Modular invariants in strongly-coupled $\mathcal{N} = 4$ SYM

Silviu S. Pufu, Princeton University

Based on:

- arXiv:1902.06263 with D. Binder, S. Chester, and Y. Wang
- arXiv:1912.13365 with S. Chester, M. Green, Y. Wang, and C. Wen
- arXiv:2003.08412 with S. Chester
- arXiv:2008.02713 with S. Chester, M. Green, Y. Wang, and C. Wen

IGST, August 26, 2020

Introduction

- This talk is on correlators of local operators in 4d $\mathcal{N} = 4$ SYM.
- Approaches:
  - integrability
  - weak coupling expansion
  - holography (large N 't Hooft limit)
  - numerical bootstrap
  - analytic bootstrap
  - supersymmetric localization
- Most studied: $1/2$-BPS single trace scalar operators $\mathcal{O}_p$ in $[0p0]$ of $SU(4)_R$ R-symmetry and dimension $\Delta = p$.

$\mathbf{20}'$ operator

- **This talk:** 4-pt function of $\mathcal{O}_2 \equiv S_{IJ} = \text{tr}(X_I X_J - \frac{\delta_{IJ}}{6} X_K X^K)$, i.e. the $\mathbf{20}'$ single trace scalar op. of dimension $\Delta = 2$.

$$\langle S(\vec{x}_1, Y_1) S(\vec{x}_2, Y_2) S(\vec{x}_3, Y_3) S(\vec{x}_4, Y_4) \rangle\,, \quad S(\vec{x}, Y) \equiv Y^I Y^J S_{IJ}(\vec{x})$$

  - S is in the same superconf. multiplet as the stress-energy tensor
  - In fact, SUSY Ward identities $\Longrightarrow$ all 4-pt correlators of stress tensor multiplet operators are determined by a single function $\mathcal{T}(U, V)$ [Belitsky, Hohenegger, Korchemsky, Sokatchev '14] . Schematically: 6 R-symmetry invariant structures, and

$$\langle SSSS \rangle = (\text{free part}) + \frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2 (V + UV\sigma^2 + U\tau^2 + \cdots)}{\vec{x}_{12}^4 \vec{x}_{34}^4} \mathcal{T}(U, V)$$

where $U \equiv \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, $V \equiv \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, and $\sigma \equiv \frac{Y_1 \cdot Y_3\, Y_2 \cdot Y_4}{Y_1 \cdot Y_2\, Y_3 \cdot Y_4}$ and $\tau \equiv \frac{Y_1 \cdot Y_4\, Y_2 \cdot Y_3}{Y_1 \cdot Y_2\, Y_3 \cdot Y_4}$.

$\mathbf{20}'$ operator

- **This talk:** 4-pt function of $\mathcal{O}_2 \equiv S_{IJ} = \mathrm{tr}(X_I X_J - \frac{\delta_{IJ}}{6} X_K X^K)$, i.e. the $\mathbf{20}'$ single trace scalar op. of dimension $\Delta = 2$.

$$\langle S(\vec{x}_1, Y_1) S(\vec{x}_2, Y_2) S(\vec{x}_3, Y_3) S(\vec{x}_4, Y_4) \rangle , \quad S(\vec{x}, Y) \equiv Y^I Y^J S_{IJ}(\vec{x})$$

  - S is in the same superconf. multiplet as the stress-energy tensor
  - In fact, SUSY Ward identities $\Longrightarrow$ all 4-pt correlators of stress tensor multiplet operators are determined by a single function $\mathcal{T}(U, V)$ [Belitsky, Hohenegger, Korchemsky, Sokatchev '14] . Schematically: 6 R-symmetry invariant structures, and

$$\langle SSSS \rangle = (\text{free part}) + \frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2 (V + UV\sigma^2 + U\tau^2 + \cdots)}{\vec{x}_{12}^4 \vec{x}_{34}^4} \mathcal{T}(U, V)$$

where $U \equiv \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, $V \equiv \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, and $\sigma \equiv \frac{Y_1 \cdot Y_3 \, Y_2 \cdot Y_4}{Y_1 \cdot Y_2 \, Y_3 \cdot Y_4}$ and $\tau \equiv \frac{Y_1 \cdot Y_4 \, Y_2 \cdot Y_3}{Y_1 \cdot Y_2 \, Y_3 \cdot Y_4}$.

$SL(2,\mathbb{Z})$ in $\mathcal{N}=4$ SYM

- Define complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2}\,, \qquad \bar\tau = \frac{\theta}{2\pi} - \frac{4\pi i}{g_{\rm YM}^2}$$

- $\mathcal{N}=4$ SYM w/ gauge algebra $\mathfrak{su}(N)$ exhibits $SL(2,\mathbb{Z})$ duality:

$$\tau \to \frac{a\tau+b}{c\tau+d}\,, \qquad a,b,c,d\in\mathbb{Z}\,, \qquad ad-bc=1\,.$$

  - S generator: $\tau \to -1/\tau$
  - T generator: $\tau \to \tau+1$, or $\theta \to \theta + 2\pi$.

