An Operator Product Expansion for Form Factors

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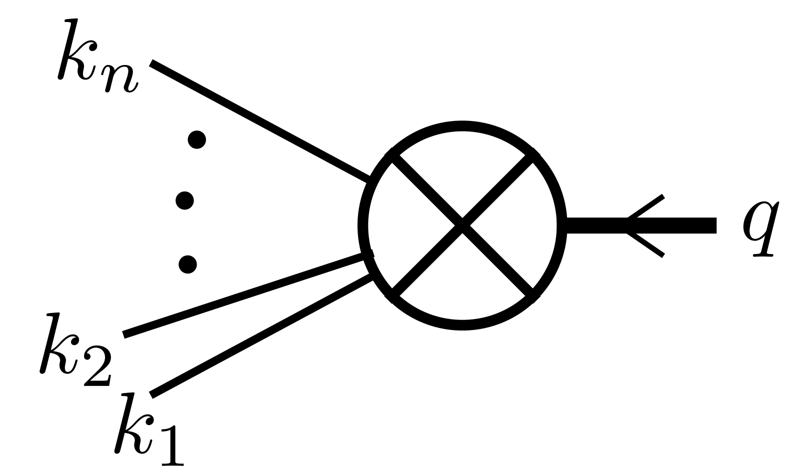
With Alexander Tumanov & Matthias Wilhelm

Introduction

Form factors (FF) = amplitudes for a local operator to create an n-particle asymptotic state

$$F_{\mathcal{O}}(k_1, \ldots, k_n) = \langle k_1, \ldots, k_n | \mathcal{O}(q) | 0 \rangle =$$



Naturally, they live in momentum space

FF in N=4 SYM have been studies a lot in perturbation theory and also at strong coupling

[van Neerven], [Brandhuber, Spence, Travaglini, Yang], [Bork, Kazakov, Vartanov], … see review by [Yang] for references

[Maldacena, Zhiboedov], [Gao, Yang]

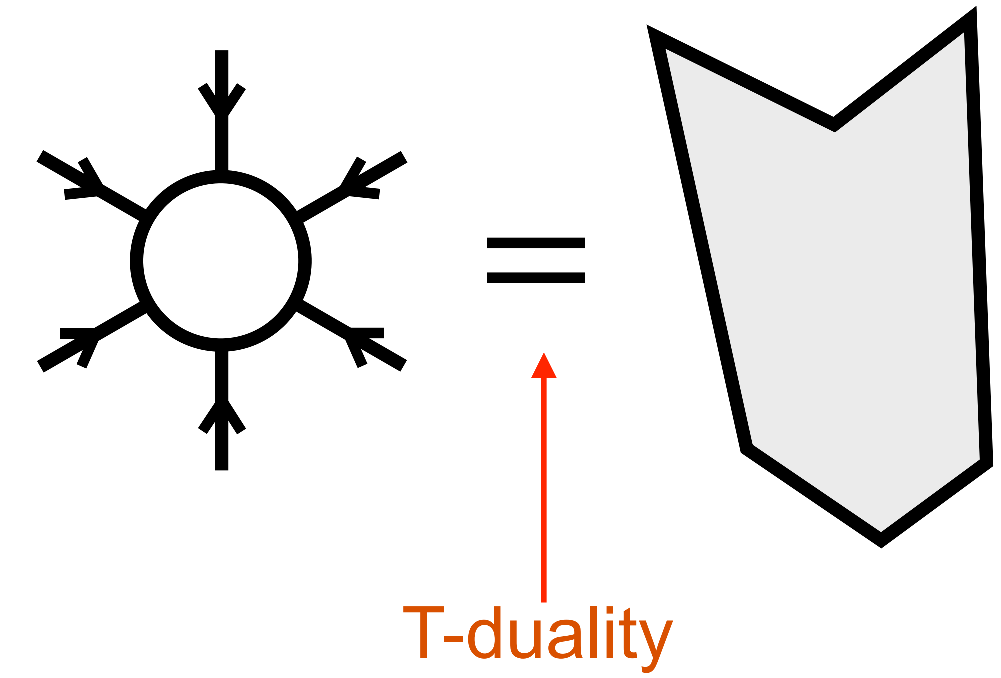
Aim — Bootstrap planar form factors at finite 't Hooft coupling

In this talk - only consider **MHV FF** of the **stress-tensor multiplet** operator in **planar** limit of **N=4 SYM** theory

