CDD deformations of 2D IQFTs
A report on non-trivial behaviour in irrelevant deformations

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Consider a theory near a RG fixed point $A_{\text{CFT}}$

$$A = \left[ A_{\text{CFT}} + \mu \int d^2 x \Phi_\Delta (x) \right] + \sum_i \alpha_i \int d^2 x O_i (x) ,$$

$\Phi_\Delta$ relevant ($d = 2\Delta < 2$); $O_i$ irrelevant ($d_i > 2$);

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Investigate this point more deeply by means of Wilson’s interpretation of RG$^1$.

---

Consider $\Sigma$, the space of quasi-local field theories

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    \[\implies \exists \ell_\ast \text{ such that } A_\ell \not\in \Sigma , -\ell > \ell_\ast;\]

    \[\implies \exists \text{ intrinsic UV scale } \Lambda_\ast = M e^{\ell_\ast} , \text{ e.g. for QED is "Landau scale";}\]
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  $\implies \exists \ell^*_\text{UV} \text{ such that } A_\ell \not\in \Sigma , -\ell > \ell^*_\text{UV}$;

  $\implies \exists$ intrinsic UV scale $\Lambda^*_\text{UV} = Me^{\ell^*_\text{UV}}$, e.g. for QED is “Landau scale”;

  - $\Sigma_{\ell^*_\text{UV} = \infty}$ space of UV complete theories: can remove cutoff consistently.
Figure: Pictorial representation of the space of quasi-local theories $\Sigma$, together with a flow. The arrow denotes the “forward RG time” direction and $-\ell_*$ the “critical RG time” before which the theory lies outside $\Sigma$. 
The $T\bar{T}$ flow

$\Sigma$

Figure: Pictorial representation of the $T\bar{T}$-flow

$$\frac{d}{d\alpha} A_\alpha = - \int d^2 x T\bar{T}_\alpha (x) ,$$

in the space of quasi-local theories $\Sigma$. At each point, the flow is tangent to the vector $T\bar{T}_\alpha (x)$. It is expected that $\ell_* = \infty$ although $\not\exists$ UV fixed point.
Introduction

What is “$\bar{T}T$”

The $\bar{T}T$ operator is defined as\(^2\)

\[
\bar{T}T(x) \equiv -\lim_{x \to x'} T(x, x') , \quad T(x, x') = \frac{1}{2} e_{\mu\lambda} e_{\nu\rho} T^{\mu\nu}(x) T^{\lambda\rho}(x') .
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\]

- expectation value is a constant:

\[
\frac{\partial}{\partial x^{\mu}} \langle T \left( x, x' \right) \rangle = - \frac{\partial}{\partial x'^{\mu}} \langle T \left( x, x' \right) \rangle = 0 ,
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and factorises

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\langle \bar{T}T \left( x \right) \rangle = - \det \langle T^{\mu\nu} \left( x \right) \rangle ,
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\(\Leftarrow \) Ward identities + spectral decomposition;

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\]

\(\iff\) Ward identities + spectral decomposition;

• singularities in collision limit are under control:

\[
T (x, x') \simeq - \bar{T}T (x') + \delta (x - x') T^\mu_\mu (x') + \sum_a C^{a, \lambda} (x - x') \frac{\partial}{\partial x'^\lambda} O_a (x') ,
\]

\(\implies\) \[
\langle T (x, x') \rangle = - \langle \bar{T}T (x) \rangle + \text{contact term} .
\]

---

Introduction

Why $\bar{T}T$ deformations

Main practical reasons

$F \sim T \to T_c F_0 + a (T - T_c)^\nu + a' (T - T_c)^\xi + \cdots$

$R^{-1}c = M \sim T \to T_c b (T - T_c)^\nu + b' (T - T_c)^\eta + \cdots$

$\xi = d \nu = 4, \eta = (d - 1) \nu = 3, b' = b a'$

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Main practical reasons

- this deformation is **universal**: (almost) any \( A_0 \) will do;

