SU(N) Principal Chiral Model at Large N and the glimpse of a new String Theory

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based on V.Kazakov, ES, K.Zarembo [1911.12860] & work in progress

IGST 2020

Plan

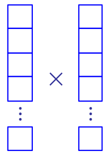
- SU(N) Principal Chiral Model. Overview
- Large N expansion
- Double-scaling limit
- 3D noncritical Strings?

Zamolodchikov,Zamolodchikov 79
Polyakov,Wiegmann 84
Wiegmann 84

2D SU(N) Principal Chiral Model

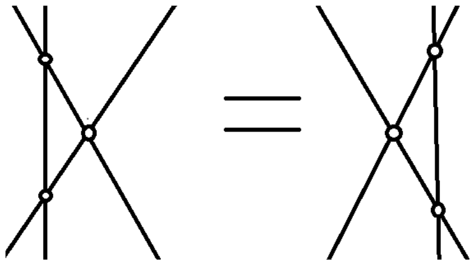
- Action : $S = \frac{N}{\lambda_0} \int d^2x \, \mathrm{tr}\, D_\mu g^\dagger D^\mu g, \quad g(t,x) \in SU(N)$

- Symmetry : $SU(N)_L \times SU(N)_R$

- Particles are massive and transform wrt bi-fundamental reps  :

$$m_l = m \frac{\sin \frac{\pi l}{N}}{\sin \frac{\pi}{N}}, \quad l = 1 \ldots N-1 \qquad\qquad m = \Lambda \frac{1}{\sqrt{\lambda_0}} e^{-\frac{4\pi}{\lambda_0}}$$

- $\mathbf{S}$-matrix is known exactly and factorised in the product of 2→2 scattering:



- Yang-Baxter (Integrability)

- Chemical potentials :

$$D_0 = \partial_0 g - i(Hg + gH)/2, \quad D_1 = \partial_1$$

$$H = \mathrm{diag}(h_1, h_2 - h_1, ..., h_{N-1} - h_{N-2}, -h_{N-1})$$

Equation for the vacuum energy

Wiegmann 84,
Japaridze, Nersesyan,
Wiegmann 84

- Bethe Ansatz equations (= periodicity) : $1 = e^{iLm_l \sinh\theta_\alpha} \prod_\beta S_{\alpha\beta}$ ← S-matrix

- BA in the thermodynamic limit $\mathcal{N}^l/L \xrightarrow[L\to\infty]{} n_l$:

$$\frac{1}{2\pi} m_l \cosh\theta = \tilde{\rho}_l(\theta) + \sum_n \int d\theta' R_{ln}(\theta-\theta')\rho_n(\theta')$$

$$E_{vac} = \min_{n_l}(E - \sum h_l n_l) = \sum_l \int d\theta (m_l \cosh\theta - h_l)\rho_l(\theta)$$

- Making Legendre transformation from densities ρ_l to pseudo-energies ϵ_l^+ we get spectral equations for the vacuum :

$$h_l - m_l \cosh\theta = \epsilon_l^-(\theta) + \sum_{n=1}^{N-1} \int_{-B_n}^{B_n} R_{ln}(\theta-\theta')\epsilon_n^+(\theta')$$

$$E_{vac} = -\sum_n \int \frac{d\theta}{2\pi} \epsilon_n^+(\theta) m_l \cosh\theta$$

$$\epsilon_n^+ \geq 0, \quad \mathrm{supp}(\epsilon_n^+) = (-B_n, B_n); \qquad \epsilon_n^- \leq 0, \quad \mathrm{supp}(\epsilon_n^-) = (-\infty, -B_n) \cup (B_n, \infty)$$