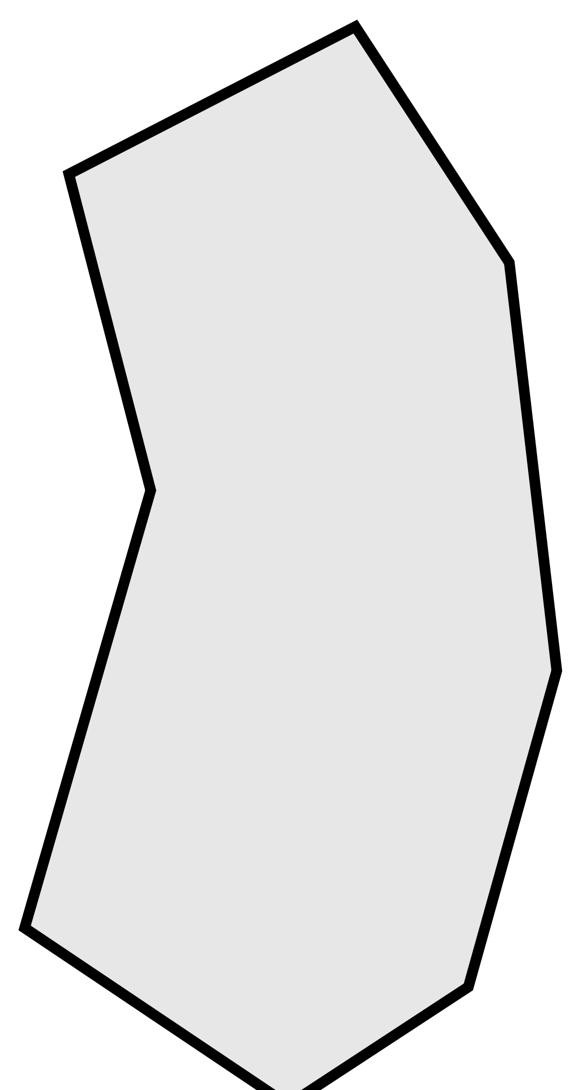
Analogy with scattering amplitudes

Duality with polygon Wilson loops

$$A_n = A_n^{\text{MHV tree}} \times \langle W_n \rangle$$



An operator product expansion around the collinear limit



Analogy with scattering amplitudes

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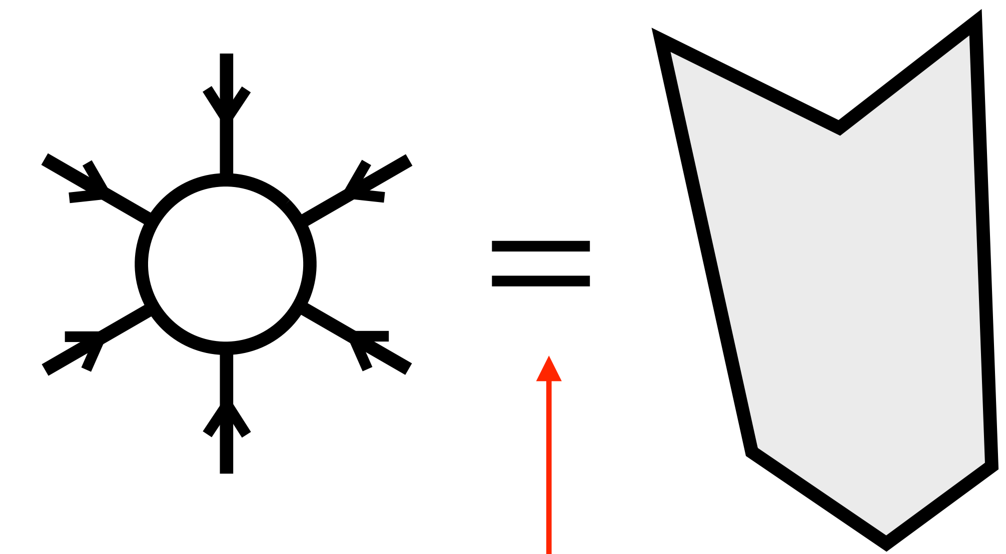
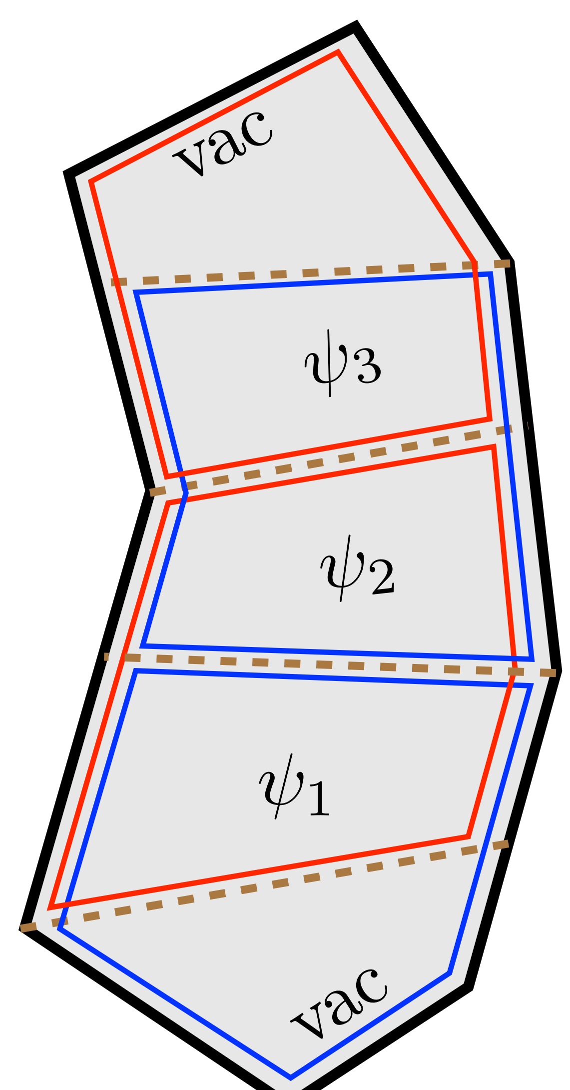
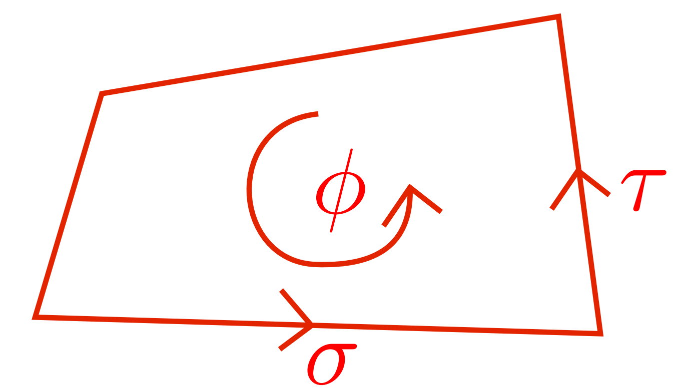


T-duality

An operator product expansion around the collinear limit





geometry

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$$

energy

momentum

angular momentum

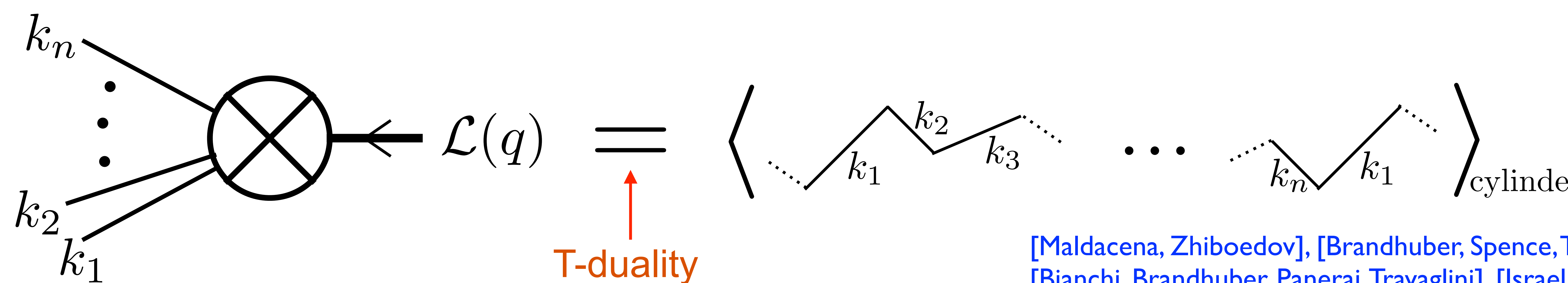
pentagon transition

can be bootstrapped
at finite coupling

Form factors

Duality with polygon **periodic** Wilson path operators

$$F_{\mathcal{L}} = F_{\mathcal{L}}^{\text{MHV tree}} \times \langle W \rangle_{\text{cylinder}}$$

