Modular invariants in strongly-coupled $\mathcal{N} = 4$ SYM

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Based on:

- arXiv:1902.06263 with D. Binder, S. Chester, and Y. Wang
- arXiv:1912.13365 with S. Chester, M. Green, Y. Wang, and C. Wen
- arXiv:2008.02713 with S. Chester, M. Green, Y. Wang, and C. Wen

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This talk is on correlators of local operators in 4d $\mathcal{N} = 4$ SYM.

Approaches:
- integrability
- weak coupling expansion
- holography (large $N$ 't Hooft limit)
- numerical bootstrap
- analytic bootstrap
- supersymmetric localization

Most studied: 1/2-BPS single trace scalar operators $O_p$ in $[0p0]$ of $SU(4)_R$ R-symmetry and dimension $\Delta = p$. 
This talk: 4-pt function of $O_2 \equiv S_{IJ} = \text{tr}(X_I X_J - \frac{\delta_{IJ}}{6} X_K X^K)$, i.e. the 20' single trace scalar op. of dimension $\Delta = 2$.

$$\langle S(\vec{x}_1, Y_1) S(\vec{x}_2, Y_2) S(\vec{x}_3, Y_3) S(\vec{x}_4, Y_4) \rangle, \quad S(\vec{x}, Y) \equiv Y^I Y^J S_{IJ}(\vec{x})$$

S is in the same superconf. multiplet as the stress-energy tensor.

In fact, SUSY Ward identities $\implies$ all 4-pt correlators of stress tensor multiplet operators are determined by a single function $\mathcal{T}(U, V)$ [Belitsky, Hohenegger, Korchemsky, Sokatchev '14]. Schematically:

$$\langle SSSS \rangle = \text{(free part)} + \frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2 (V + UV \sigma^2 + U \tau^2 + \cdots)}{\vec{x}_{12}^4 \vec{x}_{34}^4} \mathcal{T}(U, V)$$

where $U \equiv \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, $V \equiv \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$, and $\sigma \equiv \frac{Y_1 \cdot Y_3 Y_2 \cdot Y_4}{Y_1 \cdot Y_2 Y_3 \cdot Y_4}$ and $\tau \equiv \frac{Y_1 \cdot Y_4 Y_2 \cdot Y_3}{Y_1 \cdot Y_2 Y_3 \cdot Y_4}$. 
This talk: 4-pt function of $\mathcal{O}_2 \equiv S_{IJ} = \text{tr}(X_I X_J - \frac{\delta_{IJ}}{6} X_K X^K)$, i.e. the $20'$ single trace scalar op. of dimension $\Delta = 2$.

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**SL(2, ℤ) in \( \mathcal{N} = 4 \) SYM**

- Define complexified gauge coupling

  \[
  \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}, \quad \bar{\tau} = \frac{\theta}{2\pi} - \frac{4\pi i}{g_{\text{YM}}^2}
  \]

- \( \mathcal{N} = 4 \) SYM w/ gauge algebra \( \mathfrak{su}(N) \) exhibits \( SL(2, \mathbb{Z}) \) duality:

  \[
  \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.
  \]

- **S generator:** \( \tau \rightarrow -1/\tau \)

- **T generator:** \( \tau \rightarrow \tau + 1, \) or \( \theta \rightarrow \theta + 2\pi. \)

- 1/2-BPS ops, and in particular the 20\(^l\) op. S and its 4-point function, are invariant under \( SL(2, \mathbb{Z}). \)
**SL(2, ℤ) in superstring theory**

- Related topic: $SL(2, ℤ)$ in type IIB superstring theory, e.g. for superstring scattering amplitudes

\[ \tau = C + ie^{-i\phi}. \]

- Example: $R^4$ interaction $\rightarrow$ scattering amplitude $\propto$ non-holomorphic Eisenstein series $E\left(\frac{3}{2}, \tau, \bar{\tau}\right)$ (more on this later).

