Octagons in planar $\mathcal{N} = 4$ super Yang-Mills

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Outline

A lot of progress in solving planar $\mathcal{N} = 4$ super Yang–Mills theory:

- ✗ Spectrum of dilatation operator
- ✗ On-shell amplitudes/Light-like Wilson loops

Can we compute four-point correlation functions exactly, for any 't Hooft coupling?

- ✔ Correlation function of (infinitely) heavy half-BPS operators
- ✔ Exact equations from integrability
- ✔ Strong coupling expansion
- ✔ Comparison with 'experiment'

Four-point correlation functions in $\mathcal{N}=4$ SYM

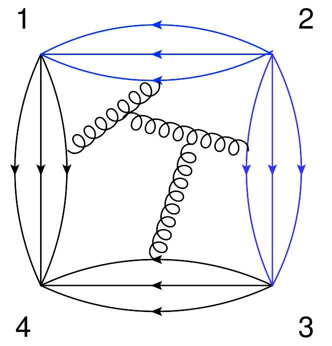
- Half-BPS operators

$$O_1 = \mathrm{tr}(Z^{K/2}\bar{X}^{K/2}) + \text{permutations}\,, \qquad O_2 = \mathrm{tr}(X^K)\,, \qquad O_3 = \mathrm{tr}(\bar{Z}^K)$$

Exact scaling dimension (R-charge) $\Delta = K$

Two– and three–point functions are protected

- "Simplest" four-point function [Coronado]



$$\langle O_1(x_1)O_2(x_2)O_1(x_3)O_3(x_4)\rangle = \frac{\mathcal{G}_K(u,v)}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{K/2}}$$

Depends on 't Hooft coupling $g^2 = g_{\mathrm{YM}}^2 N/(4\pi)^2$ and cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

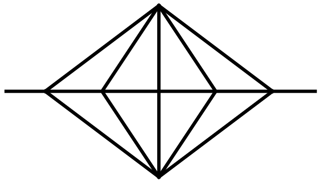
- Ultimate goal: compute $\mathcal{G}_K(u,v)$ exactly in planar $\mathcal{N}=4$ SYM for arbitrary 't Hooft coupling

Weak coupling

A lot of contributing Feynman diagrams... but their sum is remarkably simple

$$\mathcal{G}_K = \sum_\ell (g^2)^\ell \times [\text{conformal integrals at } \ell \text{ loops}]$$

- ‘Simple’ ladder integrals

$f_L =$ 

$$= \frac{1}{z-\bar{z}} \sum_{m=0}^{L} \frac{(-1)^m (2L-m)!}{L!(L-m)!m!} \ln^m(z\bar{z}) \Big[\underbrace{\mathrm{Li}_{2L-m}(z)}_{\text{polylog}} - \mathrm{Li}_{2L-m}(\bar{z}) \Big]$$

- ‘Complicated’ integrals only appear at $(K+1)-$loops [Chicherin,Drummond,Heslop,Sokatchev]

- Factorization property

$$\mathcal{G}_K = [\mathbb{O}(u,v)]^2 + O(g^{2(K+1)})$$

- $\mathbb{O}(u,v)$ is a multilinear combination of ladder integrals

$$\mathbb{O}(u,v) = \sum_\ell (g^2)^\ell \times \sum_{i_1+\dots+i_n=\ell} d_{i_1\dots i_n} f_{i_1} \dots f_{i_n}$$

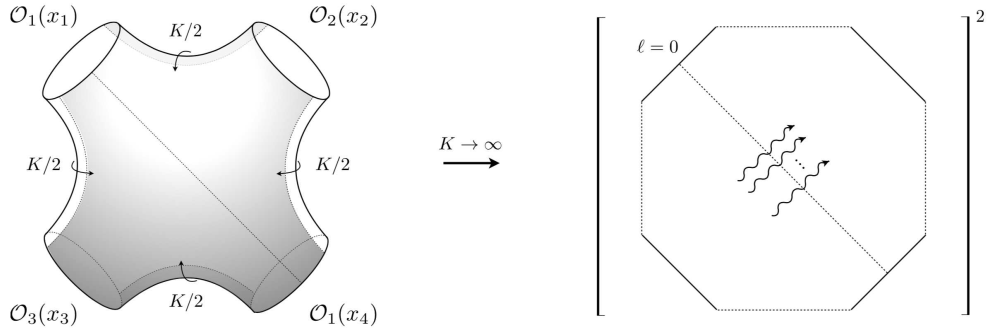
- Infinitely heavy operator limit:

$$\lim_{K\to\infty} \mathcal{G}_K = [\mathbb{O}(u,v)]^2$$

Octagon

Insight from integrability and AdS/CFT – $\mathcal{G}_K$ can be decomposed into hexagon form factors