- $1/2$-BPS ops, and in particular the $\mathbf{20}'$ op. S and its 4-point function, are invariant under $SL(2,\mathbb{Z})$.

$SL(2,\mathbb{Z})$ in superstring theory

- Related topic: $SL(2,\mathbb{Z})$ in type IIB superstring theory, e.g. for superstring scattering amplitudes

$$\tau = C + ie^{-i\phi}\,.$$

- Example: R^4 interaction $\longrightarrow$ scattering amplitude $\propto$ non-holomorphic Eisenstein series $E(\frac{3}{2},\tau,\bar{\tau})$ (more on this later).

- Connection:

CFT 4-point on function	$\overset{\text{AdS/CFT}}{\Longleftrightarrow}$	scattering amplitude in $AdS_5 \times S^5$	$\overset{\text{flat space limit}}{\longrightarrow}$	10d flat space scattering amplitude

Flat space limit:

Mellin amplitude $\mathcal{M}(s,t) \overset{\text{large } s,\, t}{\longrightarrow}$ scattering amplitude $\mathcal{A}(s,t)$

Very strong coupling expansion

- To exhibit $SL(2,\mathbb{Z})$ in $\mathcal{N}=4$ SYM correlators in holographic regime, need **"very strong coupling" limit**:

$$N \to \infty \qquad \text{with} \qquad g_{\text{YM}} \text{ fixed}\,.$$

(in the 't Hooft limit, g_{YM} is necessarily small)

- With normalization $\langle S(\vec{x}, Y_1) S(0, Y_2)\rangle = \frac{(N^2-1)(Y_1\cdot Y_2)^2}{4|\vec{x}|^4}$, schematically (suppressed position and R-symm)

$$\langle SSSS\rangle_{\text{conn}} = N^2 + f(\tau,\bar\tau)\sqrt{N} + N^0 + \frac{g(\tau,\bar\tau)}{\sqrt{N}} + \frac{h(\tau,\bar\tau)}{N} + O(N^{-3/2})\,.$$

- In terms of Witten diagrams:

N^2 : tree level SUGRA

$\sqrt{N}$: tree level R^4

N^0 : regularized 1-loop SUGRA

$N^{-1/2}$: tree level $D^4 R^4$

N^{-1} : tree level $D^6 R^4$

etc.

(Recall $L/\ell_s \sim \lambda^{1/4}$ and $\lambda = g_{\text{YM}}^2 N \quad \Longrightarrow \quad n$ derivs $\propto N^{-n/4}$)

Very strong coupling expansion

- Highlighted terms completely determined:

  N^2: [D'Hoker, Freedman, Mathur, Matusis, Rastelli '99; Arutyunov, Frolov '00; Dolan, Osborn '04; Rastelli, Zhou '16]

  $\sqrt{N}$: [Goncalves '14; Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19]

  N^0: [Alday, Bissi, Perlmutter, Heslop, Paul, ...; Chester '19]

  $N^{-\frac{1}{2}}$, N^{-1}: [Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19; Chester, SSP '20; Chester, Green, SSP, Wang, Wen '20]
- Procedure:
  - SUSY Ward identities, crossing symmetry, analyticity in Mellin space determine the position dependence at each order in $1/N$, up to a few undetermined constants.
  - SUSic localization: can compute **integrated** 4-pt functions, which can be used to determine some of the constants
  - flat space limit can be used to fix the large s, t behavior at each order in $1/N$.

Very strong coupling expansion

- Highlighted terms completely determined:
  N^2: [D'Hoker, Freedman, Mathur, Matusis, Rastelli '99; Arutyunov, Frolov '00; Dolan, Osborn '04; Rastelli, Zhou '16]
  $\sqrt{N}$: [Goncalves '14; Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19]
  N^0: [Alday, Bissi, Perlmutter, Heslop, Paul, ...; Chester '19]
  $N^{-\frac{1}{2}}$, N^{-1}: [Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19; Chester, SSP '20; Chester, Green, SSP, Wang, Wen '20]
- Procedure:
  - SUSY Ward identities, crossing symmetry, analyticity in Mellin space determine the position dependence at each order in $1/N$, up to a few undetermined constants.
  - SUSic localization: can compute **integrated** 4-pt functions, which can be used to determine some of the constants
  - flat space limit can be used to fix the large s, t behavior at each order in $1/N$.

Very strong coupling expansion

- Highlighted terms completely determined:

  N^2: [D'Hoker, Freedman, Mathur, Matusis, Rastelli '99; Arutyunov, Frolov '00; Dolan, Osborn '04; Rastelli, Zhou '16]

  $\sqrt{N}$: [Goncalves '14; Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19]

  N^0: [Alday, Bissi, Perlmutter, Heslop, Paul, . . . ; Chester '19]

  $N^{-\frac{1}{2}}$, N^{-1}: [Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19; Chester, SSP '20; Chester, Green, SSP, Wang, Wen '20]

- Procedure:
  - SUSY Ward identities, crossing symmetry, analyticity in Mellin space determine the position dependence at each order in $1/N$, up to a few undetermined constants.
  - SUSic localization: can compute **integrated** 4-pt functions, which can be used to determine some of the constants
  - flat space limit can be used to fix the large s, t behavior at each order in $1/N$.