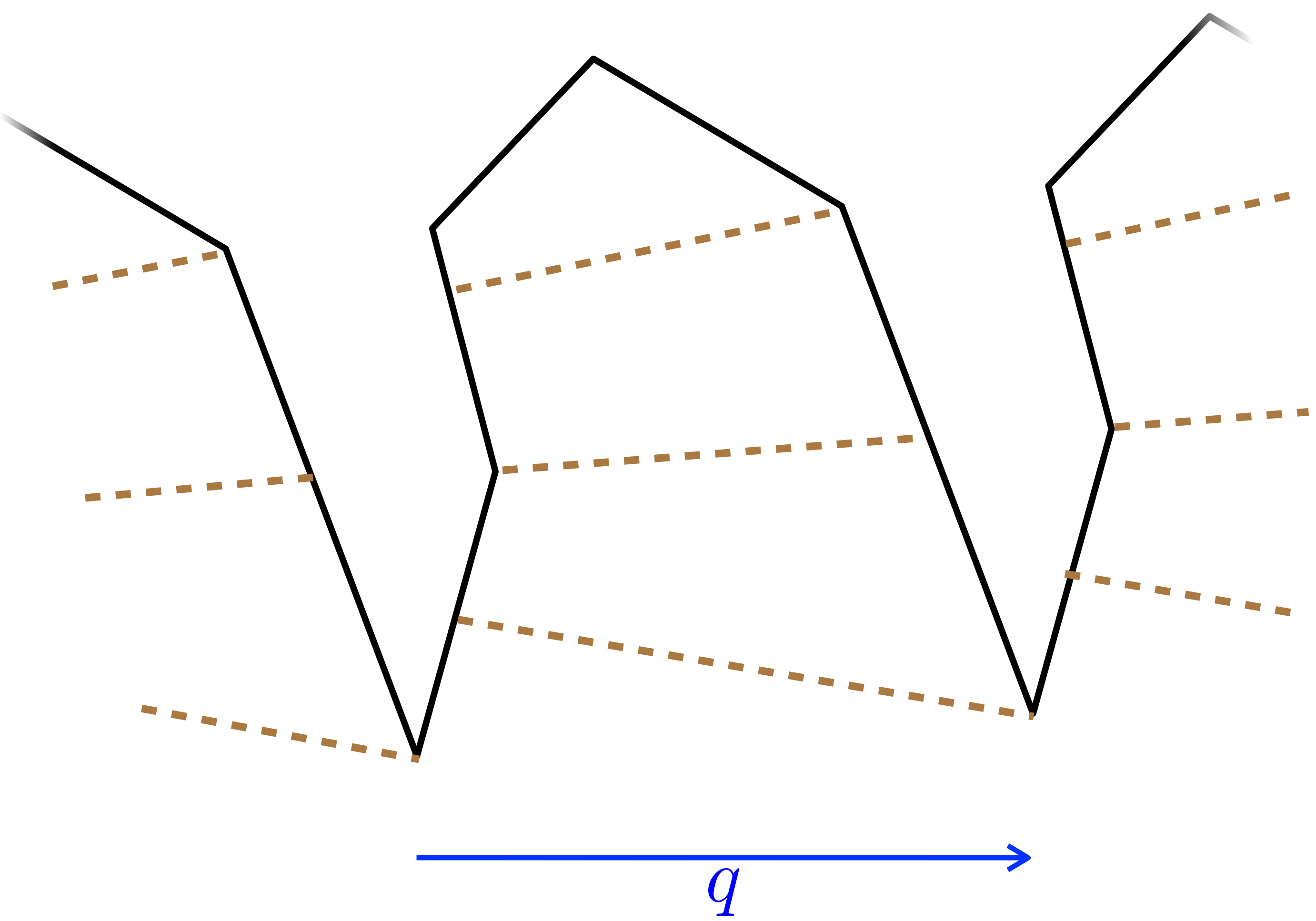


[Maldacena, Zhiboedov], [Brandhuber, Spence, Travaglini, Yang], [Bianchi, Brandhuber, Panerai, Travaglini], [Israel, Tumanov, A.S.]

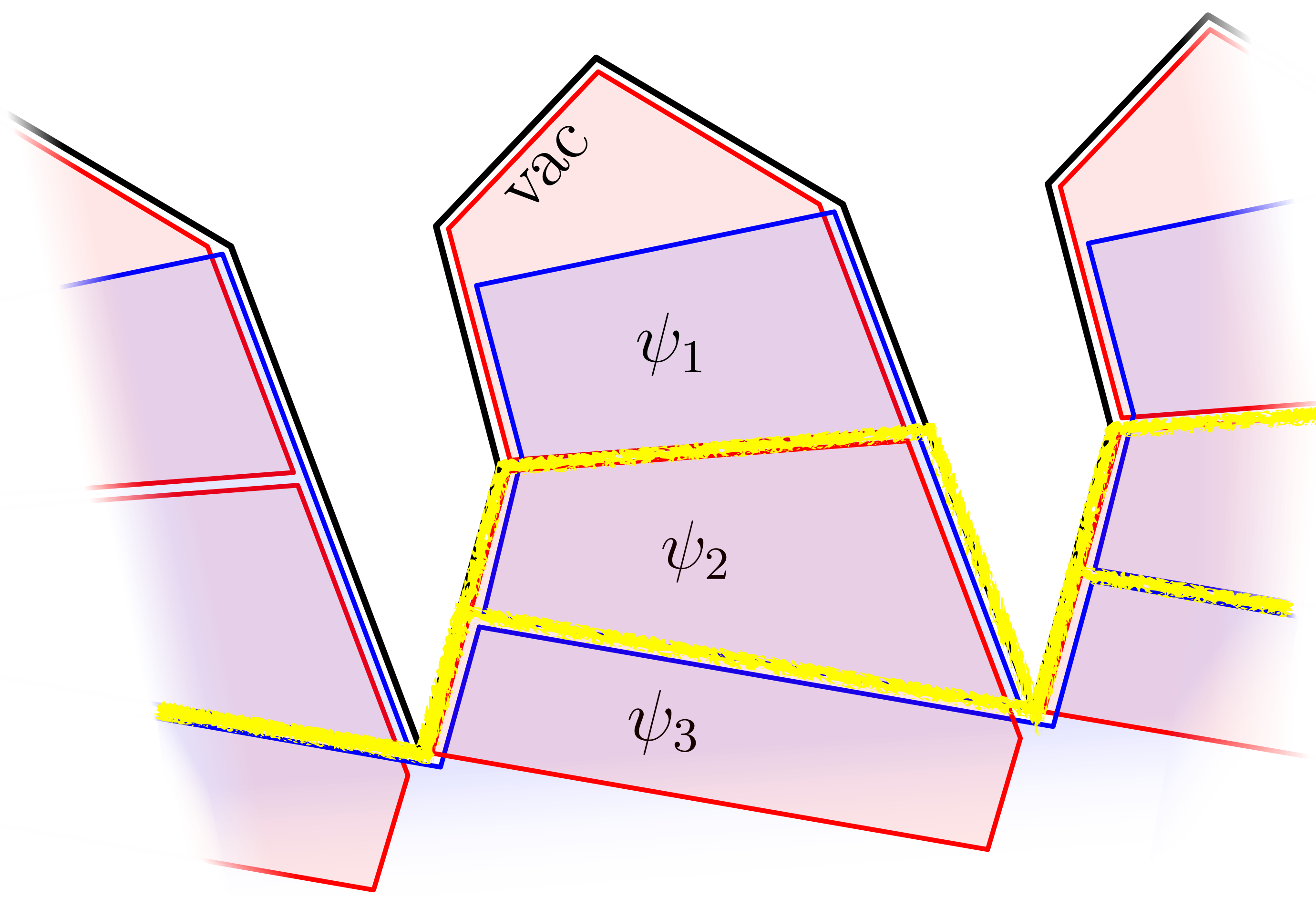
- Can be generalised to $\mathrm{N}^k\mathrm{MHV}$ FF — Wilson lines with super periodicity
- The duality has not been proven. We will analyze perturbative FF data and find a match with the Wilson loop OPE predictions.
- What is the dual of FF for general local operator?

