traditional strings is trivial because the existence of the reparametrization ghost \( b \). That is, \( Qb = T \).

For the pure spinor string, the stress-energy tensor

\[
T = -\frac{1}{2} \Pi_m \Pi^m - d_\alpha \partial \theta^\alpha - \omega_\alpha \partial \lambda^\alpha,
\]

is annihilated by \( Q \) and the \( b \) ghost is given with the help of new variables. They are the conjugate pairs

\[
(\hat{\omega}_\alpha^\alpha, \hat{\lambda}_\alpha), \quad (s^\alpha, r^\alpha),
\]

and are the non-minimal variables (they are constrained). They contribute to the BRST charge with \( \int \hat{\omega}_\alpha^\alpha r_\alpha \) and

\[
Q\hat{\lambda}_\alpha = -r_\alpha, \quad Qs^\alpha = \hat{\omega}_\alpha^\alpha, \quad Qr_\alpha = Q\hat{\omega}_\alpha^\alpha = 0.
\]

They do not change the cohomology of the BRST charge (Berkovits, hep-th/0509120).
Review of pure spinor in a flat background

The non-minimal pure spinor has action

\[ S = \int d^2z \left( \frac{1}{2} \partial X_m \bar{\partial} X^m + p_\alpha \bar{\partial} \theta^\alpha + \omega_\alpha \bar{\partial} \lambda^\alpha + Q \left( \int d^2z \ s^\alpha \bar{\partial} \hat{\lambda}_\alpha \right) \right) \]

The BRST charge is

\[ Q = \oint \lambda^\alpha d_\alpha + \hat{\omega}^\alpha r_\alpha, \]

and the complete stress-energy tensor is

\[ T = -\frac{1}{2} \Pi_m \Pi^m - d_\alpha \partial \theta^\alpha - \omega_\alpha \partial \lambda^\alpha - Q(s^\alpha \partial \hat{\lambda}_\alpha). \]

The \( b \) ghost is constructed to satisfy \( Qb = T \).
Review of pure spinor in a flat background

The $b$ ghost satisfying $Qb = T$ is

$$b = -s^\alpha \partial \hat{\lambda}_\alpha - \omega_\alpha \partial \theta^\alpha + \prod_m \bar{\Gamma}^m - \frac{1}{4(\lambda \lambda)} (\lambda \gamma_{mn} r) \bar{\Gamma}^m \bar{\Gamma}^n$$

$$+ \frac{1}{2(\lambda \lambda)} (\omega \gamma_m \hat{\lambda}) (\lambda \gamma^m \partial \theta),$$

where

$$\bar{\Gamma}^m = \frac{1}{2(\lambda \lambda)} (d \gamma^m \hat{\lambda}) + \frac{1}{8(\lambda \lambda)^2} (r \gamma^{mnp} \hat{\lambda}) \frac{1}{2} (\lambda \gamma_{np} \omega).$$

$Q\bar{\Gamma}^m$ is such that $Qb = T$.

Note that $b$ is invariant under $\delta \omega_\alpha = (\lambda \gamma^m)_\alpha \Lambda_m$.

The idea is to do this construction in a curved background.
Pure spinor string in a curved background

Given the curved superspace coordinates $Z^M$, the variable $d_\alpha$ and the pure spinor variables. The world-sheet action in curved background is

$$S = \int d^2z \left\{ \frac{1}{2} \partial Z^M \bar{\partial} Z^N (G_{NM} + B_{NM}) - \bar{\partial} Z^M E_M^\alpha d_\alpha + \omega_\alpha \bar{\nabla} \lambda^\alpha \\
+ \xi D_\xi + \alpha' r^{(2)} \Phi , \right\},$$

where

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + \lambda^\beta \bar{\partial} Z^M \Omega_{M\beta}^\alpha ,$$

$$\Omega_{M\beta}^\alpha = \delta_\beta^\alpha \Omega^M + \frac{1}{4} (\gamma^{ab})_\beta^\alpha \Omega^{Mab} .$$

The background superfields are the supervielbein $E$, the superconnection $\Omega$ and the NSNS two-form $B$. They will be constrained by the BRST invariance.
The BRST charge is $Q = \oint d\sigma \lambda^\alpha d_{\alpha}$. $Q^2 = 0$ and $\bar{\partial}(\lambda^\alpha d_{\alpha}) = 0$ put the background to satisfy constraints (Berkovits & Howe, hep-th/0112160).

$$Q^2 = 0 \Rightarrow \lambda^\alpha \lambda^\beta T_{\alpha\beta}^A = 0, \quad \lambda^\alpha \lambda^\beta H_{\alpha\beta a} = 0, \quad \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma\delta} = 0.$$

Some non-vanishing components of the background superfields are

$$T_{\alpha\beta a} = H_{\alpha\beta a} = -(\gamma_a)_{\alpha\beta}, \quad T_{\alpha ab} = 2(\gamma_{ab})_{\alpha}^\beta \Omega_{\beta}, \quad \Omega_{\alpha} = \frac{1}{4} \nabla_{\alpha} \Phi$$