Very strong coupling expansion

- Highlighted terms completely determined:

  N^2: [D'Hoker, Freedman, Mathur, Matusis, Rastelli '99; Arutyunov, Frolov '00; Dolan, Osborn '04; Rastelli, Zhou '16]

  $\sqrt{N}$: [Goncalves '14; Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19]

  N^0: [Alday, Bissi, Perlmutter, Heslop, Paul, . . . ; Chester '19]

  $N^{-\frac{1}{2}}$, N^{-1}: [Binder, Chester, SSP, Wang '19; Chester, Green, SSP, Wang, Wen '19; Chester, SSP '20; Chester, Green, SSP, Wang, Wen '20]

- Procedure:
  - SUSY Ward identities, crossing symmetry, analyticity in Mellin space determine the position dependence at each order in $1/N$, up to a few undetermined constants.
  - SUSic localization: can compute **integrated** 4-pt functions, which can be used to determine some of the constants
  - flat space limit can be used to fix the large s, t behavior at each order in $1/N$.

$\langle SSSS \rangle$ in very strong coupling expansion

- Can relate $T(U,V)$ to the Mellin transform $\mathcal{M}(s,t)$:

$$T(U,V) = \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{u}{2}-3} \Gamma^2\left(2-\frac{s}{2}\right)\Gamma^2\left(2-\frac{t}{2}\right)\Gamma^2\left(2-\frac{u}{2}\right)\mathcal{M}(s,t)\,,$$

where $u \equiv 4 - s - t$.

- Result:

$$\mathcal{M}(s,t) = \frac{2N^2}{(s-2)(t-2)(u-2)} + \frac{15E(\frac{3}{2},\tau,\bar\tau)}{8\sqrt{\pi^3}}\sqrt{N} + \mathcal{M}_{\text{1-loop}}(s,t)$$
$$+ \frac{315E(\frac{5}{2},\tau,\bar\tau)}{128\sqrt{\pi^5}}\left[s^2+t^2+u^2-3\right]\frac{1}{\sqrt{N}}$$
$$+ \frac{315\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\tau,\bar\tau)}{32\pi^3}\left[stu-\frac{1}{4}(s^2+t^2+u^2)-4\right]\frac{1}{N} + O(N^{-3/2})\,.$$

- The modular invariants $E(\frac{3}{2},\tau,\bar\tau)$, $E(\frac{5}{2},\tau,\bar\tau)$, $\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\tau,\bar\tau)$ (defined on next slide) are known from superstring scattering amplitudes [Green, Gutperle '97; Green, Sethi '98; Wang, Yin '15, ...] .

$\langle SSSS \rangle$ in very strong coupling expansion

- Can relate $T(U,V)$ to the Mellin transform $\mathcal{M}(s,t)$:

$$T(U,V) = \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{u}{2}-3} \Gamma^2\left(2-\frac{s}{2}\right)\Gamma^2\left(2-\frac{t}{2}\right)\Gamma^2\left(2-\frac{u}{2}\right)\mathcal{M}(s,t)\,,$$

where $u \equiv 4 - s - t$.

- Result:

$$\mathcal{M}(s,t) = \frac{2N^2}{(s-2)(t-2)(u-2)} + \frac{15E(\frac{3}{2},\tau,\bar\tau)}{8\sqrt{\pi^3}}\sqrt{N} + \mathcal{M}_{\text{1-loop}}(s,t)$$
$$+ \frac{315E(\frac{5}{2},\tau,\bar\tau)}{128\sqrt{\pi^5}}\left[s^2+t^2+u^2-3\right]\frac{1}{\sqrt{N}}$$
$$+ \frac{315\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\tau,\bar\tau)}{32\pi^3}\left[stu - \frac{1}{4}(s^2+t^2+u^2)-4\right]\frac{1}{N} + O(N^{-3/2})\,.$$

- The modular invariants $E(\frac{3}{2},\tau,\bar\tau)$, $E(\frac{5}{2},\tau,\bar\tau)$, $\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\tau,\bar\tau)$ (defined on next slide) are known from superstring scattering amplitudes [Green, Gutperle '97; Green, Sethi '98; Wang, Yin '15, . . .] .

Eisenstein series

- The non-holomorphic Eisenstein series $E(r, \tau, \bar{\tau})$ are modular-invariant functions defined as ($\tau = \tau_1 + i\tau_2$):

$$E(s, \tau, \bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}} .$$

They are eigenfunctions of the hyperbolic Laplacian

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - s(s-1)\right) E(s, \tau, \bar{\tau}) = 0 .$$

normalized s.t. $E(s, \tau, \bar{\tau}) \sim 2\zeta(2s)\tau_2^s$ at large τ_2.

