Sigma models with local couplings: a new connection between integrability and RG flow

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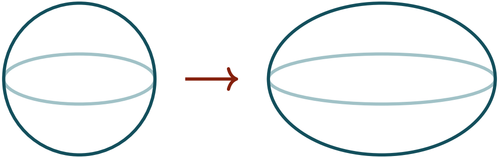
Introduction

- **Observation 1:** There is a remarkable link between the classical integrability of 2-d σ-models and their stability under RG flow at 1 loop.
- Integrable deformations of some of the simplest σ-models were originally constructed by looking for solutions to the RG equations:



$$X_1^2 + X_2^2 + X_3^2 = 1 \quad \rightarrow \quad X_1^2 + X_2^2 + (1+\eta^2)X_3^2 + \mathcal{O}(\eta^4) = 1$$

[Fateev, Onofri, Zamolodchikov]
[Lukyanov, ...]

Introduction

- Believed that 1-loop RG trajectories stay within the class of classically integrable σ-models.
- This has been checked for numerous examples including:
  - models where the global symmetry constrains the RG flow, for example, the principal chiral model;
  - models such as integrable deformations, for which renormalisability at 1 loop with finitely many couplings is non-trivial.
- Can be extended to higher loops.
- Underlies duality with integrable massive models.
- How does the classical integrability translate into simple quantum behaviour?

Introduction

- **Observation 2:** There is an interesting class of string σ-models with metric-dilaton backgrounds of the form [Tseytlin]

$$G = -2\,\mathrm{d}u\,\mathrm{d}v + G_{ij}(x,u)\,\mathrm{d}x^i\,\mathrm{d}x^j \qquad \Phi = v + \Phi(x,u)$$

  - the metric admits a covariantly constant null Killing vector;
  - the transverse metric $G_{ij}(x,u)dx^i dx^j$ can be curved;
  - the dilaton is linear in the null coordinate v.

Introduction

- The classical worldsheet Lagrangian in conformal gauge is

$$\mathcal{L} = -2\partial_+ u \partial_- v + G_{ij}(x,u)\partial_+ x^i \partial_- x^j$$

- On a general 2-d worldsheet we have Weyl invariance at 1-loop if the σ-model

$$\mathcal{L} = G_{ij}(x,u)\partial_+ x^i \partial_- x^j$$

solves the 1-loop RG equations with u formally playing the role of RG time

$$\mathrm{R}_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = 0 \quad \Rightarrow \quad \partial_u G_{ij} = R_{ij} + 2\nabla_i \nabla_j \Phi$$

Introduction

$$\mathcal{L} = -2\partial_+ u \partial_- v + G_{ij}(x, u)\partial_+ x^i \partial_- x^j$$

- In conformal gauge the equation of motion for v is $\partial_+\partial_- u = 0$.
- We fix the residual gauge symmetry using light-cone gauge by setting $u = \tau$.
- We are left with a time-dependent theory for the transverse coordinates x^i

$$\mathcal{L} = G_{ij}(x, \tau)\partial_+ x^i \partial_- x^j$$

Introduction

- **Motivated** by the goal of finding new solvable string σ-models, these observations lead us to the following question:

  Take a σ-model $\mathcal{L} = G_{ij}(x; h)\partial_+ x^i \partial_- x^j$ that is classically integrable and renormalisable at 1 loop with only the couplings h_a running.

  This σ-model can be embedded into a Weyl-invariant σ-model with $h_a(u)$ determined by the 1-loop RG flow:

$$\mathcal{L} = -2\partial_+ u \partial_- v + G_{ij}(x; h(u))\partial_+ x^i \partial_- x^j$$

Question: Is the light-cone gauge-fixed theory classically integrable?

$$\mathcal{L} = G_{ij}(x; h(\tau))\partial_+ x^i \partial_- x^j$$

Outline

- Introduction
- **Lax connections for models with local couplings**
- Conserved charges
- Massive models
- Conclusions

