Ambitwistor Strings

on $\text{AdS}_3 \times S^3$

based on: 2007.07234 K. Roehrig + O.S.
ongoing work G. Korchemsky, K. Roehrig + O.S.

see also: 2007.06574 L. Eberhardt, S. Komatsu + S. Mizera
Goal (unachieved!):

To obtain compact expressions for AdS amplitudes/boundary correlators with arbitrary multiplicity in supergravity limit.

Inspiration from flat space: (Cachazo-He-Yuan)

\[ M_{\text{super}}^{(n)}(k_i, \epsilon_i, \tilde{\epsilon}_i) = \sum_{\substack{z_i: S_i(z_i) = 0}} \frac{\text{Pf}^\prime(k_i, \epsilon_i, z_i) \, \text{Pf}^\prime(k_i, \tilde{\epsilon}_i, z_i)}{\det^\prime(\partial S_i / \partial z_j)} \]

\[ S_i(z_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} \]

scattering equations
Ambitwistor Strings

\[ S = \int_\Sigma \rho^m \bar{\partial} X^m - \frac{1}{2} e \rho^m \rho^m \]

\[ \rho^m \in \mathcal{H}^0(\Sigma) \]

\[ e \in \mathcal{H}^0(\Sigma, \bar{T}_\Sigma) \]

- chiral worldsheet theory / complexification of worldline

- \( \rho^2 = 0 \) constraint generates \( \delta X^m = \alpha \rho^m, \delta \rho = 0, \delta e = \bar{\delta} \alpha \)

- \( X \sim X + \alpha \rho \) so target is really space of null \( \rho \) \( \text{"ambitwistor space"} \)
Vertex qps ~ $e^{i k \cdot x}$ BRST invariant if $k^2 = 0$.
Since $X(z)X(0) \sim 0$ no massive states in spectrum

With n vertex operators, integrate out $X$ to find $\sum k_i = 0$ and
$$\tilde{\rho} = \sum_i k_i \delta(z - z_i) \Rightarrow \rho(z) = \sum_i \frac{k_i}{z - z_i} \Rightarrow \rho^2 = \sum_{i,j} \frac{k_i \cdot k_j}{(z - z_i)(z - z_j)}$$

Usual basis of Beltrami differentials $e_i \in H^0, 1(\Sigma, \mathcal{T}_z (-\infty, -\infty))$ gives
$$\int_{\Sigma} \rho^2 = \frac{1}{2} \sum_i e_i \text{Res}_i (\rho^2) = \sum_i e_i \left( \sum_{j,i} \frac{k_i \cdot k_j}{z_i - z_j} \right)$$
and integrating over moduli $e_i$ imposes scattering eqs.
- Anomaly free in $d=10$ if also include $\int \bar{\psi} \delta \psi \bar{\epsilon} + \frac{i}{2} \bar{\epsilon} \bar{\psi} \delta \bar{\psi}$ and also gauging $\bar{\psi} \equiv \rho \bar{\psi}$ and $\bar{\epsilon} \equiv \rho \bar{\epsilon}$ along with $H \equiv \frac{1}{2} \rho^2$

\[ \text{SL}(11) \text{ worldsheet algebra} \]

\[ \Phi(\vec{z}) \Phi(\vec{u}) \sim \frac{2H}{|z-u|} \quad \tilde{\Phi}(\vec{z}) \tilde{\Phi}(\vec{u}) \sim \frac{2H}{|z-u|} \quad \Phi(\vec{z}) \tilde{\Phi}(\vec{u}) \sim 0 \]

- Complete $(510+)$ spectrum $\Rightarrow$ Type II supergravity

- Pfaffians in CHY come from $\psi$, $\bar{\psi}$

Since we obtain correct, $n$-particle supergravity tree amplitudes, will be a consistent theory on any background solving the (super-) Einstein eq's
Ambitwistor Strings on $AdS_3 \times S^3(\times \mathbb{M}_y)$

- $AdS_3 \times S^3 \cong SL(2,\mathbb{R}) \times SU(2)$ (we need complexification)
- Pure NS flux $H_{abc} = -f_{abc}$ background solves Einstein eq's

\[ S = \int \mathcal{L} (g^{-1} \bar{\nabla} g)^c + \frac{i}{2} m (\bar{\chi} \bar{\nabla} \chi) + \frac{i}{2} m (\bar{\chi} \bar{\nabla} \bar{\chi}) \]
\[ \Sigma - eH + \chi \bar{\chi} + \bar{\chi} \bar{\chi} \]