- Expanded at large τ_2 (i.e. small g_{YM}), they have a *finite number* of perturbative contributions + non-perturbative contributions. Example for $\tau_1 = 0$ and $\tau_2 = 4\pi/g_{\text{YM}}^2$:

$$E(\tfrac{3}{2}, \tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\tau_2^{1/2}} + e^{-2\pi\tau_2}\left[4\pi + \frac{3}{4\tau_2} + \cdots\right] + e^{-4\pi\tau_2}\left[\cdots\right] + \cdots$$

Eisenstein series

- The non-holomorphic Eisenstein series $E(r, \tau, \bar{\tau})$ are modular-invariant functions defined as ($\tau = \tau_1 + i\tau_2$):

$$E(s, \tau, \bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}} .$$

They are eigenfunctions of the hyperbolic Laplacian

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - s(s-1)\right) E(s, \tau, \bar{\tau}) = 0 .$$

normalized s.t. $E(s, \tau, \bar{\tau}) \sim 2\zeta(2s)\tau_2^s$ at large τ_2.

- Expanded at large τ_2 (i.e. small g_{YM}), they have a *finite number* of perturbative contributions + non-perturbative contributions.
Example for $\tau_1 = 0$ and $\tau_2 = 4\pi/g_{\text{YM}}^2$:

$$E(\tfrac{3}{2}, \tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\tau_2^{1/2}} + e^{-2\pi\tau_2}\left[4\pi + \frac{3}{4\tau_2} + \cdots\right] + e^{-4\pi\tau_2}\left[\cdots\right] + \cdots$$

Eisenstein series

- The non-holomorphic Eisenstein series $E(r, \tau, \bar{\tau})$ are modular-invariant functions defined as ($\tau = \tau_1 + i\tau_2$):

$$E(s, \tau, \bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}} .$$

  They are eigenfunctions of the hyperbolic Laplacian

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - s(s-1)\right) E(s, \tau, \bar{\tau}) = 0 .$$

  normalized s.t. $E(s, \tau, \bar{\tau}) \sim 2\zeta(2s)\tau_2^s$ at large τ_2.

- Expanded at large τ_2 (i.e. small $g_{\rm YM}$), they have a *finite number* of perturbative contributions + non-perturbative contributions. Example for $\tau_1 = 0$ and $\tau_2 = 4\pi/g_{\rm YM}^2$:

$$E(\tfrac{3}{2}, \tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\tau_2^{1/2}} + e^{-2\pi\tau_2}\left[4\pi + \frac{3}{4\tau_2} + \cdots\right] + e^{-4\pi\tau_2}\left[\cdots\right] + \cdots$$

Generalized Eisenstein series

- The coefficient of $1/N$ (contact $D^6 R^4$ interaction) is a **"generalized Eisenstein series"** $\mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})$.
- For general $r, s_1, s_2 \geq 0$, with $r \geq s_1 + s_2$, define $\mathcal{E}(r, s_1, s_2, \tau, \bar{\tau})$ as the unique $SL(2, \mathbb{Z})$-invariant solution of inhomogeneus eigenvalue equation

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - r(r+1)\right) \mathcal{E}(r, s_1, s_2, \tau, \bar{\tau}) = -E(s_1, \tau, \bar{\tau}) E(s_2, \tau, \bar{\tau}) .$$

  that grows slower than τ_2^{r+1} at large τ_2.
- Also has a finite number of perturbative terms:

$$\mathcal{E} = a_1 \tau_2^{s_1+s_2} + a_2 \tau_2^{1+s_1-s_2} + a_3 \tau_2^{1+s_2-s_1} + a_4 \tau_2^{2-s_1-s_2} + \beta_r \tau_2^{-r} + \text{non-pert.}$$

Generalized Eisenstein series

- The coefficient of $1/N$ (contact $D^6 R^4$ interaction) is a **"generalized Eisenstein series"** $\mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})$.
- For general $r, s_1, s_2 \geq 0$, with $r \geq s_1 + s_2$, define $\mathcal{E}(r, s_1, s_2, \tau, \bar{\tau})$ as the unique $SL(2, \mathbb{Z})$-invariant solution of inhomogeneus eigenvalue equation

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - r(r+1)\right) \mathcal{E}(r, s_1, s_2, \tau, \bar{\tau}) = -E(s_1, \tau, \bar{\tau}) E(s_2, \tau, \bar{\tau}) \, .$$

  that grows slower than τ_2^{r+1} at large τ_2.
- Also has a finite number of perturbative terms:

$$\mathcal{E} = a_1 \tau_2^{s_1+s_2} + a_2 \tau_2^{1+s_1-s_2} + a_3 \tau_2^{1+s_2-s_1} + a_4 \tau_2^{2-s_1-s_2} + \beta_r \tau_2^{-r} + \text{non-pert.}$$