PCM with local coupling

- As a working example we consider the principal chiral model (PCM) for the simple Lie group G

$$\mathcal{L} = -h \, \mathrm{Tr}[g^{-1}\partial_+ g \, g^{-1}\partial_- g]$$

- 2-d σ-model whose target space is the Lie group G with the round metric:
  - $\partial_\pm = \partial_\tau \pm \partial_\sigma$ are light-cone derivatives on the worldsheet;
  - g is a G-valued field so that $g^{-1}\partial_\pm g \in \mathrm{Lie(G)}$ are pull-backs of the Maurer-Cartan 1-form;
  - $-\mathrm{Tr}$ is the positive-definite ad-invariant bilinear form on Lie(G);
  - $\mathrm{G}_L \times \mathrm{G}_R$ global symmetry.
  - h is the σ-model coupling, proportional to the radius squared;

PCM with local coupling

- The PCM is integrable: its equations of motion are encoded in the zero-curvature of the Lax connection [Zakharov, Mikhailov]

$$L_\pm = \frac{1 + z^{\pm 1}}{2}\, g^{-1}\partial_\pm g$$

where $z \in \mathbb{C}$ is the constant spectral parameter.

- The PCM for a simple Lie group G is renormalisable at 1 loop with only the coupling h running: [McKane, Stone; Friedan]

$$\frac{\mathrm{d}}{\mathrm{d}t} h = c \quad \Rightarrow \quad h = ct$$

where c is the dual Coxeter number of G and $t = \log \mu$ is the RG time.

(from now on $c = 1$ for simplicity)

PCM with local coupling

- The PCM can be used to construct the conformal gauge string σ-model

$$\mathcal{L} = -2\partial_+ u \partial_- v - u \, \mathrm{Tr}[g^{-1}\partial_+ g \, g^{-1}\partial_- g]$$

whose light-cone gauge fixing is

$$\mathcal{L} = -\tau \, \mathrm{Tr}[g^{-1}\partial_+ g \, g^{-1}\partial_- g]$$

- This is the PCM with time-dependent local coupling.

Question: Is this model classically integrable?

Lax connection: RG $\Rightarrow$ Lax

- Let us try to construct a Lax connection for this model

$$\mathcal{L} = -\tau \, \mathrm{Tr}[g^{-1}\partial_+ g \, g^{-1}\partial_- g]$$

- The equations of motion in first-order form are ($J_\pm = g^{-1}\partial_\pm g$)

$$\partial_+(\tau J_-) + \partial_-(\tau J_+) = 0 \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

- We assume the following ansatz for the Lax connection

$$L_\pm = \frac{1 + z^{\pm 1}}{2} J_\pm \qquad z = z(w; \tau, \sigma)$$

where $w \in \mathbb{C}$ is now the constant spectral parameter.

Lax connection: RG ⇒ Lax

- It turns out that if

$$L_\pm = \frac{1+z^{\pm 1}}{2} J_\pm \qquad z = \pm\sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$$

then the corresponding curvature is

$$\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = \frac{z^{-1}-z}{4}\frac{1}{\tau}\Big[\partial_+(\tau J_-) + \partial_-(\tau J_+)\Big] + \frac{(1+z)(1+z^{-1})}{4}\Big[\partial_+ J_- - \partial_- J_+ + [J_+, J_-]\Big]$$

- Demanding that this vanishes for all w indeed implies the equations of motion.

Lax connection: RG ⇒ Lax

$$L_\pm = \frac{1+z^{\pm 1}}{2} J_\pm \qquad z = \pm\sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$$

- This Lax connection has a number of unusual new features:
  - z is now a function of τ and σ, while w is constant;
  - the σ-model coupling only depends on τ; however the Lax connection depends explicitly on the spatial coordinate σ;
  - w and σ always come in the combination $w+\sigma$;
  - complicated analytic properties including position-dependent branch cuts.

Lax connection: Lax $\Rightarrow$ RG

- So far we have established that when the local coupling is determined by the 1-loop RG flow we can construct a Lax connection.