Fateev, Kazakov, Wiegmann 94

- Let’s choose chemical potentials along first discrete Fourier mode :

$$h_l = h\frac{\sin\frac{\pi}{N}l}{\sin\frac{\pi}{N}}$$

- Discrete Fourier transform along Dynkin diagram diagonalises the equations collapising them to one :

$$\int_{-B}^{B} d\theta'\, K(\theta-\theta')\epsilon(\theta') = h - m\cosh\theta, \quad \epsilon(\pm B) = 0$$

$$K(\theta) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\mathrm{e}^{-i\omega\theta}R(\omega), \quad R(\omega) = \frac{\pi}{2N}\frac{\sinh\frac{\pi|\omega|}{N}}{\cosh\frac{\pi\omega}{N} - \cos\frac{\pi}{N}}\,.$$

$$E_{vac} \equiv -N^2h^2 f = -\frac{m}{8\sin^2\frac{\pi}{N}}\int_{-B}^{B} d\theta\epsilon(\theta)\cosh\theta$$

Large N expansion

- LO solution at large-N has the form of semicircle :

Fateev, Kazakov, Wiegmann 94

$$\epsilon_0(\theta) = h\sqrt{B_0^2 - \theta^2} \qquad \frac{m}{h} = B_0 K_1(B_0)$$

- Numerics :

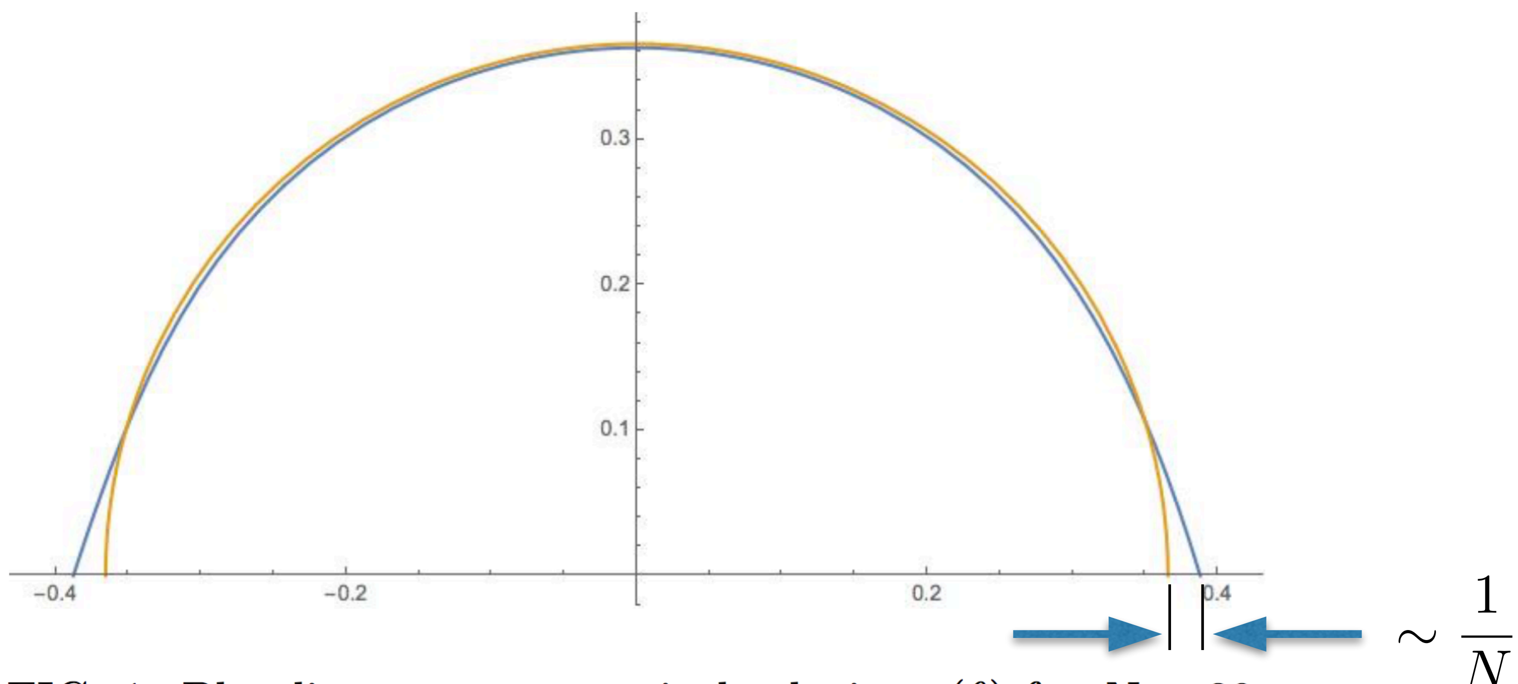


FIG. 1. Blue line: exact numerical solution $\epsilon(\theta)$ for $N = 30$, $\Delta = 2^{-3}$ and $h = 1$. Yellow line: solution $\epsilon_0(\theta)$ in the leading order - semicircle of radius $B_0(\Delta)$.