Generalized Eisenstein series

- The coefficient of $1/N$ (contact D^6R^4 interaction) is a **"generalized Eisenstein series"** $\mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})$.
- For general $r, s_1, s_2 \geq 0$, with $r \geq s_1 + s_2$, define $\mathcal{E}(r, s_1, s_2, \tau, \bar{\tau})$ as the unique $SL(2, \mathbb{Z})$-invariant solution of inhomogeneus eigenvalue equation

$$\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - r(r+1)\right) \mathcal{E}(r, s_1, s_2, \tau, \bar{\tau}) = -E(s_1, \tau, \bar{\tau}) E(s_2, \tau, \bar{\tau}) \,.$$

  that grows slower than τ_2^{r+1} at large τ_2.
- Also has a finite number of perturbative terms:

$$\mathcal{E} = a_1 \tau_2^{s_1+s_2} + a_2 \tau_2^{1+s_1-s_2} + a_3 \tau_2^{1+s_2-s_1} + a_4 \tau_2^{2-s_1-s_2} + \beta_r \tau_2^{-r} + \text{non-pert.}$$

Supersymmetric localization results

- Specific setup [Pestun '07, Hama, Hosomichi '12] :

$\mathcal{N} = 4$ SYM on $\mathbb{R}^4$	$\longrightarrow$	$\mathcal{N} = 4$ SYM on S^4	$\longrightarrow$	SUSY-preserving deformation by mass parameter m and squashing parameter b

  $(b, m) = (1, 0)$ corresp. to $\mathcal{N} = 4$ SYM on round sphere.

- Can compute $Z(m, b, \tau, \bar{\tau})$ exactly [Pestun '07, Hama, Hosomichi '12] .
- Two messages:
  - Derivatives of $Z(m, b, \tau, \bar{\tau})$ evaluated at $(b, m) = (1, 0)$ give integrated correlators in $\mathcal{N} = 4$ SYM. (Take 4 ders for 4-pt function.)
  - Can obtain Eisenstein and generalized Eisenstein series by expanding in the very strong coupling limit.

Supersymmetric localization results

- Specific setup [Pestun '07, Hama, Hosomichi '12] :

  $\mathcal{N}=4$ SYM on $\mathbb{R}^4$ $\longrightarrow$ $\mathcal{N}=4$ SYM on S^4 $\longrightarrow$ SUSY-preserving deformation by mass parameter m and squashing parameter b

  $(b,m)=(1,0)$ corresp. to $\mathcal{N}=4$ SYM on round sphere.

- Can compute $Z(m,b,\tau,\bar{\tau})$ exactly [Pestun '07, Hama, Hosomichi '12] .
- Two messages:
  - Derivatives of $Z(m,b,\tau,\bar{\tau})$ evaluated at $(b,m)=(1,0)$ give integrated correlators in $\mathcal{N}=4$ SYM. (Take 4 ders for 4-pt function.)
  - Can obtain Eisenstein and generalized Eisenstein series by expanding in the very strong coupling limit.

Supersymmetric localization results

- Specific setup [Pestun '07, Hama, Hosomichi '12] :

$\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ $\longrightarrow$ $\mathcal{N} = 4$ SYM on S^4 $\longrightarrow$ SUSY-preserving deformation by mass parameter m and squashing parameter b

$(b, m) = (1, 0)$ corresp. to $\mathcal{N} = 4$ SYM on round sphere.

- Can compute $Z(m, b, \tau, \bar{\tau})$ exactly [Pestun '07, Hama, Hosomichi '12] .
- Two messages:
  - Derivatives of $Z(m, b, \tau, \bar{\tau})$ evaluated at $(b, m) = (1, 0)$ give integrated correlators in $\mathcal{N} = 4$ SYM. (Take 4 ders for 4-pt function.)
  - Can obtain Eisenstein and generalized Eisenstein series by expanding in the very strong coupling limit.

Integrated correlators

For $b = 1$ (round sphere), we derived [Binder, Chester, SSP, Wang '19] :

$$\tau_2^2 \partial_m^2 \partial_\tau \partial_{\bar\tau} \log Z \Big|_{m=0} = -\frac{8}{\pi} \int dr\, d\theta\, r^3 \sin^2\theta \, \frac{\mathcal{T}(U,V)}{U^2} \Bigg|_{\substack{U=1+r^2-2r\cos\theta \\ V=r^2}}$$

and [Chester, SSP '20]

$$\partial_m^4 \log Z \Big|_{m=0} = 12\zeta(3)(N^2-1) + \frac{32}{\pi} \int dr\, d\theta\, r^3 \sin^2\theta \,(1+U+V) \times \bar{D}_{1111}(U,V) \frac{\mathcal{T}(U,V)}{U^2} \Bigg|_{\substack{U=1+r^2-2r\cos\theta \\ V=r^2}}$$

where $\bar{D}_{1111}(U,V)$ is $\propto \vec{x}_{13}^2 \vec{x}_{24}^2$ times the Witten diagram for the contact interaction of four $\Delta = 1$ scalar operators.

- Similar relations for derivs w.r.t. squashing not available yet.