  Natural expectation — modified periodicity constraint (T-dual to other charges)

Form factors OPE



Form factors OPE



Only need to bootstrap one new ingredient

$$= \underbrace{\sum_{\psi_i}\left[\prod_i e^{-E_i\tau_i+ip_i\sigma_i+im_i\phi_i}\right] P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)}_{\text{same as before (does not “know” about the periodicity)}}\boxed{\times F(\psi_3)}$$

new
“FF transition”

Geometry

- 2-particle FF has no ccr (like the pentagon)
- 3-particle FF has two ccr - $\{\tau, \sigma\}$ $(\phi = 0)$
- n-particle FF has 3n-7 independent ccr

analogy with correlation functions

$$E \leftrightarrow \Delta$$

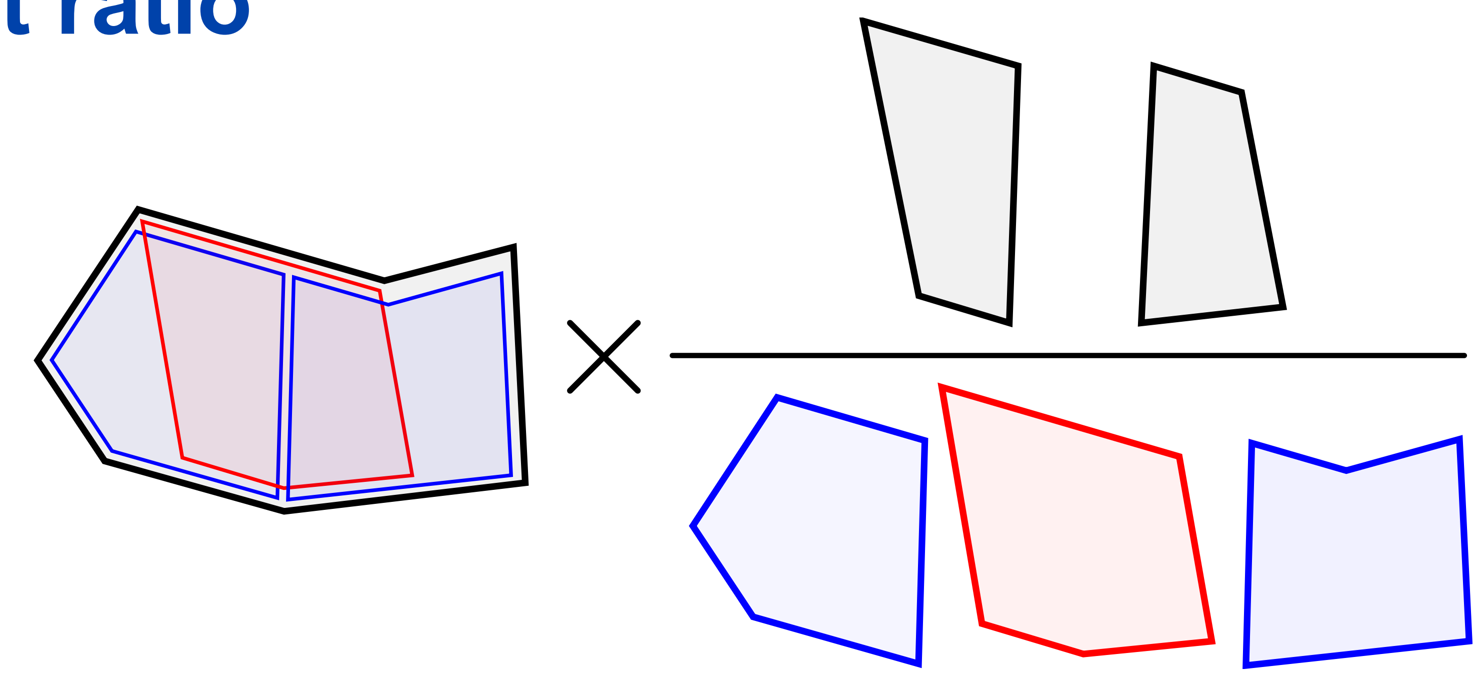
$$P(\psi|\varphi) \leftrightarrow C_{123}$$

$$F(\psi) \leftrightarrow \langle \mathcal{O} \rangle_{\text{defect}}$$

The conformal invariant finite ratio

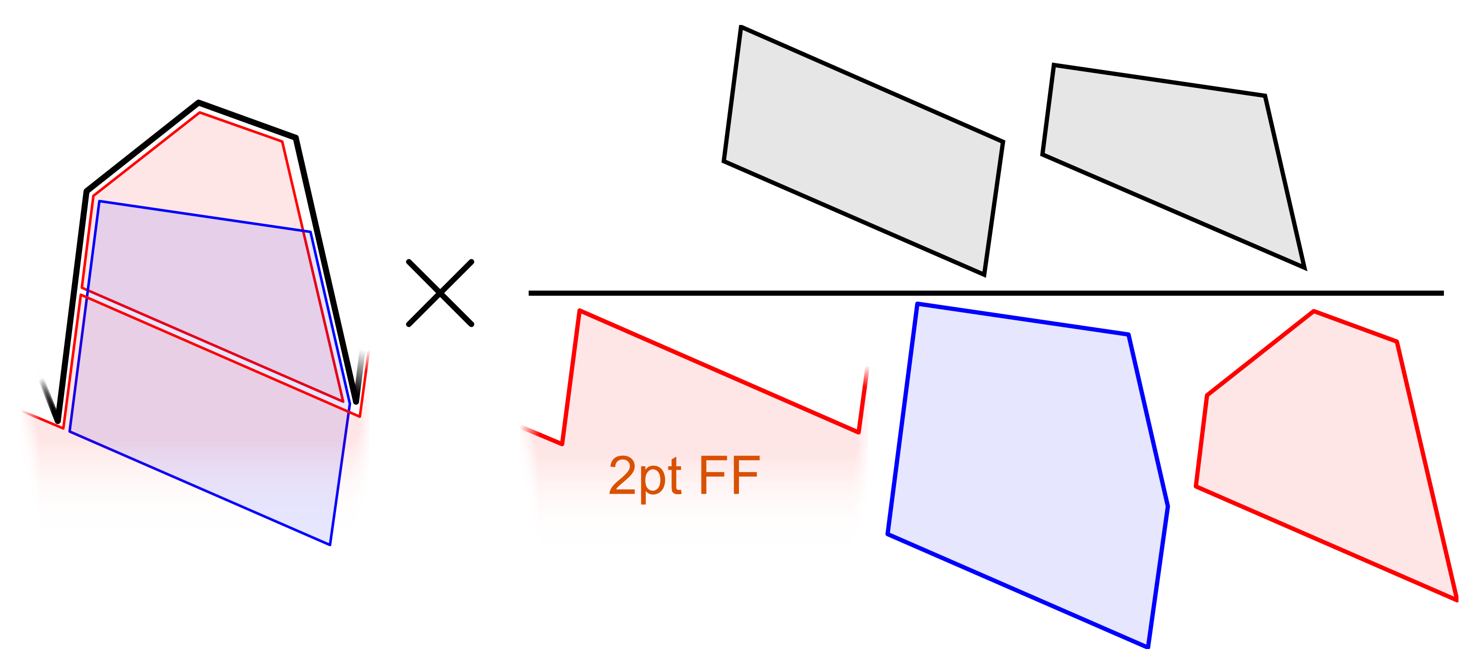
For a given tesselation of a **closed polygon** in terms of null squares we define the following **finite conformal invariant ratio**