- Connection:

  \[
  \text{CFT} \quad \text{4-point on function} \quad \xleftrightarrow{\text{AdS/CFT}} \quad \text{scattering amplitude in } AdS_5 \times S^5 \quad \xrightarrow{\text{flat space limit}} \quad 10d \text{ flat space scattering amplitude}
  \]

**Flat space limit:**

Mellin amplitude $\mathcal{M}(s, t) \xrightarrow{\text{large } s, t} \text{scattering amplitude } \mathcal{A}(s, t)$
Very strong coupling expansion

To exhibit $SL(2,\mathbb{Z})$ in $\mathcal{N} = 4$ SYM correlators in holographic regime, need “very strong coupling” limit:

$$N \to \infty \quad \text{with} \quad g_{YM} \text{ fixed}.$$  

(in the ’t Hooft limit, $g_{YM}$ is necessarily small)

With normalization

$$\left\langle S(\vec{x}, Y_1)S(0, Y_2) \right\rangle = \frac{(N^2-1)(Y_1 \cdot Y_2)^2}{4|\vec{x}|^4},$$

schematically (suppressed position and R-symm)

$$\left\langle SSSS \right\rangle_{\text{conn}} = N^2 + f(\tau, \bar{\tau})\sqrt{N} + N^0 + \frac{g(\tau, \bar{\tau})}{\sqrt{N}} + \frac{h(\tau, \bar{\tau})}{N} + O(N^{-3/2}).$$

In terms of Witten diagrams:

- $N^2$: tree level SUGRA
- $\sqrt{N}$: tree level $R^4$
- $N^0$: regularized 1-loop SUGRA
- $N^{-1/2}$: tree level $D^4R^4$
- $N^{-1}$: tree level $D^6R^4$
- $N^{-n}$: $n$ derivs $\propto N^{-n/4}$

(Recall $L/\ell_s \sim \lambda^{1/4}$ and $\lambda = g_{YM}^2 N$)
**Very strong coupling expansion**

- **Highlighted terms completely determined:**
  - $N^2$: [D’Hoker, Freedman, Mathur, Matusis, Rastelli ’99; Arutyunov, Frolov ’00; Dolan, Osborn ’04; Rastelli, Zhou ’16]
  - $\sqrt{N}$: [Goncalves ’14; Binder, Chester, SSP, Wang ’19; Chester, Green, SSP, Wang, Wen ’19]
  - $N^0$: [Alday, Bissi, Perlmutter, Heslop, Paul, . . . ; Chester ’19]
  - $N^{-\frac{1}{2}}$, $N^{-1}$: [Binder, Chester, SSP, Wang ’19; Chester, Green, SSP, Wang, Wen ’19; Chester, SSP ’20; Chester, Green, SSP, Wang, Wen ’20]

- **Procedure:**
  - SUSY Ward identities, crossing symmetry, analyticity in Mellin space determine the position dependence at each order in $1/N$, up to a few undetermined constants.
  - SUSic localization: can compute **integrated** 4-pt functions, which can be used to determine some of the constants.
  - Flat space limit can be used to fix the large $s$, $t$ behavior at each order in $1/N$. 
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\[ \langle \text{SSSS} \rangle \text{ in very strong coupling expansion} \]

- Can relate \( T(U, V) \) to the Mellin transform \( \mathcal{M}(s, t) \):

\[
T(U, V) = \int \frac{ds \, dt}{(4\pi i)^2} U^s_2 V^{u_2-3} \Gamma^2 \left(2 - \frac{s}{2}\right) \Gamma^2 \left(2 - \frac{t}{2}\right) \Gamma^2 \left(2 - \frac{u}{2}\right) \mathcal{M}(s, t),
\]

where \( u \equiv 4 - s - t \).