Matrix model for $\mathcal{N} = 2^*$ partition function

Pestun computed the S^4 partition function of the $\mathcal{N} = 2^*$ theory [Pestun '07; Russo, Zarembo '13] :

$$Z = \int d^{N-1}a \, \frac{\prod_{i<j}(a_i - a_j)^2 H^2(a_i - a_j)}{H(m)^{N-1} \prod_{i \neq j} H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2} |Z_{\text{inst}}(m, \tau)|^2$$

where H is the product of two Barnes G-functions, and Z_{inst} represents the contribution of instantons (Nekrasov partition function) localized at the N and S poles of S^4.

- Perturbatively in $1/N$ and $1/\lambda$ (in the 't Hooft limit) one can ignore the instanton contributions, but they are very important in the very strong coupling limit.
- Taking derivatives w.r.t. m, τ, $\bar{\tau}$ evaluated at $m = 0 \implies$ insertions in the Gaussian matrix model.

Matrix model for $\mathcal{N} = 2^*$ partition function

Pestun computed the S^4 partition function of the $\mathcal{N} = 2^*$ theory [Pestun '07; Russo, Zarembo '13] :

$$Z = \int d^{N-1}a \, \frac{\prod_{i<j}(a_i - a_j)^2 H^2(a_i - a_j)}{H(m)^{N-1} \prod_{i \neq j} H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2} |Z_{\text{inst}}(m, \tau)|^2$$

where H is the product of two Barnes G-functions, and Z_{inst} represents the contribution of instantons (Nekrasov partition function) localized at the N and S poles of S^4.

- Perturbatively in $1/N$ and $1/\lambda$ (in the 't Hooft limit) one can ignore the instanton contributions, but they are very important in the very strong coupling limit.
- Taking derivatives w.r.t. m, τ, $\bar{\tau}$ evaluated at $m = 0 \Longrightarrow$ insertions in the Gaussian matrix model.

Matrix model for $\mathcal{N} = 2^*$ partition function

Pestun computed the S^4 partition function of the $\mathcal{N} = 2^*$ theory [Pestun '07; Russo, Zarembo '13] :

$$Z = \int d^{N-1}a \, \frac{\prod_{i<j}(a_i - a_j)^2 H^2(a_i - a_j)}{H(m)^{N-1} \prod_{i \neq j} H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2} |Z_{\text{inst}}(m, \tau)|^2$$

where H is the product of two Barnes G-functions, and Z_{inst} represents the contribution of instantons (Nekrasov partition function) localized at the N and S poles of S^4.

- Perturbatively in $1/N$ and $1/\lambda$ (in the 't Hooft limit) one can ignore the instanton contributions, but they are very important in the very strong coupling limit.
- Taking derivatives w.r.t. m, τ, $\bar{\tau}$ evaluated at $m = 0 \implies$ insertions in the Gaussian matrix model.

Derivatives of the S^4 partition function

- For $\tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\big|_{m=0}$, the mass derivatives either both act on H ($\Longrightarrow$ pert terms, harder to compute), both act on Z_{inst}, or both act on Z^*_{inst} ($\Longrightarrow$ non-pert terms, easier to compute):

$$\tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\big|_{m=0} = \frac{N^2}{4} - \frac{3\sqrt{N}}{2^4\,\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + \frac{45}{2^8\sqrt{N}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau)$$
$$+ \frac{1}{N^{\frac{3}{2}}}\left[-\frac{39}{2^{13}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + \frac{4725}{2^{15}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau)\right]$$
$$+ \frac{1}{N^{\frac{5}{2}}}\left[-\frac{1125}{2^{16}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) + \frac{99225}{2^{18}\pi^{\frac{9}{2}}} E(\tfrac{9}{2},\tau,\bar\tau)\right]$$
$$+ \frac{1}{N^{\frac{7}{2}}}\left[\frac{4599}{2^{22}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) - \frac{2811375}{2^{25}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau) + \frac{245581875}{2^{27}\pi^{\frac{11}{2}}} E(\tfrac{11}{2},\tau,\bar\tau)\right]$$
$$+ O(N^{-\frac{9}{2}}),$$

- No integer powers of $1/\sqrt{N}$. (Probably due to SUSY.)
- Eisenstein series all the way! (No generalized Eisensteins)

Derivatives of the S^4 partition function

- For $\tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\big|_{m=0}$, the mass derivatives either both act on H ($\Longrightarrow$ pert terms, harder to compute), both act on Z_{inst}, or both act on Z_{inst}^* ($\Longrightarrow$ non-pert terms, easier to compute):

$$\begin{aligned}
\tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\big|_{m=0} &= \frac{N^2}{4} - \frac{3\sqrt{N}}{2^4\,\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + \frac{45}{2^8\sqrt{N}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) \\
&+ \frac{1}{N^{\frac{3}{2}}}\left[-\frac{39}{2^{13}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + \frac{4725}{2^{15}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^{\frac{5}{2}}}\left[-\frac{1125}{2^{16}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) + \frac{99225}{2^{18}\pi^{\frac{9}{2}}} E(\tfrac{9}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^{\frac{7}{2}}}\left[\frac{4599}{2^{22}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) - \frac{2811375}{2^{25}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau) + \frac{245581875}{2^{27}\pi^{\frac{11}{2}}} E(\tfrac{11}{2},\tau,\bar\tau)\right] \\
&+ O(N^{-\frac{9}{2}}) \,,
\end{aligned}$$