- $\mathcal{L} \in \Omega^1 (\Sigma, \mathbb{C}^*)$ independent field (not $g^{-1} \bar{\nabla} g$). No \textit{not} \textit{tt} term, as would break chirality here

- $m$ a bi-invariant metric on $SL(2) \times SL(2)$, with $\kappa^{ab} m_{ab} = 0$

\[ \bar{\nabla} \chi = \bar{\nabla} \chi^a + f_{abc} (g^{-1} \bar{\nabla} g)^b \chi^c \]

\textit{covariant, but }$\chi$ \textit{non-cov}
Currents become
\[ H = \frac{1}{2} m (\tilde{J}, \tilde{J}) , \quad \varphi = J \cdot \varphi + \frac{1}{6} m (\varphi, (\varphi, \varphi)) , \quad \tilde{\varphi} = J \cdot \tilde{\varphi} - \frac{1}{6} m (\tilde{\varphi}, (\varphi, \varphi)) \]
and obey same $SL(1,1)$ algebra as in flat space
\[ \varphi (z) \varphi (\bar{z}) \sim \tilde{\varphi} (z) \tilde{\varphi} (\bar{z}) \sim \frac{2H}{z - \bar{z}} \]
anomaly free iff $m_{ab} \kappa^{ab} = 0 \Leftrightarrow$ Einstein eq.'s

\[ Q = \oint c T + \varepsilon H + \delta \varphi + \tilde{\delta} \tilde{\varphi} + \text{ghosts} \]
\[ \uparrow \text{hol. stress tensor on } \Sigma \]
\[ T = \int (g^{-1} \partial g)^a \wedge \text{fermions} \]

$Q^2 = 0$ provided $d = 10$ and $m_{ab} \kappa^{ab} = 0$
• Spectrum is Type II supergravity on $\text{AdS}_5 \times S^5(\times \mathbb{M}_4)$

$$U = \text{ee} \delta^2(\lambda) \, \bar{\psi} \, \psi \, \bar{V}_{ab}(g)$$

no normal ordering since $gg \text{ OPE trivial}$

• For bulk-boundary propagators, choose

$$V_{\bar{a}b}^+(g) = \frac{\langle \lambda / t_a / \lambda \rangle \langle \lambda / t_b / \lambda \rangle}{\langle \lambda / g(\xi) / \lambda \rangle} \quad V_{ab}^- = \frac{[\lambda / g^{-1}(\lambda)] [\lambda / g^{-1}(\lambda)]}{[\lambda / g(\xi) / \lambda]}$$

where $\langle \lambda / \lambda \rangle = (\vec{x}) (x \lambda) = (\bar{x} \bar{x})$ describes the boundary point

• Worldsheet correlator (of flat space Pfaffians) can be computed explicitly using $T\bar{T}$ and $T\bar{X}$ OPEs, even at $n$ points (see paper)
The Gadlin Model

- AdS scattering eq's come from moduli of \( \int e^{-iJ^2} \)

- OPE of \( J \) with vertex operator shows \( H \) acts on bulk-boundary propagators as

\[
H(z) = \frac{1}{i} \sum_i \frac{\Delta_i (\Delta_i - 2)}{(z - z_i)^2} + \sum_i \frac{1}{z - z_i} \left( \sum_{j \neq i} e_i f_j + e_j f_i + \frac{i}{2} \Delta_i \Delta_j \right)
\]

vanishes since massless

BRST closed

\( z_i \to \) worldsheet co-ord / inhomogeneity parameter

rep's "spins" determined by dual boundary operators

This is \( sl_2 \) Gadlin Hamiltonian acting on boundary coords
Also appears in K-sized connection for standard U(2) model

\[
\left( \frac{\partial}{\partial z_i} - \frac{1}{k+h^\nu} \sum_{j \neq i} \frac{t_i}{z_i - z_j} \right) \psi(x, z) = 0
\]

Here instead we evolve correlator \( \psi(x, z) \) through \('times' e_i:\]

\[
\exp(-seH) |\psi> = \int \exp(-\Sigma \epsilon_i E_i(z_i)) |E> <E| \psi> dE
\]

\( \epsilon \) eigenvalue of \( H_i = \text{Res}_z H(z) \)

Integrating over these n-3 moduli \( e_i \) again imposes scattering eq's

\[
\int d^n \epsilon_i \ e^{-\int eH} |\psi> = \int \prod \delta(E_i(z_i)) |E> <E| \psi> d\epsilon
\]

Analogous relation to "AdS Gross-Mende limit" as \( k \to h^\nu \)