$\mathcal{W}_{\rm hep} \equiv$



Similarly, for a given tessellation of a **periodic polygon** in terms of null squares we define the following **finite conformal invariant ratio**

$\mathcal{W}_{\rm 4pt\ ff} =$



Note - the conformal transformation also acts on the periodicity constraint

The FF transition

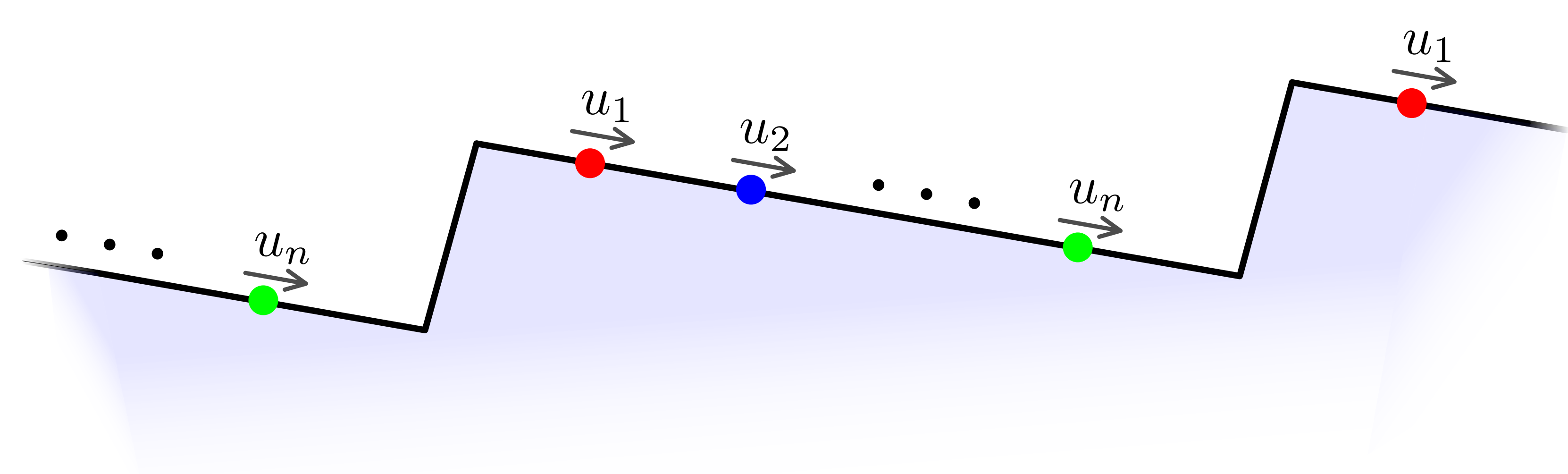
GKP **states** are characterized by - {**number of excitations**, their **type**, their **momentum**}

$$|\psi\rangle = |(u_1, a_1), \ldots, (u_n, a_n)\rangle$$

momentum or rapidity   flavour $\{\phi, \psi, F_a\}$

Correspondingly

$$F(\psi) = F\left((u_1, a_1), \ldots, (u_n, a_n)\right) =$$



The FF transition is neutral under transverse rotations and internal symmetries

$\Rightarrow$ $|\psi\rangle$ is a singlet $\Rightarrow$ $n \geq 2$ $\Rightarrow$ the FF OPE is very hard to check!

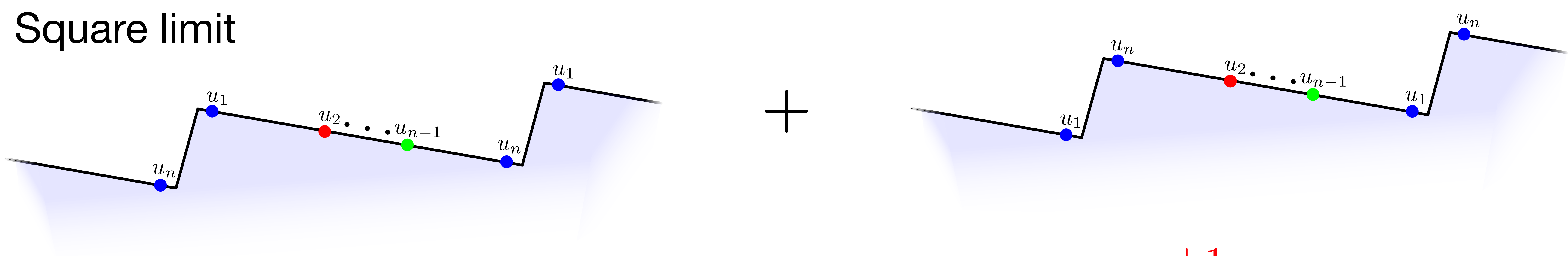
- It starts with two particle singlet
- σ only couples to the c.o.m. of the two excitations
- In perturbation theory, the singlet state mixes all type of fields - {gauge fields, scalars, fermions} and has not been constructed before + there is degeneracy

Non-perturbative properties

I. Watson $$F\,(\ldots,(u_i,a_i),(u_{i+1},a_{i+1}),\ldots) = S_{a_i\,a_{i+1}}(u_i,u_{i+1}) \times F\,(\ldots,(u_{i+1},a_{i+1}),(u_i,a_i),\ldots)$$

(property of the state)

II. Square limit



± 1

$$\lim_{u_1\to u_n} F\,((u_1,a),\ldots,(u_n,\bar{a})) = \frac{i}{\mu_a(u_1)}\left(\frac{1}{u_1-u_n+i\epsilon} \mp S_{a\bar{a}}(u_1,u_1)\frac{1}{u_1-u_n-i\epsilon}\right)$$