- Result:

\[
\mathcal{M}(s, t) = \frac{2N^2}{(s-2)(t-2)(u-2)} + \frac{15E(\frac{3}{2}, \tau, \bar{\tau})}{8\sqrt{\pi^3}} \sqrt{N} + \mathcal{M}_{1\text{-loop}}(s, t)
\]
\[
+ \frac{315E(\frac{5}{2}, \tau, \bar{\tau})}{128\sqrt{\pi^5}} \left[ s^2 + t^2 + u^2 - 3 \right] \frac{1}{\sqrt{N}}
\]
\[
+ \frac{315E(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{32\pi^3} \left[ stu - \frac{1}{4} (s^2 + t^2 + u^2) - 4 \right] \frac{1}{N} + O(N^{-3/2}).
\]

- The modular invariants \( E(\frac{3}{2}, \tau, \bar{\tau}), E(\frac{5}{2}, \tau, \bar{\tau}), \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau}) \) (defined on next slide) are known from superstring scattering amplitudes

[Green, Gutperle '97; Green, Sethi '98; Wang, Yin '15, …].
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The non-holomorphic Eisenstein series $E(r, \tau, \bar{\tau})$ are modular-invariant functions defined as ($\tau = \tau_1 + i\tau_2$):

$$E(s, \tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}}.$$ 

They are eigenfunctions of the hyperbolic Laplacian

$$\left(4\tau_2^2 \partial_\tau \partial_\bar{\tau} - s(s - 1)\right) E(s, \tau, \bar{\tau}) = 0.$$ 

normalized s.t. $E(s, \tau, \bar{\tau}) \sim 2\zeta(2s)\tau_2^s$ at large $\tau_2$.

Expanded at large $\tau_2$ (i.e. small $g_{YM}$), they have a finite number of perturbative contributions + non-perturbative contributions. Example for $\tau_1 = 0$ and $\tau_2 = 4\pi/g_{YM}^2$:

$$E\left(\frac{3}{2}, \tau, \bar{\tau}\right) = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3\tau_2^{1/2}} + e^{-2\pi\tau_2} \left[4\pi + \frac{3}{4\tau_2} + \cdots \right] + e^{-4\pi\tau_2} \left[\cdots\right] + \cdots$$
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Generalized Eisenstein series

- The coefficient of $1/N$ (contact $D^6R^4$ interaction) is a “generalized Eisenstein series” $\mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})$.

- For general $r, s_1, s_2 \geq 0$, with $r \geq s_1 + s_2$, define $\mathcal{E}(r, s_1, s_2, \tau, \bar{\tau})$ as the unique $SL(2, \mathbb{Z})$-invariant solution of inhomogeneous eigenvalue equation

\[
\left(4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - r(r + 1)\right) \mathcal{E}(r, s_1, s_2, \tau, \bar{\tau}) = -E(s_1, \tau, \bar{\tau})E(s_2, \tau, \bar{\tau}).
\]

that grows slower than $\tau_2^{r+1}$ at large $\tau_2$.

- Also has a finite number of perturbative terms:

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\mathcal{E} = a_1 \tau_2^{s_1+s_2} + a_2 \tau_2^{1+s_1-s_2} + a_3 \tau_2^{1+s_2-s_1} + a_4 \tau_2^{2-s_1-s_2} + \beta_r \tau_2^{-r} + \text{non-pert.}
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Supersymmetric localization results

Specific setup \cite{Pestun '07, Hama, Hosomichi '12}:

\[ \mathcal{N} = 4 \text{ SYM on } \mathbb{R}^4 \rightarrow \mathcal{N} = 4 \text{ SYM on } S^4 \rightarrow \]

SUSY-preserving deformation by mass parameter \( m \) and squashing parameter \( b \)

\( (b, m) = (1, 0) \) corresp. to \( \mathcal{N} = 4 \) SYM on round sphere.

Can compute \( Z(m, b, \tau, \bar{\tau}) \) exactly \cite{Pestun '07, Hama, Hosomichi '12}.