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- seemingly well-defined (UV completeness) paired with non-trivial UV behaviour (e.g. Hagedorn singularity, non-locality, etc...);
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- the term “UV fragility” introduced\(^3\) to denote this phenomenon;

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Some important motivations

- seemingly well-defined (UV completeness) paired with non-trivial UV behaviour (e.g. Hagedorn singularity, non-locality, etc...);
- the term “UV fragility” introduced$^3$ to denote this phenomenon;
- describe sub-leading critical behaviour;

$$F \underset{T \rightarrow T_c}{\sim} F_0 + a \left( T - T_c \right)^{2\nu} + a' \left( T - T_c \right)^{\xi} + \cdots$$

$$R_c^{-1} = M \underset{T \rightarrow T_c}{\sim} b \left( T - T_c \right)^{\nu} + b' \left( T - T_c \right)^{\eta} + \cdots$$

$\bar{T}T$ lowest $d(= 4)$ irrelevant $\Rightarrow \xi = d_{\bar{T}T} \nu = 4\nu \ , \ \eta = (d_{\bar{T}T} - 1) \nu = 3\nu \ , \ \frac{b'}{a'} = \frac{b}{a} \ .$

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• Finite size spectrum (cylinder) obeys Burgers equation\(^4\)

\[
\frac{\partial}{\partial \alpha} E_n (R, \alpha) + E_n (R, \alpha) \frac{\partial}{\partial R} E_n (R, \alpha) + \frac{1}{R} P_n (R)^2 = 0 ;
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\[ E(R, 0) \quad E(R, \alpha) \quad E(R, \alpha) \]

\[ \alpha = 0 \quad \alpha > 0 \quad \alpha < 0 \]

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To derive, use \( \langle T \bar{T} \rangle = -\text{det}_{\mu\nu} \langle T^{\mu\nu} \rangle \) and standard identifications

\[
\langle n | T^{xx} | n \rangle = -\frac{1}{R} E_n(R), \quad \langle n | T^{xy} | n \rangle = \frac{i}{R} P_n(R), \quad \langle n | T^{yy} | n \rangle = -\frac{d}{dR} E_n(R).
\]

The finite-size spectrum

- Functional form (zero momentum sector)

\[ E(R, \alpha) = E(R - \alpha E(R, \alpha), 0) \ . \]

\[ \text{see e.g. for CFT Barbón and Rabinovici, arXiv:2004.10138.} \]
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- Entropy density is constant in vanishing volume \( s(R = 0, \alpha) \propto \sqrt{\frac{\epsilon}{\alpha}} \).

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  \[ s(R, -|\alpha|) \sim \frac{c}{6} \left( R^2 - R_H^2 \right)^{-1/2} , \]

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- Hagedorn-type high energy spectrum\(^5\)
  \[ \mathcal{N}(E) \sim e^{ER_H} \]

\(^5\)see e.g. for CFT Barbón and Rabinovici, arXiv:2004.10138.
The $\bar{T}T$ deformation implies for $S$-matrix\textsuperscript{6}

$$\frac{\delta S_{N \rightarrow M} (\{p_i\}, \{q_k\}, \alpha)}{S_{N \rightarrow M} (\{p_i\}, \{q_k\}, \alpha)} = \frac{i}{2} \delta \alpha \left[ \sum_{p_i < p_j} \vec{p}_i \wedge \vec{p}_j + \sum_{q_k < q_l} \vec{q}_k \wedge \vec{q}_l \right].$$


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- In integrable case $S$ matrix deformation can be taken as definition

$$S_{2 \to 2} (\theta, \alpha) = e^{i \alpha m^2 \sinh(\theta)} S_{2 \to 2} (\theta, 0) .$$

Action flow via TBA/NLIE. Gravitational phase shift $\Delta t = -\alpha E$

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Action flow via TBA/NLIE\(^7\). Gravitational phase shift\(^8\) $\Delta t = -\alpha E$

- $\alpha < 0$: healthy theory, no local observables (probably);
- $\alpha > 0$: superluminal propagation, $S$-matrix well defined.