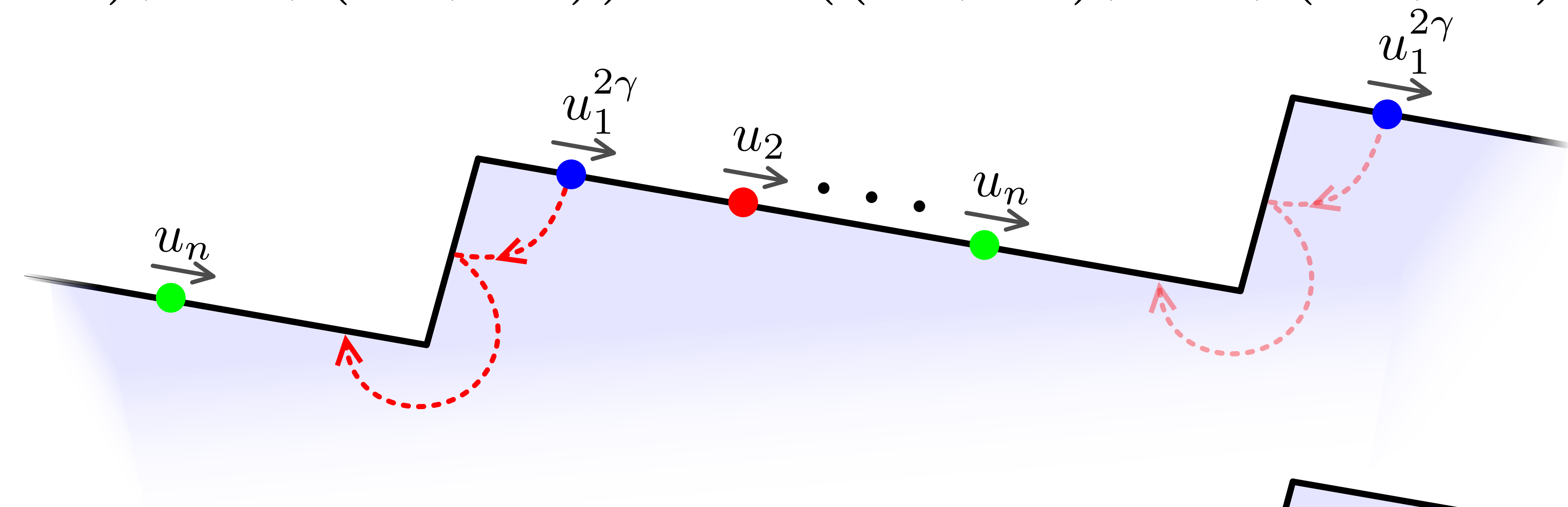
fermions/bosons

$\Rightarrow$ δ-function for gluons and fermions, principle part for scalars

Non-perturbative properties

III. Reflection symmetry $$F\left((u_1,a_1),\ldots,(u_n,a_n)\right)=F\left((-u_n,a_n),\ldots,(-u_1,a_1)\right)$$

IV. Miror axiom $$F((u_1^{2\gamma},a_1),\ldots,(u_n,a_n))=F((u_2,a_2),\ldots,(u_n,a_n),(u_1,a_1))$$



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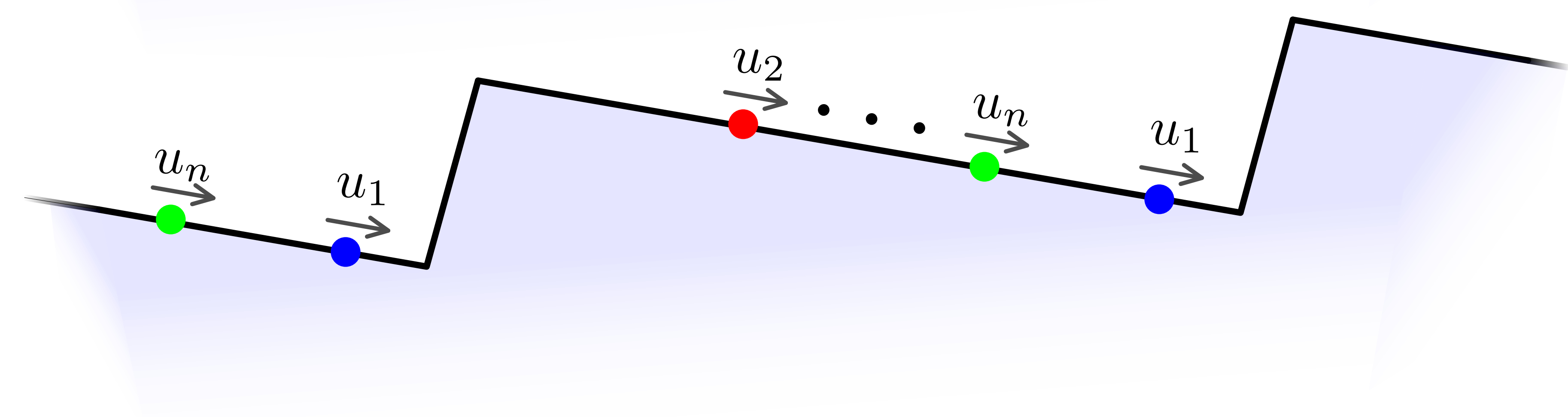


V. Factorization a factorized ansatz is consistent with all properties

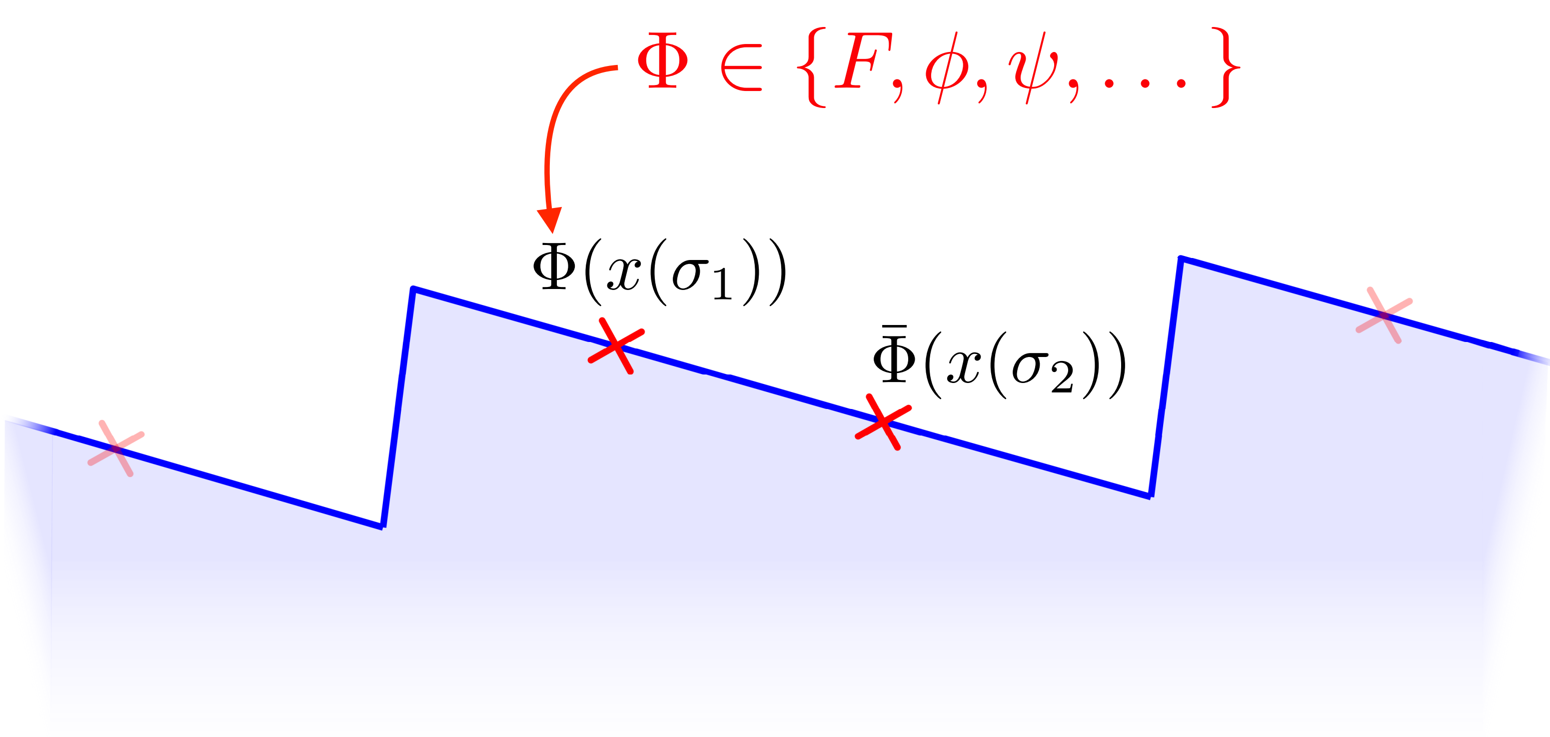
$$F\left((u_1,a_1),\ldots,(u_n,a_n)\right)=\prod_{i<j}F((u_i,a_i),(u_j,a_j))$$