Kazakov, ES
Zarembo '19

- Method : Use ansatz $$\epsilon_\bullet(\theta) = h \sum_{k,s=0}^{\infty} \frac{\alpha_{k,s}}{N^{k+s}} (B^2 - \theta^2)^{\frac{1}{2}-k}$$

  and match the bulk and boundary asymptotic using Wiener-Hopf method.

$$\epsilon_\bullet(k) = G_+(k) \operatorname*{res}_{p=0} \frac{G_-(p) R_\bullet(p) \epsilon_\bullet(p)}{k-p}$$

WH factorisation : $$G_\pm(k) = \frac{2^{\pm ik+1}\, k^{\mp 1}}{\sqrt{k \pm i\varepsilon}\, B\left(1 - \frac{1}{2N} \mp \frac{ik}{2}, \frac{1}{2N} \mp \frac{ik}{2}\right)}$$

It gives us any 1/N order in an algorithmic way.

- Vacuum energy in the first 3 orders :

$$f = \sum N^{-i} f_i = \frac{B_0^2 I_1 K_1}{8\pi} + \frac{\pi B_0^2 K_1 (7 I_1 K_0 - I_0 K_1)}{192 K_0 N^2} + \frac{\zeta(3) K_1}{64 \pi K_0 N^3}$$

Weak coupling

- Weak coupling corresponds to $h \gg m$ or equivalently $B \gg 1$. First 3 orders in 2 loops:

$$f = \frac{B_0}{16\pi} + \frac{(6B_0 - 1)\pi}{384N^2} + \frac{\zeta(3)}{64\pi N^3} - i e^{-2B_0} \frac{4B_0 + 3}{64\pi}\left(1 + \frac{\pi^2}{3N^2}\right)$$

Match direct 2-loop calculation

Renormalon
(singularity in Borel plane)

- Our method with mild modifications can be used to generate perturbation series at finite N (and not only in PCM)

see also Volin 09
Marino,Reis 19

- There will be also exponential contributions of the type $\sim e^{-2kNB}$ which in weak coupling matches the k-uniton classical solutions of PCM. Long-term goal - write the full trans-series :

Work in progress

$$\lambda(h) \equiv 4\pi/B(h) \qquad f = \sum_{l,m} \#\lambda^l e^{-\frac{m\#}{\lambda}} \to \sum_{l,m} \#B^{-l} e^{-mB\#}$$

and resum it

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$$f^{2-loop} = \frac{1}{16\pi\cos^2\frac{\pi}{2N}}\left(B_0 + \log\left(\frac{2N}{\pi}\sin\frac{\pi}{2N}\right) - \frac{1}{2N}\left(2\gamma + \psi\left(1+\frac{1}{2N}\right) + \psi\left(1-\frac{1}{2N}\right)\right)\right)$$

Balog,Naik,Niedermayer,Weisz 1992 & from the week coupling version (at finite N) of our method

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Strong coupling

- Strong coupling corresponds to $\Delta = h/m - 1 \to 0$ or equivalently $B \to 0$

$$f = \frac{\Delta}{4\pi|\ln\Delta|} - \frac{\pi}{96|\ln\Delta|N^2} + \frac{\zeta(3)}{64\pi\sqrt{\Delta|\ln\Delta|}\,N^3}$$

Nonanalytic Stringy Behaviour!

$$B = 2\sqrt{\frac{\Delta}{|\ln\Delta|}} + \frac{\ln 2}{N} - \frac{\pi^2}{24\sqrt{\Delta|\ln\Delta|}\,N^2} - \frac{3\zeta(3)|\ln\Delta|}{128\Delta N^3}$$