[Basso,Komatsu,Vieira],[Fleury,Komatsu],[Sfondrini,Eden]



[see Thiago's talk]

At large K the correlator factorizes into the product of two octagon patches [Coronado]

$$\lim_{K\to\infty} \mathcal{G}_K = [\mathbb{O}(u,v)]^2 \,,$$

$$\mathbb{O}(u,v) = \sum_\psi \underbrace{\langle \mathcal{H}_1|\psi\rangle \, \langle\psi|\mathcal{H}_2\rangle}_{\text{hexagon form fact.}}$$

sum over intermediate mirror states

Octagon at finite coupling $=$ an infinite sum of multiple integrals

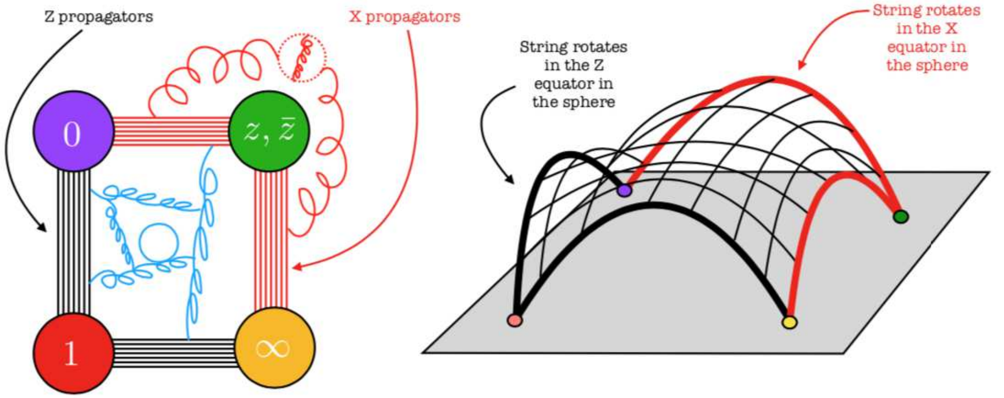
Weak coupling expansion

$$\mathbb{O}(u,v) = 1 + g^2 f_1 - 2g^4 f_2 + 6g^6 f_3 + g^8(-20 f_4 - \tfrac{1}{2} f_2^2 + f_1 f_3) + \dots$$

Bootstrapped to all loops from asymptotics of $\mathcal{G}_K$ in different OPE limits

Octagon at strong coupling

Scattering amplitude of four closed strings on $\mathrm{AdS}_5\times\mathrm{S}^5$ [Bargheer, Coronado, Vieira]



$$\mathbb{O}(z,\bar{z})^2 = \mathbb{O}\times\mathbb{O} = \texttt{inside graphs}\times\texttt{outside graphs}$$
$$= \texttt{top of folded string}\times\texttt{bottom of folded string}$$

[picture from 1909.04077]

Leading asymptotics of the octagon

$$\mathbb{O} = \mathrm{e}^{-g\,A_0 + A_1\log g + \ldots}$$

A_0 – the minimal area of a string that ends on four BMN geodesics, A_1 – quadratic fluctuations

$$A_0(y,\xi) = \int_{-\infty}^{\infty}\frac{d\theta}{\pi}\,\xi\,\cosh\theta\log(1+Y(\theta))$$

$$Y(\theta) = -\frac{\cosh\left(\frac{1}{2}(\xi+y)\right)\cosh\left(\frac{1}{2}(\xi-y)\right)}{\cosh\left(\frac{1}{2}(\xi\cosh\theta+y)\right)\cosh\left(\frac{1}{2}(\xi\cosh\theta-y)\right)}$$

Kinematical variables : $z = -\,\mathrm{e}^{-y-\xi}$, $\bar{z} = -\,\mathrm{e}^{+y-\xi}$