- No integer powers of $1/\sqrt{N}$. (Probably due to SUSY.)
- Eisenstein series all the way! (No generalized Eisensteins)

Derivatives of the S^4 partition function

- $\partial_m^4 \log Z\big|_{m=0}$ is much harder b/c the m derivatives can act on different factors. Evidence for [Chester, Green, SSP, Wang, Wen '20] :

$$
\begin{aligned}
\partial_m^4 \log Z\big|_{m=0} &= 6N^2 + \frac{6\sqrt{N}}{\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + C_0 - \frac{9}{2\sqrt{N}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) \\
&- \frac{27}{2^3\pi^3 N}\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) + \frac{1}{N^{\frac{3}{2}}}\left[\frac{117}{2^8\pi^{\frac{3}{2}}}E(\tfrac{3}{2},\tau,\bar\tau) - \frac{3375}{2^{10}\pi^{\frac{7}{2}}}E(\tfrac{7}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^2}\left[C_1 + \frac{14175}{704\pi^4}\mathcal{E}(6,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau) - \frac{1215}{88\pi^4}\mathcal{E}(4,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^{\frac{5}{2}}}\left[\frac{675}{2^{10}\pi^{\frac{5}{2}}}E(\tfrac{5}{2},\tau,\bar\tau) - \frac{33075}{2^{12}\pi^{\frac{9}{2}}}E(\tfrac{9}{2},\tau,\bar\tau)\right] \\
&+ \frac{1}{N^3}\sum_{r=3,5,7,9}\left[\alpha_r\mathcal{E}(r,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) + \beta_r\mathcal{E}(r,\tfrac{5}{2},\tfrac{5}{2},\tau,\bar\tau) + \gamma_r\mathcal{E}(r,\tfrac{7}{2},\tfrac{3}{2},\tau,\bar\tau)\right] \\
&+ O(N^{-\frac{7}{2}}),
\end{aligned}
$$

- Eisenstein series at odd orders in $N^{-1/2}$, generalized Eisenstein series at even orders in $N^{-1/2}$.

Derivatives of the S^4 partition function

- $\partial_m^4 \log Z\big|_{m=0}$ is much harder b/c the m derivatives can act on different factors. Evidence for [Chester, Green, SSP, Wang, Wen '20] :

$$\begin{aligned}\partial_m^4 \log Z\big|_{m=0} &= 6N^2 + \frac{6\sqrt{N}}{\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + C_0 - \frac{9}{2\sqrt{N}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) \\ &- \frac{27}{2^3\pi^3 N}\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) + \frac{1}{N^{\frac{3}{2}}}\left[\frac{117}{2^8\pi^{\frac{3}{2}}}E(\tfrac{3}{2},\tau,\bar\tau) - \frac{3375}{2^{10}\pi^{\frac{7}{2}}}E(\tfrac{7}{2},\tau,\bar\tau)\right] \\ &+ \frac{1}{N^2}\left[C_1 + \frac{14175}{704\pi^4}\mathcal{E}(6,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau) - \frac{1215}{88\pi^4}\mathcal{E}(4,\tfrac{5}{2},\tfrac{3}{2},\tau,\bar\tau)\right] \\ &+ \frac{1}{N^{\frac{5}{2}}}\left[\frac{675}{2^{10}\pi^{\frac{5}{2}}}E(\tfrac{5}{2},\tau,\bar\tau) - \frac{33075}{2^{12}\pi^{\frac{9}{2}}}E(\tfrac{9}{2},\tau,\bar\tau)\right] \\ &+ \frac{1}{N^3}\sum_{r=3,5,7,9}\left[\alpha_r\mathcal{E}(r,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) + \beta_r\mathcal{E}(r,\tfrac{5}{2},\tfrac{5}{2},\tau,\bar\tau) + \gamma_r\mathcal{E}(r,\tfrac{7}{2},\tfrac{3}{2},\tau,\bar\tau)\right] \\ &+ O(N^{-\frac{7}{2}}),\end{aligned}$$

- Eisenstein series at odd orders in $N^{-1/2}$, generalized Eisenstein series at even orders in $N^{-1/2}$.

Comments

- Up to $1/N$, the two integrated correlators + flat space limit are sufficient to determine the $\langle SSSS \rangle$ at separated points (including position dependence).
- Coeffs of $\sqrt{N}$ and $1/\sqrt{N}$ can be determined entirely from localization, and they agree in the flat space limit with superstring amplitude $\Longrightarrow$ precision test of AdS/CFT beyond SUGRA.
- CFT data obtained from $\langle SSSS \rangle$ has the same structure of the expansion in the very strong coupling limit. For example, anomalous dimensions of double trace op $S_{IJ}S^{IJ}$

$$\gamma = \frac{a_1}{N^2} + \frac{a_2 E(\frac{3}{2}, \tau, \bar{\tau})}{N^{7/2}} + \frac{a_3}{N^4} + \frac{a_4 E(\frac{5}{2}, \tau, \bar{\tau})}{N^{9/2}} + \frac{a_5 \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{N^5} + \cdots$$

- Relation to 't Hooft expansion: the terms $N^2 (g_{\text{YM}}^2 N)^{-k}$ from the very strong coupling limit recombine into $N^2 f(\lambda)$, etc.