This model is one-loop conformal invariant (Chandía, Vallilo, hep-th/0401226).
Pure spinor string in a curved background

Define the one-forms $E^A = dZ^M E_M{}^A$ and $\Omega^B{}_A = dZ^M \Omega_{MA}{}^B$, here $A = (a, \alpha)$. The torsion two-form is

$$T^A = \nabla E^A = dE^A + E^B \Omega^A{}_B = \frac{1}{2} E^B E^C T_{CB}{}^A,$$

the curvature two-form is

$$R^A{}_B = d\Omega^A{}_B + \Omega^C{}_B \Omega^A{}_C = \frac{1}{2} E^C E^D R_{DCB}{}^A$$

and

$$H = dB = \frac{1}{6} E^C E^B E^A H_{ABC}.$$

The Bianchi identities are

$$\nabla T^A = T^B R^A{}_B, \quad \nabla R^A{}_B = 0, \quad dH = 0.$$

The covariant derivative on a super p-form $\Psi^A{}_B$ is

$$\nabla \Psi^A{}_B = d \Psi^A{}_B + \Psi^C{}_A \Omega^A{}_C + (-1)^{p+1} \Omega^A{}_C \Psi^C{}_B.$$
Pure spinor string in a curved background

The transformation of the variables under $Q$
(Chandía, hep-th/0604115)

$$QZ^M = \lambda^\alpha E_\alpha^M, \quad Q\lambda^\alpha = -\lambda^\beta \Sigma_\beta^\alpha, \quad Q\omega_\alpha = d_\alpha + \Sigma_\alpha^\beta \omega_\beta,$$

$$Qd_\alpha = - (\lambda \gamma_a)_\alpha \Pi^a + \lambda^\beta \lambda^\gamma \omega_\delta R_{\alpha\beta\gamma\delta} + \Sigma_\alpha^\beta d_\beta,$$

where $\Pi^A = \partial Z^M E_M^A$ and $\Sigma_\alpha^\beta = \lambda^\gamma \Omega_{\gamma\alpha}^\beta$ gives a local Lorentz rotation. As a check

$$Q\Pi^\alpha = \nabla \lambda^\alpha + \Pi^\beta \Sigma_\beta^\alpha, \quad Q\Pi^a = (\lambda \gamma^a \Pi) + \lambda^\alpha \Pi^b T_{\alpha b}^a - \Pi^b \Sigma_b^a,$$

where $\Sigma_b^a = \lambda^\gamma \Omega_{\gamma a}^b$.

The combination $(\lambda \hat{\lambda}) = \lambda^\alpha \hat{\lambda}_\alpha$ appears in the $b$ ghost $\Rightarrow$
non-minimal variables are not inert under BRST transformations.
Non-minimal variables in a curved background

The BRST transformation of \((\hat{\lambda}_\alpha, \hat{\omega}^\alpha, r_\alpha, s^\alpha)\)

\[
Q\hat{\lambda}_\alpha = -r_\alpha + \lambda^\gamma X_{\gamma\alpha}^\beta \hat{\lambda}_\alpha + \Sigma_\alpha^\beta \hat{\lambda}_\alpha, \quad Q\hat{\omega}^\alpha = -\hat{\omega}^\beta \lambda^\gamma X_{\gamma\beta}^\alpha - \hat{\omega}^\beta \Sigma_\beta^\alpha
\]

\[
Qs^\alpha = \hat{\omega}^\alpha + s^\beta \lambda^\gamma X_{\gamma\beta}^\alpha + s^\beta \Sigma_\beta^\alpha, \quad Qr_\alpha = \lambda^\gamma X_{\gamma\alpha}^\beta r_\beta + \Sigma_\alpha^\beta r_\beta,
\]

where \(X\) is constrained by \(Q^2 = 0\) and it is given by

\[
X_{\gamma\alpha}^\beta = 3\delta_\beta^\alpha \Omega_\gamma - \frac{1}{4} (\gamma^{ab})_\alpha^\beta T_{\gamma ab}.
\]

Let me call \(\tilde{\Sigma} = X + \Sigma\), and the BRST transformations are simplified to

\[
Q\hat{\lambda}_\alpha = -r_\alpha + \tilde{\Sigma}_\alpha^\beta \hat{\lambda}_\alpha, \quad Q\hat{\omega}^\alpha = -\hat{\omega}^\beta \tilde{\Sigma}_\beta^\alpha
\]

\[
Qs^\alpha = \hat{\omega}^\alpha + s^\beta \tilde{\Sigma}_\beta^\alpha, \quad Qr_\alpha = \tilde{\Sigma}_\alpha^\beta r_\beta.
\]

It is the transformation in flat spacetime plus a Lorentz rotation by \(\tilde{\Sigma}\).
Non-minimal variables in a curved background

The non-minimal pure spinor variables contribute to the world-sheet action with

\[ Q \left( \int d^2 z \ s^\alpha \nabla \hat{\lambda}_\alpha - 3(\Pi^\alpha \Omega_\alpha)(s^\beta \hat{\lambda}_\beta) + \frac{1}{4} \Pi^A T_{Aab}(s_\gamma^{ab} \hat{\lambda}) \right) \]