Octagon at finite coupling

Two-point functions in integrable models are given by Fredholm determinants [Itoyama,Thacker,Korepin]

$$\langle 0|J_1 J_2|0\rangle = \sum_{\psi} \langle J_1|\psi\rangle\langle\psi|J_2\rangle = \det(1-H)$$

The same applies to the octagon for an arbitrary coupling [Kostov,Petkova,Serban]

$$\mathbb{O}(u,v) = \sum_{\psi} \langle \mathcal{H}_1|\psi\rangle\langle\psi|\mathcal{H}_2\rangle = \sqrt{\det(1-\lambda C K)}$$

Semi-infinite matrices

$$K_{nm} = \frac{g}{2i}\int_{|\xi|}^{\infty} dt\, \frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cosh y + \cosh t} \underbrace{J_m(2g\sqrt{t^2-\xi^2})}_{\text{Bessel function}} J_n(2g\sqrt{t^2-\xi^2})$$

$$C_{nm} = \delta_{n+1,m} - \delta_{n,m+1}\,, \qquad \lambda = 2(\cosh y + 1)$$

Weak coupling expansion – multi-linear sum of ladders

$$\mathbb{O}(u,v) = \sum_{\ell} (g^2)^{\ell} \times \sum_{i_1+\cdots+i_n=\ell} d_{i_1\ldots i_n} f_{i_1}\cdots f_{i_n}$$

Hidden simplicity

Similarity transformation for $H = \lambda C K$ [Belitsky,GK]

$$H = \Omega^{-1} \begin{bmatrix} \mathbb{k}_- & 0 \\ 0 & \mathbb{k}_+ \end{bmatrix} \Omega \,, \qquad \mathbb{k}_+ = \mathbb{U}^{-1}\, \mathbb{k}_-\, \mathbb{U}$$

Semi-infinite matrix $\mathbb{k}_- \implies$ Integral operator $\mathbb{K}_\chi$

Octagon for arbitrary coupling

$$\mathbb{O} = \sqrt{\det(1-H)} = \det(1-\mathbb{K}_\chi)$$

$$\mathbb{K}_\chi f(x_1) = \int_0^\infty dx_2\, K(x_1,x_2)\chi(x_2) f(x_2)$$

$$\chi(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x/(2g)^2 + \xi^2})}$$

depends on g, y, ξ

Integrable Bessel kernel

$$K(x_1,x_2) = \frac{\sqrt{x_1} J_1\left(\sqrt{x_1}\right) J_0\left(\sqrt{x_2}\right) - \sqrt{x_2} J_0\left(\sqrt{x_1}\right) J_1\left(\sqrt{x_2}\right)}{2(x_1 - x_2)}$$

Appeared in random matrix models, KPZ equation, BES equation, ...

Method of differential equations

A powerful method for computing correlators in integrable models [Its,Izergin,Korepin,Slavnov]

Logarithmic derivative of the octagon

$$U(g,y,\xi) = -2g\partial_g \log \mathbb{O}$$

Satisfies the system of *exact* integro-differential equations [Belitsky,GK]

$$\begin{aligned}
\partial_y U &= \int_0^\infty dx\, Q^2(x)\partial_y \chi(x)\,, \\
g\partial_g U &= -2\int_0^\infty dx\, Q^2(x) x\partial_x \chi(x)\,, \\
\partial_\xi U &= 8g^2\xi \int_0^\infty dx\, Q^2(x)\partial_x \chi(x) + \frac{\sinh\xi}{\cosh y + \cosh\xi}\int_0^\infty dx\, Q^2(x)\chi(x) \\
\chi &= \frac{\cosh y + \cosh\xi}{\cosh y + \cosh(\sqrt{x/(2g)^2 + \xi^2})}
\end{aligned}$$

Auxiliary function $Q(x)$ obeys a PDE

$$(g\partial_g + 2x\partial_x)^2\, Q(x) + (x - g\partial_g U + U)\, Q(x) = 0$$

Octagon in the null limit

Four operators are at the vertices of a null rectangle $x^2_{i,i+1} \to 0$

$$g = \text{fixed}\,, \qquad y \to \infty\,, \qquad \xi = \text{fixed}$$

Prediction for the null octagon

$$\log \mathbb{O} = -\frac{y^2}{2\pi^2}\Gamma(g) + \frac{1}{8}C(g) + g^2\xi^2$$

Exact solution

$$\Gamma(g) = \log(\cosh(2\pi g))\,, \qquad C(g) = -\log\left(\frac{\sinh(4\pi g)}{4\pi g}\right)$$

Agrees with weak coupling expansion

(Unconventional) strong coupling limit: $y \gg g \gg 1$

$$\Gamma(g)/\pi^2 = \frac{2g}{\pi} - \frac{\log 2}{\pi^2} - \sum_{n\geq 1} \frac{(-1)^n}{\pi^2 n}\,\mathrm{e}^{-4\pi g n}$$

Large g expansion does not contain $1/g$ corrections !