Comments

- Up to $1/N$, the two integrated correlators + flat space limit are sufficient to determine the $\langle SSSS \rangle$ at separated points (including position dependence).
- Coeffs of $\sqrt{N}$ and $1/\sqrt{N}$ can be determined entirely from localization, and they agree in the flat space limit with superstring amplitude $\Longrightarrow$ precision test of AdS/CFT beyond SUGRA.
- CFT data obtained from $\langle SSSS \rangle$ has the same structure of the expansion in the very strong coupling limit. For example, anomalous dimensions of double trace op $S_{IJ}S^{IJ}$

$$\gamma = \frac{a_1}{N^2} + \frac{a_2 E(\frac{3}{2}, \tau, \bar{\tau})}{N^{7/2}} + \frac{a_3}{N^4} + \frac{a_4 E(\frac{5}{2}, \tau, \bar{\tau})}{N^{9/2}} + \frac{a_5 \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{N^5} + \cdots$$

- Relation to 't Hooft expansion: the terms $N^2(g_{\text{YM}}^2 N)^{-k}$ from the very strong coupling limit recombine into $N^2 f(\lambda)$, etc.

Comments

- Up to $1/N$, the two integrated correlators + flat space limit are sufficient to determine the $\langle SSSS \rangle$ at separated points (including position dependence).
- Coeffs of $\sqrt{N}$ and $1/\sqrt{N}$ can be determined entirely from localization, and they agree in the flat space limit with superstring amplitude $\Longrightarrow$ precision test of AdS/CFT beyond SUGRA.
- CFT data obtained from $\langle SSSS \rangle$ has the same structure of the expansion in the very strong coupling limit. For example, anomalous dimensions of double trace op $S_{IJ}S^{IJ}$

$$\gamma = \frac{a_1}{N^2} + \frac{a_2 E(\frac{3}{2}, \tau, \bar{\tau})}{N^{7/2}} + \frac{a_3}{N^4} + \frac{a_4 E(\frac{5}{2}, \tau, \bar{\tau})}{N^{9/2}} + \frac{a_5 \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{N^5} + \cdots$$

- Relation to 't Hooft expansion: the terms $N^2(g_{YM}^2 N)^{-k}$ from the very strong coupling limit recombine into $N^2 f(\lambda)$, etc.

Comments

- Up to $1/N$, the two integrated correlators + flat space limit are sufficient to determine the $\langle SSSS \rangle$ at separated points (including position dependence).
- Coeffs of $\sqrt{N}$ and $1/\sqrt{N}$ can be determined entirely from localization, and they agree in the flat space limit with superstring amplitude $\Longrightarrow$ precision test of AdS/CFT beyond SUGRA.
- CFT data obtained from $\langle SSSS \rangle$ has the same structure of the expansion in the very strong coupling limit. For example, anomalous dimensions of double trace op $S_{IJ}S^{IJ}$

$$\gamma = \frac{a_1}{N^2} + \frac{a_2 E(\frac{3}{2}, \tau, \bar{\tau})}{N^{7/2}} + \frac{a_3}{N^4} + \frac{a_4 E(\frac{5}{2}, \tau, \bar{\tau})}{N^{9/2}} + \frac{a_5 \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{N^5} + \cdots$$

- Relation to 't Hooft expansion: the terms $N^2 (g_{\text{YM}}^2 N)^{-k}$ from the very strong coupling limit recombine into $N^2 f(\lambda)$, etc.

Conclusion

- A combination of techniques (supersymmetric localization, analytic bootstrap in Mellin space) can be used to study holographic correlators beyond the SUGRA approximation in $\mathcal{N} = 4$ SYM (and other theories).
- In the very strong coupling limit, $\mathcal{N} = 4$ SYM correlators can be written in terms of Eisenstein series and generalized Eisenstein.

For the future:

- Connection with integrability?
- Use integrated constraints away from strong coupling limit, e.g. in numerical bootstrap (ongoing).
- Convergence / resummation of $1/N$ expansion.

Conclusion

- A combination of techniques (supersymmetric localization, analytic bootstrap in Mellin space) can be used to study holographic correlators beyond the SUGRA approximation in $\mathcal{N} = 4$ SYM (and other theories).
- In the very strong coupling limit, $\mathcal{N} = 4$ SYM correlators can be written in terms of Eisenstein series and generalized Eisenstein.

For the future:

- Connection with integrability?
- Use integrated constraints away from strong coupling limit, e.g. in numerical bootstrap (ongoing).
- Convergence / resummation of $1/N$ expansion.