"critical" level in U(2) model \([\text{Feigin, Frenkel, Reshetikhin}]\)
• Since $H(z) \in \mathcal{O}^0(\Sigma, \mathcal{K}_z^2(z_1, \ldots, z_n))$ and $H_i = \text{Res}_z H \in \mathcal{O}^0(\Sigma, \mathcal{K}_z(z_1, \ldots, z_n))$
the eigenvalues $E_i(z)$ must take the form

$$E(z) = \sum_{i,j} \frac{c_{ij}}{(z - z_i)(z - z_j)}$$  

$$E_i(z) = \text{Res}_z E = \sum_{j \neq i} \frac{c_{ij}}{z - z_j}$$

where $c_{ij} = c_{ji} \in \mathbb{C}$ and

$c_{ii} = 0 \quad \text{(no double poles - massless particles on AdS)}$

$\sum_j c_{ij} = 0 \quad \text{(no poles at } z = \infty \text{ - well-defined CFT on } \Sigma; \text{ invariant under global } SL(2) \text{ on boundary)}$

• AdS scattering eq's $E_i = 0$ same form as in flat space, replacing $s_{ij} \rightarrow c_{ij}$. Related to (but $\neq$) Mellin parameters
• What sort of eigenstates should we consider?
\[ [H, \sum S_i] = 0 \]
- invariant under \textit{worldsheet SL}(2)
- \textit{global SL}(2)

Bethe states not enough (Bethe vacuum breaks \textit{global SL}(2))

• Use soV [Sklyanin]: look for zeros (in \( z \)) of operator
\[ f(z) = \sum_i \frac{\delta \Delta_i}{z - z_i} \Rightarrow \text{diagonalize by Fourier transform} \]

**Problem**: This FT does not sit well with \textit{global SL}(2) transformations of boundary. Perhaps resolve by choosing 1 particle to live at \( z = \infty \) on \( \Sigma \) [ongoing w/ Korchemsky, Koch Huang, DD]
Simplification: only need to know eigenstates on support of scattering eq's, since

\[
\int \frac{d\vec{z}_1 \ldots d\vec{z}_n}{\text{vol } S^1(\vec{z})} \prod_i \delta(E_i(\vec{z})) \left| E(\vec{z}, \lambda) \right\rangle \left\langle E(\vec{z}, \lambda) \right| \Psi(\vec{z}, \lambda) \right> = \sum_{\vec{z}_*: E_*(\vec{z}_*) = 0} \int \frac{d\vec{z}}{\det'(\delta E_i/\delta \vec{z}_j)} \left| E(\vec{z}_*, \lambda) \right\rangle \left\langle E(\vec{z}_*, \lambda) \right| \Psi(\vec{z}_*, \lambda) \right>
\]

eg \( n = 4 \), in general, eigenstate \( \phi(x, z) \) obeys ode

\[
\frac{d^2 \Phi}{dX^2} + \left[ \frac{1}{X} + \frac{1}{X-1} + \frac{3}{X-z} \right] \frac{d\Phi}{dX} + \left[ \frac{4X - 2 - E(\vec{z})}{X(X-1)(X-z)} \right] \Phi = 0
\]

\( X = \frac{x_{12} x_{34}}{x_{12} x_{44}}, \quad Z = \frac{Z_{12} Z_{34}}{x_{23} x_{44}}. \) General soln involves Heun f's, but when \( E(\vec{z}_*) = 0 \) reduces to

\[
\frac{1}{(X-Z)^2} \left[ a + b \ln(X^2(1-X)^2) \right]_{Z = Z_*}
\]
Conclusions

• There exist compact expressions for n-particle AdS amplitudes in supergravity limit (also for YM, bi-adjoint scalar...)

• In AdS, scattering eqs take same form as in flat space

\[ E_{i}(z) = \sum_{j} \frac{c_{ij}}{z - z_{j}} = 0 \quad c_{ij} = c_{ji} \quad c_{ii} = 0 \quad \sum_{j} c_{ij} = 0 \]

where \( E_{i}(z) \) are eigenvalues of \( sl_{2} \times su_{2} \) Gaudin Hamiltonian

• Amplitude formula

\[ \int \sum_{E_{i}(z) = 0} \frac{1}{\det(\frac{\partial E_{i}}{\partial z_{j}})} \left| E(\bar{z}, \bar{x}) \right| \left< E(\bar{z}, \bar{x}) \right| \left| \Phi(\bar{z}, \bar{x}) \right> d\bar{z} \]

somewhat reminiscent of scattering wavepackets in flat space

• Bi-adjoint scalar formula proved [Eberhardt, Komatsu, Mite]

by unpacking back into Witten diagrams
Thank You!