- or rewriting vacuum energy f through the exact B :

$$N^2 f = \frac{B^2N^2}{16\pi} - \frac{BN\ln 2}{8\pi} + \frac{\ln^2 2}{16\pi} + \frac{3\zeta(3)}{256\pi BN}$$

We see the emergence of the **double scaling limit** :

$$N \to \infty,\ B \to 0,\ b = BN \ - \ \text{fixed}$$

Double scaling

- Making rescaling $t = N\theta,\ k = \omega/N,\ \epsilon_{DS}(t) = \pi N/2m\epsilon(t/N)$ and keeping the leading $1/N$ terms we get DS equation :

$$\epsilon_{DS}(t) + \int_{-b}^{b} ds K_{DS}(t-s)\epsilon_{DS}(s) = \delta(b) - \frac{t^2}{2} \qquad b \equiv \lim_{\substack{N\to\infty \\ B\to 0}} BN$$

$$K_{DS}(t) = \frac{1}{\pi^2}\left[2\psi(1) - \psi\left(1+\frac{it}{\pi}\right) - \psi\left(1-\frac{it}{\pi}\right)\right]$$

energy :

$$f_{DS} \equiv \lim_{N\to\infty} N^2 f = \frac{1}{4\pi^3}\int_{-b}^{b} dt\, \epsilon_{DS}(t)$$

the relation to the original parameters $N \to \infty$ and $\Delta \to 0$:

$$N^2\Delta - 8\pi f_{DS} \ln\frac{N}{\pi} = \epsilon_{DS}(0) + \int_{-b}^{b} K_{DS}(t)\epsilon_{DS}(t)$$

- The limit of small $b \to 0$ just reproduces ordinary strong coupling :

$$f_{DS} = \frac{\sqrt{2}}{3\pi^3}\delta^{\frac{3}{2}} + \frac{128\zeta(3)}{45\pi^7}\delta^3 + \ldots \qquad b = \sqrt{2}\delta^{\frac{1}{2}} + \frac{32\zeta(3)}{5\pi^4}\delta^2 + \ldots$$

- Expansion (first 5 orders) in $b \to \infty$:

This strong coupling looks as a weak coupling

$$f_{DS} = \frac{\tilde{b}^2}{16\pi} + \frac{3\zeta(3)}{256\pi\tilde{b}} + \frac{135\zeta(5)}{16384\pi\tilde{b}^3} \qquad \tilde{b} = b - \log 2$$

Partition function for 3d String?

$\tilde{b}$ is related to original parameters as (first 3 orders) :

$$N^2\Delta = \frac{\tilde{b}^2}{2}\ln\frac{2N\mathrm{e}^{\frac{1}{2}-\gamma}}{\tilde{b}} + \frac{\pi^2}{24} + \frac{3\zeta(3)}{32\tilde{b}}\ln\frac{2N\mathrm{e}^{-\frac{4}{3}-\gamma}}{\tilde{b}} + \ldots$$

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Partition function for 3d String?

It looks very similar to DS in c=1 MQM dual to 2D strings!

Is PCM in the regime of double scaling dual to 3D noncritical Strings?

c=1 MQM dual to 2d Strings

MQM

copy-paste from Klebanov's lctrs 91

surfaces embedded in 1d ⟷

$$Z \sim \int D^{N^2}\Phi(x)\exp\left[-N\int_{-T/2}^{T/2} dx\ \mathrm{Tr}\left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2+\frac{1}{2\alpha'}\Phi^2-\frac{\kappa}{3!}\Phi^3\right)\right]$$

$$\lim_{T\to\infty}\ln Z=\sum_h N^{2-2h}\sum_\Lambda \kappa^V\prod_{i=1}^{V}\int_{-\infty}^{\infty}dx_i\prod_{\langle ij\rangle}e^{-|x_i-x_j|/\alpha'}$$