Two messages:

- Derivatives of \( Z(m, b, \tau, \bar{\tau}) \) evaluated at \( (b, m) = (1, 0) \) give integrated correlators in \( \mathcal{N} = 4 \) SYM. (Take 4 ders for 4-pt function.)
- Can obtain Eisenstein and generalized Eisenstein series by expanding in the very strong coupling limit.
Supersymmetric localization results

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  $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ → $\mathcal{N} = 4$ SYM on $S^4$ → SUSY-preserving deformation by mass parameter $m$ and squashing parameter $b$

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  - Can obtain Eisenstein and generalized Eisenstein series by expanding in the very strong coupling limit.
Integrated correlators

For \( b = 1 \) (round sphere), we derived [Binder, Chester, SSP, Wang ’19]:

\[
\tau_2^2 \partial_m^2 \partial_\tau \partial_{\bar{\tau}} \log Z \bigg|_{m=0} = -\frac{8}{\pi} \int dr d\theta \ r^3 \sin^2 \theta \frac{T(U, V)}{U^2} \bigg|_{U=1+r^2-2r \cos \theta} \quad V=r^2
\]

and [Chester, SSP ’20]

\[
\partial_m^4 \log Z \bigg|_{m=0} = 12 \zeta(3)(N^2 - 1) + \frac{32}{\pi} \int dr d\theta \ r^3 \sin^2 \theta (1 + U + V) \times \bar{D}_{1111}(U, V) \frac{T(U, V)}{U^2} \bigg|_{U=1+r^2-2r \cos \theta} \quad V=r^2
\]

where \( \bar{D}_{1111}(U, V) \) is \( \propto \bar{x}_{13}^2 \bar{x}_{24}^2 \) times the Witten diagram for the contact interaction of four \( \Delta = 1 \) scalar operators.

- Similar relations for derivs w.r.t. squashing not available yet.
Matrix model for $\mathcal{N} = 2^*$ partition function

Pestun computed the $S^4$ partition function of the $\mathcal{N} = 2^*$ theory

[Pestun ’07; Russo, Zarembo ’13] :

$$Z = \int d^{N-1}a \frac{\prod_{i<j}(a_i - a_j)^2 H^2(a_i - a_j)}{H(m)^{N-1} \prod_{i\neq j} H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2 |Z_{\text{inst}}(m, \tau)|^2}$$

where $H$ is the product of two Barnes $G$-functions, and $Z_{\text{inst}}$ represents the contribution of instantons (Nekrasov partition function) localized at the $N$ and $S$ poles of $S^4$.

- Perturbatively in $1/N$ and $1/\lambda$ (in the ’t Hooft limit) one can ignore the instanton contributions, but they are very important in the very strong coupling limit.

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Matrix model for $\mathcal{N} = 2^*$ partition function

Pestun computed the $S^4$ partition function of the $\mathcal{N} = 2^*$ theory

$[\text{Pestun '07; Russo, Zarembo '13}] :$

$$Z = \int d^{N-1} a \frac{\prod_{i<j}(a_i - a_j)^2 H^2(a_i - a_j)}{H(m)^{N-1} \prod_{i \neq j} H(a_i - a_j + m)} e^{-\frac{8\pi^2}{\lambda} \sum_i a_i^2 |Z_{\text{inst}}(m, \tau)|^2}$$

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Derivatives of the $S^4$ partition function

For $\tau_2^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z \big|_{m=0}$, the mass derivatives either both act on $H$ ($\implies$ pert terms, harder to compute), both act on $Z_{\text{inst}}$, or both act on $Z_{\text{inst}}^*$ ($\implies$ non-pert terms, easier to compute):

$$\tau_2^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z \big|_{m=0} = \frac{N^2}{4} - \frac{3\sqrt{N}}{2^4 \pi^{3/2}} E\left(\frac{3}{2}, \tau, \bar{\tau}\right) + \frac{45}{2^8 \sqrt{N} \pi^{5/2}} E\left(\frac{5}{2}, \tau, \bar{\tau}\right)$$

$$+ \frac{1}{N^{3/2}} \left[ - \frac{39}{2^{13} \pi^{3/2}} E\left(\frac{3}{2}, \tau, \bar{\tau}\right) + \frac{4725}{2^{15} \pi^{7/2}} E\left(\frac{7}{2}, \tau, \bar{\tau}\right) \right]$$

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$$+ \frac{1}{N^{7/2}} \left[ \frac{4599}{2^{22} \pi^{3/2}} E\left(\frac{3}{2}, \tau, \bar{\tau}\right) - \frac{2811375}{2^{25} \pi^{7/2}} E\left(\frac{7}{2}, \tau, \bar{\tau}\right) + \frac{245581875}{2^{27} \pi^{11/2}} E\left(\frac{11}{2}, \tau, \bar{\tau}\right) \right]$$

$$+ O(N^{-9/2})$$

No integer powers of $1/\sqrt{N}$. (Probably due to SUSY.)