Perturbative check and predictions

The first singlet state above the vacuum has two excitations and energy $E = 2$

two asymptotic {scalars, fermions, gauge fields}

$$|u_1, u_2; \sigma_1, \sigma_2\rangle^{\xi}_{\text{singlet}} \quad = \psi^{\xi}_{\phi\bar{\phi}}\,|\phi(\sigma_1)\,\bar{\phi}(\sigma_2)\rangle + \psi^{\xi}_{\psi\bar{\psi}}|\psi(\sigma_1)\,\bar{\psi}(\sigma_2)\rangle + \psi^{\xi}_{F\bar{F}}|F(\sigma_1)\,\bar{F}(\sigma_2)\rangle$$



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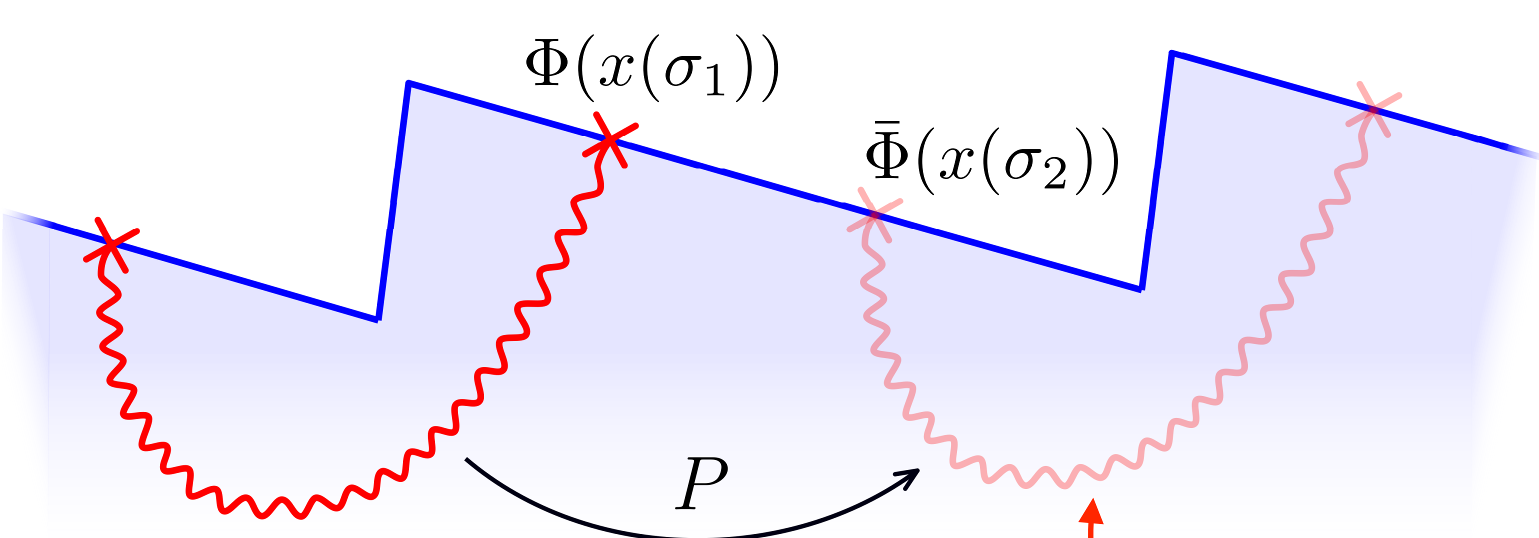
FF transition at Born level

$$\langle FF|\Phi_s(\sigma_1)\,\bar{\Phi}_s(\sigma_2)\rangle = \left[\frac{\sqrt{\partial_{\sigma_1}x_-(\sigma_1)}\sqrt{|\partial_{\sigma_2}x_-(\sigma_2)}}{x_-(\sigma_1) - x_-(\sigma_2)}\right]^{2s} = \frac{1}{\left(e^{\sigma_1+\sigma_2} + e^{-\sigma_1-\sigma_2} + 2\,e^{\sigma_1-\sigma_2}\right)^{2s}}$$



periodic image of the same propagator

tree level propagator

Perturbative check and predictions

$$F_\xi(u_1,u_2) = \int d\sigma_1 d\sigma_2 \, \langle FF|u_1,u_2;\sigma_1,\sigma_2\rangle^{\xi}_{\text{singlet}}$$