- What about the converse?
  - For which local couplings can we construct a Lax connection?

- Consider the PCM with a coupling that is a general function of τ and σ:

(ignoring boundary/periodicity conditions for now)

$$\mathcal{L} = -h(\tau,\sigma)\,\mathrm{Tr}[g^{-1}\partial_+ g\, g^{-1}\partial_- g]$$

- The equations of motion in first-order form are ($J_\pm = g^{-1}\partial_\pm g$)

$$\partial_+(h(\tau,\sigma)J_-) + \partial_-(h(\tau,\sigma)J_+) = 0 \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

Lax connection: Lax ⇒ RG

- We again assume the following ansatz for the Lax connection

(this can be further relaxed to $L_\pm = z_\pm J_\pm$, $z_\pm = z_\pm(w;\tau,\sigma)$)

$$L_\pm = \frac{1+z^{\pm 1}}{2} J_\pm \qquad z = z(w;\tau,\sigma)$$

- The corresponding curvature is

$$\begin{aligned}\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = {} & \frac{z^{-1}-z}{4}\big(\partial_+ J_- + \partial_- J_+\big) \\ & - \frac{\partial_+ \log z}{2z} J_- - \frac{z\partial_- \log z}{2} J_+ \\ & + \frac{(1+z)(1+z^{-1})}{4}\Big[\partial_+ J_- - \partial_- J_+ + [J_+, J_-]\Big]\end{aligned}$$

- The term in red should vanish on the equations of motion.

Lax connection: Lax ⇒ RG

- Comparing coefficients of $J_\pm$ and its derivatives, a Lax connection exists if

$$h(\tau,\sigma) = f^+(\xi^+) + f^-(\xi^-) \qquad \xi^\pm = \frac{1}{2}(\tau \pm \sigma)$$

  - For $f^\pm(\xi^\pm) = \xi^\pm$, $h(\tau,\sigma) = \tau$ and we recover the time-dependent case.
  - Different $h(\tau,\sigma)$ correspond to 2-d conformal transformations of this case.
  - This freedom is expected as they preserve the form of the Lagrangian.
  - Can also be understood from the conformal gauge string σ-model: to fix the residual gauge symmetry we can take $u = f^+(\xi^+) + f^-(\xi^-)$.

- Therefore, the existence of a Lax connection implies that the local coupling is determined by the 1-loop RG flow (up to conformal transformations).

Lax connection: Lax ⇔ RG

- There are various special cases:

$$\mathcal{L} = -\big(f^+(\xi^+) + f^-(\xi^-)\big)\,\mathrm{Tr}[g^{-1}\partial_+ g g^{-1}\partial_- g]$$

  - $f^+ + f^- =$ const: the PCM with constant coupling,

    invariant under conformal transformations;
  - $f^- =$ const: the "chiral model,"

    invariant under "half" the conformal transformations;
  - $f^+ + f^- = \tau$: the time-dependent model,

    natural in string context, spatial momentum is conserved;
  - $f^+ + f^- = \sigma$: the space-dependent model,

    energy is conserved.

Lax connection: Lax $\Leftrightarrow$ RG – further examples

- The Lax connection for the PCM with local coupling is not accidental.
- The same construction works for a number of other integrable σ-models:
  - the symmetric space σ-model; [Eichenherr, Forger]
  - the PCM plus WZ term; [Veselov, Takhtajan]
  - the q-deformation of the PCM;
    (otherwise known as the inhomogeneous Yang-Baxter deformation or η-deformation, generalises the squashed 3-sphere and is a 2-coupling model) [Klimčík]
  - the isotropic current-current deformation of the WZW model;
    (otherwise known as the λ-deformation) [Sfetsos]
  - an analogous deformation of gauged WZW with no global symmetry.
- For many of these the RG flow is more complicated and the existence of a Lax connection is particularly non-trivial.