The Noether charge of the BRST transformations is

\[ Q = \int \lambda^\alpha d_\alpha + \hat{\omega}^\alpha r_\alpha \]

And the energy-momentum tensor is

\[ T = -\frac{1}{2} \Pi_a \Pi^a - d_\alpha \Pi^\alpha - \omega_\alpha \nabla \lambda^\alpha - Q \left( s^\alpha \nabla \hat{\lambda}_\alpha - 3(\Pi^\alpha \Omega_\alpha)(s^\beta \hat{\lambda}_\beta) + \frac{1}{4} \Pi^A T_{Aab}(s_\gamma^{ab} \hat{\lambda}) \right) \]

Next step: find $b$ ghost.
Construction of the $b$ ghost

In flat spacetime we had

$$\tilde{\Gamma}^m = \frac{1}{2(\lambda \hat{\lambda})}(d\gamma^m \hat{\lambda}) + \frac{1}{8(\lambda \hat{\lambda})^2}(r\gamma^{mnp} \hat{\lambda}) \frac{1}{2}(\lambda \gamma_{np} \omega),$$

with

$$Qd_{\alpha} = -(\lambda \gamma_m)_{\alpha} \Pi^m, \quad Q\omega_{\alpha} = d_{\alpha},$$

$$Q\lambda_{\alpha} = 0, \quad Qr_{\alpha} = 0, \quad Q\hat{\lambda}_{\alpha} = -r_{\alpha}.$$ 

With this,

$$Q\tilde{\Gamma}^m = -\frac{1}{2(\lambda \hat{\lambda})} \Pi^n(\hat{\lambda} \gamma^m \gamma_n \lambda) - \frac{1}{4(\lambda \hat{\lambda})^2}(\lambda \gamma^{np} r)(\hat{\lambda} \gamma_p \gamma^m \lambda)\tilde{\Gamma}_n$$
Construction of the $b$ ghost

In curved spacetime we have

$$\tilde{\Gamma}^a = \frac{1}{2(\lambda \hat{\lambda})}(D \gamma^a \hat{\lambda}) + \frac{1}{8(\lambda \hat{\lambda})^2}(r \gamma^{abc} \hat{\lambda}) \frac{1}{2}(\omega \gamma_{bc} \lambda),$$

where

$$D_\alpha = d_\alpha - \lambda \gamma \chi_{\gamma \alpha} \beta \omega_\beta,$$ well defined under $\delta \omega_\alpha = (\lambda \gamma^a)_\alpha \Lambda_a.$

And

$$QD_\alpha = -(\lambda \gamma_a)_\alpha \Pi^a + \tilde{\Sigma}_\alpha^\beta D_\beta, \quad Q\omega_\alpha = D_\alpha + \tilde{\Sigma}_\alpha^\beta \omega_\beta,$$

$$Q\lambda^\alpha = 8(\lambda \Omega) \lambda^\alpha - \lambda^\beta \tilde{\Sigma}_\beta^\alpha.$$ 

It turns out that

$$Q\tilde{\Gamma}^a = -\frac{1}{2(\lambda \hat{\lambda})}\Pi^b(\hat{\lambda} \gamma_a \gamma_b \lambda) - \frac{1}{4(\lambda \hat{\lambda})^2}(\lambda \gamma^{bc} r)(\hat{\lambda} \gamma_c \gamma^a \lambda) \tilde{\Gamma}_b - \tilde{\Sigma}_a^b \tilde{\Gamma}_b.$$
Construction of the $b$ ghost

The $b$ ghost becomes

$$
b = - \left( s^\alpha \nabla \hat{\lambda}_\alpha - 3(\Pi^\alpha \Omega_\alpha)(s^\beta \hat{\lambda}_\beta) + \frac{1}{4} \Pi^A T_{Aab}(s_{\gamma}^{ab} \hat{\lambda}) \right)$$

$$- \omega_\alpha \Pi^\alpha + \Pi_a \bar{\Gamma}^a - \frac{1}{4(\lambda \hat{\lambda})}(\lambda \gamma_{ab} r) \bar{\Gamma}^a \bar{\Gamma}^b + \frac{1}{2(\lambda \hat{\lambda})}(\omega \gamma_a \hat{\lambda})(\lambda \gamma^a \Pi).$$

Using

$$Q \Pi^a = (\lambda \gamma^a \Pi) - \Pi^b \tilde{\Sigma}_b^a,$$

$$Q \Pi^\alpha = \nabla \lambda^\alpha - 3(\lambda \Omega) \Pi^\alpha - \frac{1}{2}(\lambda \gamma_{ab} \Omega)(\Pi \gamma^{ab})^\alpha + \Pi^\beta \tilde{\Sigma}_\beta \alpha,$$

one obtains

$$Qb = T.$$
Final remarks

The background fields have an effect on the BRST transformations of the non-minimal pure spinor variables.

The construction of the $b$ ghost in a curved background is possible for the case shown here.

The corresponding analysis for the type II superstring in curved background has been problematic (for me).
Thank you!!