+1d from Liouville mode

vacuum state reduces the problem to free fermions

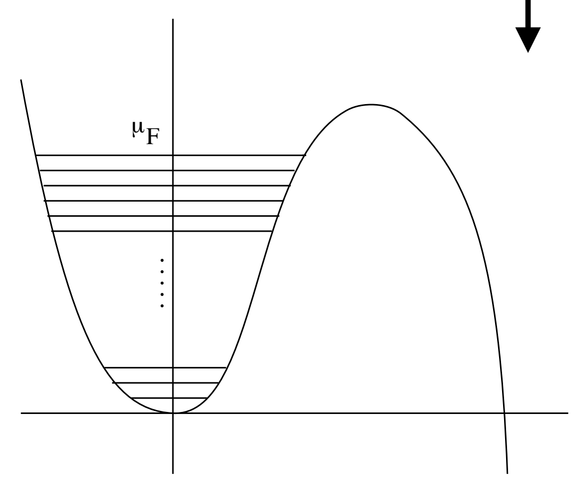


Fig. 2a) N fermions in the asymmetric potential arising directly from the triangulated surfaces.

WKB

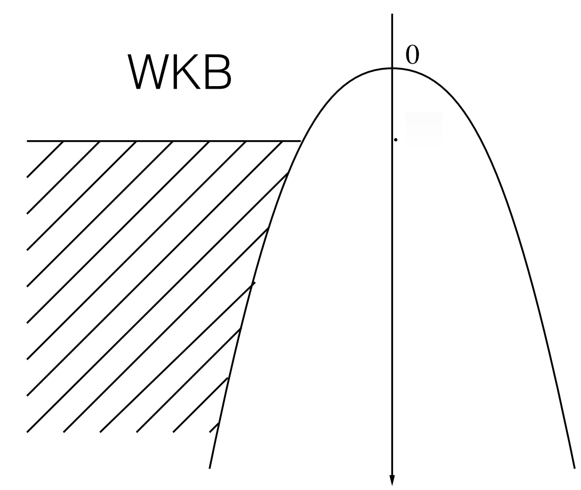


Fig. 2b) The double-scaling limit magnifies the quadratic local maximum.

2d c=1 strings

Das, Jevicky 90

Double Scaling

$$\chi=\frac{\partial^2 E_0}{\partial\Delta^2}=\frac{1}{\log\Delta}+\frac{1}{24N^2\Delta^2}+\frac{7\log^2\Delta}{2800N^4\Delta^4}$$

Kazakov, Migdal 88

$\Delta=(\mu_c-\mu)\log(\mu_c-\mu)$ is analogue of our Δ

Hundreds of papers in one slide!

c=1 MQM

0+1 MQM

↓

Vacuum → Schrödinger eqn for free fermions

↓

Large N expansion

↓

DS emerges from the combination of near-threshold regime with large N and it governs by the WKB asymptotic

↓

2d c=1 String

2d = time × eigenvalue

↕

Liouville field

SU(N) PCM

1+1 Matrix QFT

↓ ✓

Vacuum → Integral eqn for fermions

↓ ✓

Large N expansion

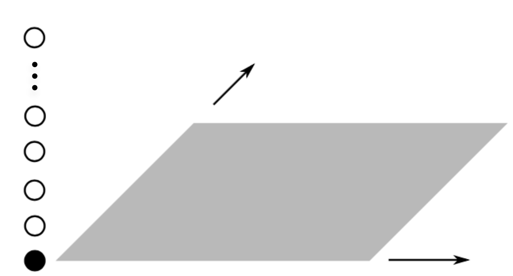
↓ ✓

DS emerges from the combination of strong coupling regime with large N and it governs by the WKB-type asymptotic

↓ ?

3d String ?

3d = time × space × Dynkin diagram ?



Further questions

- Exponential corrections $\to$ Transseries $\to$ Resurgence
- Form-factors and correlation functions in DS limit? Playground for SU(N) SoV?

  F-fs in large N : Cubero, Orland

  SU(N) SoV for spin chains : F.L.M talk
- Excite higher discrete Fourier modes along Dynkin diagram $\to$ 3rd dimension.
- String Field Theory as an effective action for pseudonergies (or densities) à la Das-Jevicki collective theory for c=1 MM?
- Reformulation in terms of scheme independent observables?

  in c=1 : Migdal, Kazakov
- Finite temperate (finite cylinder with twisted boundary conditions)?

  Leurent, ES 15

  Kazakov, Leurent 10
- Vortices, "Black holes", ...?

Thank you!