Lax connection: Lax ⇔ RG – further examples

- As a more involved example, consider the q-deformation of the PCM

[Klimčík; Delduc, Magro, Vicedo]
[Valent, Klimčík, Squellari]

$$\mathcal{L} = -h\,\mathrm{Tr}[g^{-1}\partial_+ g \, \frac{1}{1-\eta R} \, g^{-1}\partial_- g]$$

where R annihilates Cartan generators and acts on roots as $R : e_\pm \to \mp i e_\pm$.

- This is a 2-coupling model:
  - the radius squared h;
  - the deformation parameter η.
- Symmetry is broken to $\mathrm{G}_L \times \mathrm{U}(1)^{\mathrm{rank}\,\mathrm{G}}$.

Lax connection: Lax ⇔ RG – further examples

- The model is integrable with Lax connection

$$L_\pm = -\frac{1+z^{\pm 1}}{2}(1+\eta^2)\,\mathrm{Ad}_g\,\frac{1}{1\pm\eta R}\,J_\pm$$

- The model is renormalisable at 1 loop with the couplings η and h running

$$\frac{\mathrm{d}}{\mathrm{d}t}h = c(1+\eta^2)^2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\eta h^{-1}) = 0$$

$$\Rightarrow$$

$$\nu \equiv \eta(t)h(t)^{-1} = \mathrm{const} \qquad \arctan\eta(t) + \frac{\eta(t)}{1+\eta(t)^2} = 2c\nu t$$

Lax connection: Lax ⇔ RG – further examples

- Using these functions we construct the time-dependent model

$$\mathcal{L} = -h(\tau)\,\mathrm{Tr}[g^{-1}\partial_+ g\,\frac{1}{1-\eta(\tau)R}\,g^{-1}\partial_- g]$$

- Again admits a Lax connection of the same form as the original model with

$$\eta \to \eta(\tau) \qquad h \to h(\tau)$$

$$\arctan\Big(\frac{1}{\eta(t)}\,\frac{1-z}{1+z}\Big) - \frac{1-z}{1+z}\,\frac{\eta(\tau)}{1+\eta(\tau)^2} = -2c\nu(w+\sigma)$$

- Again, the existence of a Lax connection implies that the local couplings are determined by the 1-loop RG flow (up to conformal transformations).

Lax connection: Lax $\Leftrightarrow$ RG

Take a classically integrable σ-model and let its couplings be time-dependent functions; then the Lax connection generalises to the time-dependent theory if the coupling functions are determined by the 1-loop RG flow of the original model.

- We have tested this construction on a number of integrable 2-d σ-models.
- Establishes a new connection between classical integrability and RG flow.

Non-local charges

- Consider the time-dependent PCM on 2-d Minkowski space and choose the following branch of the Lax connection:

$$L_\pm = \frac{1+z^{\pm 1}}{2} J_\pm \qquad z = -\sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$$

- At spatial infinity $\sigma \to \pm\infty$, $z \to -1$, and $L_\pm$ vanish if $J_\pm$ are bounded.
- Therefore, the monodromy matrix

$$M(\tau) = \mathrm{P}\overrightarrow{\exp} \int_{-\infty}^{\infty} \mathrm{d}\sigma \, L \qquad L = \frac{1}{2}(L_+ - L_-)$$

is conserved: $\partial_\tau M = M L_\tau \big|_{\sigma\to+\infty} - L_\tau \big|_{\sigma\to-\infty} M = 0.$ [Lüscher, Pohlmeyer, ...]

Non-local charges

- Expanding the monodromy around $w = -\infty$ we find

$$
\begin{aligned}
M(\tau) = 1 + \frac{1}{2w}\int_{-\infty}^{\infty} \mathrm{d}\sigma \, (\tau J_\tau) \\
- \frac{1}{4w^2}\Big(\int_{-\infty}^{\infty} \mathrm{d}\sigma \, (\tau^2 J_\tau + 2\tau\sigma J_\sigma) + \text{bi-local term}\Big) + \dots
\end{aligned}
$$

- The $\mathcal{O}(w^{-1})$ term is the Noether charge from one of the global G symmetries.
- At higher orders we find generalisations of the multi-local Yangian charges.
- The integrals explicitly depend on σ and for term-by-term convergence we require increasingly strong decaying boundary conditions on $J_\pm$.