Octagon at strong coupling

- (Conventional) strong coupling limit: $g \to \infty$ with y , $\xi =$ fixed

$$\log \mathbb{O} = \underbrace{-gA_0 + \frac{1}{2}A_1^2 \log g + B}_{\text{Strong Szegő theorem}} + \frac{A_2}{4g} + \frac{A_3}{12g^2} + \frac{A_4}{24g^3} + \dots$$

- Exact expressions for the expansion coefficients [Belitsky,GK]

$$A_0 = 2I_0 \,, \qquad A_1 = 1 \,,$$

$$A_2 = -\frac{3I_1}{4}, \qquad A_3 = -\frac{9I_1^2}{16},$$

$$A_4 = -\frac{3I_1^3}{8} + \frac{15I_2}{128}, \qquad A_5 = -\frac{15I_1^4}{64} + \frac{75I_1I_2}{256}, \qquad \dots$$

Dependence on y and ξ enters through *profile functions*

$$I_n(y,\xi) = \int_0^\infty \frac{dz}{\pi} \frac{(z^{-1}\partial_z)^n}{(2n-1)!!} z\partial_z \log\left(\frac{\cosh(\sqrt{z^2+\xi^2}) - \cosh\xi}{\cosh(\sqrt{z^2+\xi^2}) + \cosh y}\right)$$

A_0 agrees with the semiclassical result of [Bargheer,Coronado,Vieira]

A_1 is universal, generated by Fisher-Hartwig singularity

- A new Szegő-Akhiezer-Kac formula for the Bessel operator (arXiv.org > math > math.FA)

Comparison with "experiment"

Reference kinematical point: $\xi = 1/10$ and $y = 2\pi i/3$

✔ Numerical evaluation of the octagon

$$\mathbb{O} \approx \det\left[1 - \mathbb{k}_-\right]_{M\times M}$$

$$(\mathbb{k}_-)_{nm} = (2n+1)\int_0^\infty \frac{dx}{x} J_{2m+1}(\sqrt{x}) J_{2n+1}(\sqrt{x})\chi(x)$$