FF transition at Born-level

(at this order the Wilson line is computed by integrating the gluon propagator along the path)

$$F_{\text{gauge field}}(u_1,u_2) = -2\left(u_1^2+\frac{1}{4}\right)\cosh(\pi u_1)\,\delta(u_1-u_2)$$

$$F_{\text{fermions}}(u_1,u_2) = 8i\,u_1\sinh(\pi u_1)\,\delta(u_1-u_2)$$

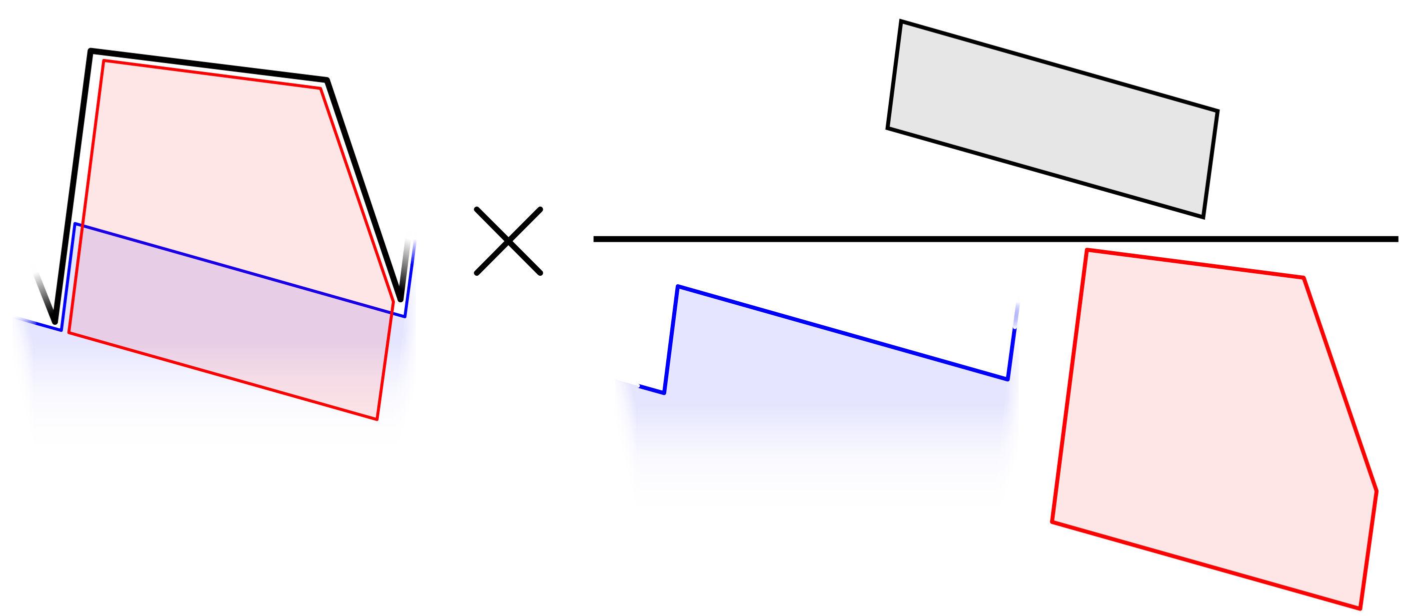
$$F_{\text{scalar}}(u_1,u_2) = -\frac{12}{(u_1-u_2-2i)\,(u_1-u_2-i)}\,\frac{\Gamma\left(iu_1-iu_2\right)}{\Gamma\left(\frac{1}{2}+iu_1\right)\Gamma\left(\frac{1}{2}-iu_2\right)}$$

This combine into

$$\mathcal{W}^{\text{1-loop}}_{\text{3pt ff}} = e^{-2\tau}\sum_{\xi\in\{\phi,\psi,F\}}\int\frac{du_1}{2\pi}\frac{du_2}{2\pi}\,e^{2i(u_1+u_2)\sigma}\,P_\xi(0|u_1,u_2)\,\mu_\xi(u_1)\,\mu_\xi(u_2)\,F_\xi(u_1,u_2)\;+\;\mathcal{O}(e^{-4\tau})$$

$$= 2e^{-2\tau}\left(1+\sigma^2 e^{2\sigma}-2\cosh^2(\sigma)\log\left(1+e^{2\sigma}\right)\right)\;+\;\mathcal{O}(e^{-4\tau})$$

Perturbative check and predictions

$\mathcal{W}_{\text{3pt ff}} =$ 

$$\mathcal{W}^{\text{1-loop}}_{\text{3pt ff}} = \sigma^2 - 2\,\mathrm{Li}_2\left(-e^{2\tau}\right) + 2\,\mathrm{Li}_2\left(-e^{2\tau} - e^{-2\sigma}\right) + 2\,\mathrm{Li}_2\left(-e^{2\tau} - e^{2\sigma}\left(1+e^{2\tau}\right)^2\right)$$

$$= 2e^{-2\tau}\left(1 + \sigma^2 e^{2\sigma} - 2\cosh^2(\sigma)\log\left(1+e^{2\sigma}\right)\right) \;+\; \mathcal{O}(e^{-4\tau})$$

$\checkmark$

Perturbative check and predictions

2-loop check $$\mathcal{W}^{\text{2-loop}}_{\text{3pt ff}} = e^{-2\tau}\left(A(\sigma) + \tau \times \boxed{B(\sigma)}\right) + \mathcal{O}(e^{-4\tau})$$

The term linear in τ comes from 1-loop correction to the particles energy

$$B(\sigma) = \sum_{\xi\in\{\phi,\psi,F\}} \int \frac{du_1}{2\pi}\frac{du_2}{2\pi} e^{2i(u_1+u_2)\sigma}\left(E_\xi^{(1)}(u_1) + E_\xi^{(1)}(u_2)\right) P_\xi(0|u_1,u_2)\,\mu_\xi(u_1)\,\mu_\xi(u_2)\,F_\xi(u_1,u_2)$$

$$= 8\left(1-\left(1+e^{-2\sigma}\right)\log\left(1+e^{2\sigma}\right)\right)\left(1-\left(1+e^{2\sigma}\right)\log\left(1+e^{-2\sigma}\right)\right) \quad \checkmark$$

3-loop prediction $$\mathcal{W}^{\text{3-loop}}_{\text{3pt ff}} = e^{-2\tau}\left(A_3(\sigma) + \tau B_3(\sigma) + \tau^2 C_3(\sigma)\right) + \mathcal{O}(e^{-4\tau})$$

$$\begin{aligned} C_3(\sigma) = &-8e^{-2\sigma}\left(e^{2\sigma}+1\right)^2 \mathrm{Li}_3\left(-e^{2\sigma}\right) - \frac{64}{3}e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log^3\left(e^{2\sigma}+1\right) + \frac{32\left(e^{2\sigma}+1\right)^2\log^2\left(e^{2\sigma}+1\right)}{e^{2\sigma}} \\ &+\left(64\left(e^{2\sigma}+1\right) + 64e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log^2\left(e^{2\sigma}+1\right) - 64e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log\left(e^{2\sigma}+1\right)\right)\sigma \\ &-32e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log\left(e^{2\sigma}+1\right) + \pi^2\left(\frac{4}{3} - \frac{4}{3}e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log\left(e^{2\sigma}+1\right)\right) + 24 \; + \left(48 - 48e^{-2\sigma}\left(e^{2\sigma}+1\right)^2\log\left(e^{2\sigma}+1\right)\right)\sigma^2 \end{aligned}$$

Future directions

- FF transition at finite coupling (in progress with B. Basso)
- Strong coupling
- $N^k MHV$ and charged FF transition
- Other operators
- FF transition and QSC