Non-local charges

- What about if we consider the model on the cylinder, $\sigma \sim \sigma + 2\pi$.

$$L_\pm = \frac{1 + z^{\pm 1}}{2} J_\pm \qquad z = -\sqrt{\frac{w + \sigma - \tau}{w + \sigma + \tau}}$$

- The explicit σ dependence in the Lax connection breaks periodicity and it is not clear how to construct conserved charges.
- Could the fact that w and σ always appear in the combination $w + \sigma$ help: can $\sigma \to \sigma + 2\pi$ be compensated by a shift in w?
- Suggests that w and σ should be treated on a more equal footing.

Local charges

- Working with a general local coupling, we can also try to construct "higher-spin" local charges

- Motivated by the form these charges take in the PCM, we consider the ansatz

[Evans, Hassan, MacKay, Mountain, ...]

$$Q^{(n)} = -\frac{1}{2}\int d\sigma \ \mathrm{Tr}\Big(\sum_{i=0}^{n}\mu_{i,n}(\tau,\sigma)J_\sigma^{n-i}J_\tau^{i}\Big)$$

- Let us comment on the two simplest cases: $n = 2$ and $n = 3$.

Local charges

- For the general case, $f^{\pm\prime} \neq 0$, we find that:
  - taking $n = 2$ we can construct one quadratic conserved charge;
    (for $f^+ + f^- = \tau$ this is the spatial momentum, for $f^+ + f^- = \sigma$ it is the energy)
  - taking $n = 3$ the ansatz does not give a cubic conserved charge.
- As expected, for the case of constant coupling we can construct both quadratic and cubic charges.
- For the "chiral" case of $f^{+\prime} = 0$ (or $f^{-\prime} = 0$) we can construct "half" the number of quadratic and cubic charges.

Massive models

- The construction also works for the sine-Gordon model:

$$\mathcal{L} = \frac{1}{g^2}\left(\partial_+ x \partial_- x - m^2 \sin^2 x\right)$$

- Replacing the couplings g and m by functions of τ, the existence of a Lax connection (assuming a natural ansatz) requires that these should solve the 1-loop RG equations: $m^2(\tau) = e^{(-2+g^2)\tau} m_0^2 \qquad g(\tau) = g.$

- The resulting time-dependent model indeed appears to be integrable

$$\mathcal{L} = \frac{1}{g^2}\left(\partial_+ x \partial_- x - e^{(-2+g^2)\tau} m_0^2 \sin^2 x\right)$$

as the time dependence can be eliminated by a conformal transformation.

Massive models

- What about other time-dependent integrable massive models: complex sine-Gordon, Toda models, etc.?
- For sine-Gordon integrability is more direct than for σ-models: massive models may help to better understand this relation.
- 1-d reductions of integrable massive models and σ-models can coincide: massive models may provide specific insights into certain σ-models.
- For example, the time-dependent analogue of sine-Gordon mechanics:

$$\mathcal{L}_{\text{1-d}} = \tau(\dot{\theta}^2 - m^2 \sin^2\theta)$$

can be found from the time-dependent SU(2) PCM or sine-Gordon.

Conclusions

- We have established a new, rather surprising, connection between classical integrability and RG flow in 2-d σ-models:

  > *Take a classically integrable σ-model and let its couplings be time-dependent functions; then the Lax connection generalises to the time-dependent theory if the coupling functions are determined by the 1-loop RG flow of the original model.*

- The Lax connections have unusual new features:
  - they depend on both τ and σ;
  - they have a more complicated analytic structure.
- Gives rise to a classical origin of quantum RG flow.

Conclusions

- Is a general proof of the relation possible?
- Can the construction be used to solve string σ-models?
- Is there an underlying algebraic structure?
- Can we understand these theories at the quantum level?
- Is there an origin from 4-d Chern-Simons? [Costello, Yamazaki, (Witten); ...]