Truncate the size of (semi-infinite) matrix $\mathbb{k}_-$ to $M \sim 10^2$

✔ Strong coupling expansion of the octagon

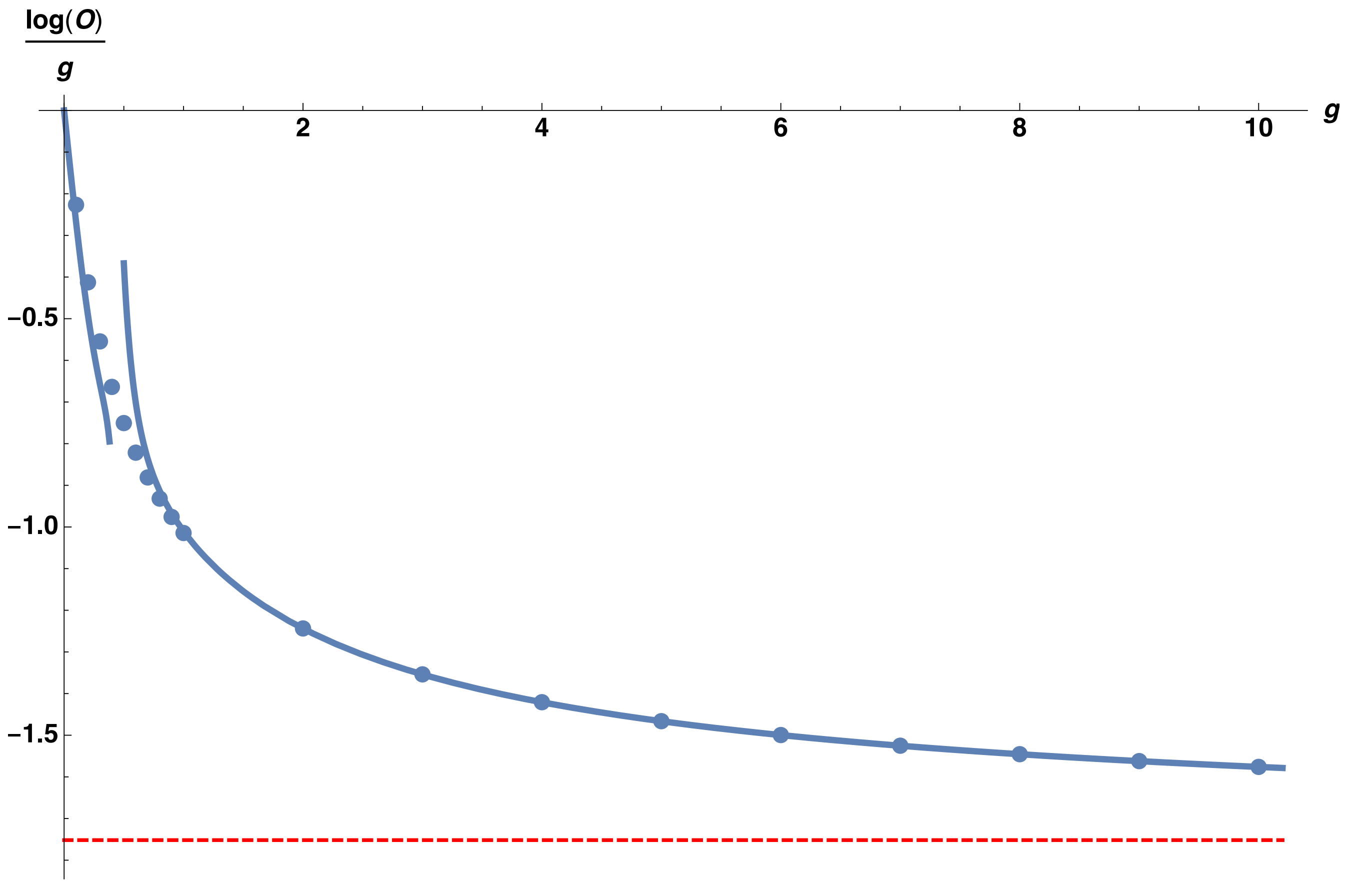
$$\begin{aligned}\log\mathbb{O} = &-1.752g + 0.5\log(g) + \frac{0.1803}{g} - \frac{0.04334}{g^2} + \frac{0.01809}{g^3} - \frac{0.01107}{g^4} + \frac{0.009247}{g^5}\\ &- \frac{0.009990}{g^6} + \frac{0.01330}{g^7} - \frac{0.02108}{g^8} + \frac{0.03857}{g^9} - \frac{0.08006}{g^{10}} + \frac{0.1849}{g^{11}} - \frac{0.4717}{g^{12}}\\ &+ \frac{1.309}{g^{13}} - \frac{3.940}{g^{14}} + \frac{12.74}{g^{15}} - \frac{44.14}{g^{16}} + \frac{162.4}{g^{17}} - \frac{635.1}{g^{18}} + \dots\end{aligned}$$

Sign alternating series with factorially growing coefficients

The strong coupling expansion of the octagon is Borel summable !

Comparison with “experiment” II

Dependence of $(\log \mathbb{O})/g$ on the coupling constant for $\xi = 1/10$ and $y = 2\pi i/3$



Weak and strong coupling expansion against numerical values

Dashed line – the leading term at strong coupling

Conclusions and open questions

Correlation functions of heavy BPS operators have interesting properties:

- ✔ Satisfy integro-differential equations, admit exact solution in the null limit and at strong coupling
- ✔ The same *exact* anomalous dimensions control asymptotics of 6-point MHV amplitude at the origin [Basso, Dixon,Papathanasiou] , why? [see Georgios's talk]
- ✔ What is the reason for remarkable simplification of the Fredholm determinant?
- ✔ Strong coupling expansion is Borel summable, why?
- ✔ Generalization to correlators involving light operators?
- ✔ Nonplanar corrections?

Thank you for your attention!