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\[ \partial_m^4 \log Z \bigg|_{m=0} \] is much harder b/c the $m$ derivatives can act on different factors. Evidence for [Chester, Green, SSP, Wang, Wen '20]:

\[
\partial_m^4 \log Z \bigg|_{m=0} = 6N^2 + \frac{6\sqrt{N}}{\pi^{\frac{3}{2}}} E\left(\frac{3}{2}, \tau, \bar{\tau}\right) + C_0 - \frac{9}{2\sqrt{N}\pi^{\frac{5}{2}}} E\left(\frac{5}{2}, \tau, \bar{\tau}\right)
\]

\[
- \frac{27}{2^3\pi^3 N} E\left(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau}\right) + \frac{1}{N^\frac{3}{2}} \left[ \frac{117}{2^8\pi^{\frac{3}{2}}} E\left(\frac{3}{2}, \tau, \bar{\tau}\right) - \frac{3375}{2^{10}\pi^{\frac{7}{2}}} E\left(\frac{7}{2}, \tau, \bar{\tau}\right) \right]
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+ \frac{1}{N^2} \left[ C_1 + \frac{14175}{704\pi^4} E\left(6, \frac{5}{2}, \frac{3}{2}, \tau, \bar{\tau}\right) - \frac{1215}{88\pi^4} E\left(4, \frac{5}{2}, \frac{3}{2}, \tau, \bar{\tau}\right) \right]
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+ \frac{1}{N^3} \sum_{r=3,5,7,9} \left[ \alpha_r E\left(r, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau}\right) + \beta_r E\left(r, \frac{5}{2}, \frac{5}{2}, \tau, \bar{\tau}\right) + \gamma_r E\left(r, \frac{7}{2}, \frac{3}{2}, \tau, \bar{\tau}\right) \right]
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\[ + O(N^{-\frac{7}{2}}), \]

Eisenstein series at odd orders in $N^{-1/2}$, generalized Eisenstein series at even orders in $N^{-1/2}$. 

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Up to $1/N$, the two integrated correlators $+$ flat space limit are sufficient to determine the $\langle SSSS \rangle$ at separated points (including position dependence).

Coeffs of $\sqrt{N}$ and $1/\sqrt{N}$ can be determined entirely from localization, and they agree in the flat space limit with superstring amplitude $\rightarrow$ precision test of AdS/CFT beyond SUGRA.

CFT data obtained from $\langle SSSS \rangle$ has the same structure of the expansion in the very strong coupling limit. For example, anomalous dimensions of double trace op $S_{\mu \nu} S^{\mu \nu}$

$$\gamma = \frac{a_1}{N^2} + \frac{a_2 E(\frac{3}{2}, \tau, \bar{\tau})}{N^{7/2}} + \frac{a_3}{N^4} + \frac{a_4 E(\frac{5}{2}, \tau, \bar{\tau})}{N^{9/2}} + \frac{a_5 \mathcal{E}(3, \frac{3}{2}, \frac{3}{2}, \tau, \bar{\tau})}{N^5} + \ldots$$

Relation to 't Hooft expansion: the terms $N^2(g_{YM}^2 N)^{-k}$ from the very strong coupling limit recombine into $N^2 f(\lambda)$, etc.
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Conclusion

- A combination of techniques (supersymmetric localization, analytic bootstrap in Mellin space) can be used to study holographic correlators beyond the SUGRA approximation in $\mathcal{N} = 4$ SYM (and other theories).
- In the very strong coupling limit, $\mathcal{N} = 4$ SYM correlators can be written in terms of Eisenstein series and generalized Eisenstein.

For the future:
- Connection with integrability?
- Use integrated constraints away from strong coupling limit, e.g. in numerical bootstrap (ongoing).
- Convergence / resummation of $1/N$ expansion.
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