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The $\bar{T}T$ flow

The $\bar{T}T$ deformation implies for $S$-matrix\(^6\)

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Generalize (for integrable systems) $S$-matrix deformation as

\[
S_{\alpha} \rightarrow 2 \left( \theta, \theta' \right) = e^{i \sum_{s \in \mathbb{Z}^+} \alpha s \sinh (s \theta - s \theta')} S_0 \rightarrow 2 \left( \theta, \theta' \right),
\]

or general R-CDD factors

\[
S_{\alpha} \rightarrow 2 \left( \theta, \theta' \right) = \prod_{s \in \mathbb{Z}^+} \sinh (s \theta - s \theta') - i b s \sinh (s \theta - s \theta') + i b s.
\]

Use TBA/NLIE to analyse finite-size spectra ($L = \log (1 + e^{-\epsilon})$, $r = mR$)

\[
\epsilon \alpha \left( \theta \right) = \nu_0 \left( \theta \right) - \left( \Phi^* \alpha \right) \left( \theta \right),
\]

\[
E \left( r \right) = -\int_{-\infty}^{\infty} dt \cosh t L \left( t \right)
\]

where $\Phi \left( \theta, \theta' \right) \propto \partial / \partial \theta' \log S \left( \theta, \theta' \right)$, and, e.g. $\nu_0 \left( \theta \right) = mR \cosh \theta$.

Search for non-trivial behaviour, such as Hagedorn temperature in $T$
Generalize (for integrable systems) $S$-matrix deformation as

- instead of $\exp\left[\frac{i}{\alpha}m^2 \sinh \theta\right]$ choose a general (relativistic) E-CDD factor

$$S_{2\rightarrow 2}^{\alpha} (\theta, \theta') = e^{i \sum_{s \in 2\mathbb{Z}+1} \alpha_s \sinh(s\theta - s\theta')} S_{2\rightarrow 2}^{0} (\theta, \theta'),$$
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$$S_{2\rightarrow 2}^{\alpha} (\theta, \theta') = e^{i \sum_{s \in 2\mathbb{Z}+1} \alpha_s \sinh(s\theta - s\theta') \left( S_{2\rightarrow 2}^0 (\theta, \theta') \right)},$$

- or general R-CDD factors

$$S_{2\rightarrow 2}^{\alpha} (\theta, \theta') = \left[ \prod_{s \in 2\mathbb{Z}+1} \frac{\sinh(s\theta - s\theta') - i b_s}{\sinh(s\theta - s\theta') + i b_s} \right] S_{2\rightarrow 2}^0 (\theta, \theta').$$
CDD deformations

CDD & TBA

Generalize (for integrable systems) $S$-matrix deformation as

- instead of $\exp \left[ i \alpha m^2 \sinh \theta \right]$ choose a general (relativistic) E-CDD factor

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- use TBA/NLIE to analyse finite-size spectra ($L = \log (1 + e^{-\epsilon})$, $r = mR$)

$$\epsilon_\alpha (\theta) = \nu_0 (\theta) - (\Phi_\alpha \ast L_\alpha) (\theta), \quad E (r) = - \int_{-\infty}^{\infty} dt \cosh tL (t),$$

where $\Phi (\theta, \theta') \propto \partial / \partial \theta' \log S (\theta, \theta')$, and, e.g. $\nu_0 (\theta) = mR \cosh \theta$;
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I will present (partial) results for the special case of 2 R-CDDs

$$\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma,\sigma' = \pm 1} \frac{1}{\cosh(\theta + \sigma \theta_0 + i\sigma' \gamma)} , \quad \theta_0 \in \mathbb{R}_{\geq 0} , \quad \gamma \in [0, \frac{\pi}{2}) .$$
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$$\varepsilon(\theta) = r \cosh \theta - (\Phi \ast L)(\theta):$$

1. $\varepsilon \sim r$ this implies $L \sim e^{-\varepsilon}$ and $\varepsilon > 0$ (if $\varepsilon < 0$ somewhere then $L \sim r$);

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   - subcases
     2.1 $\varepsilon \gg 1$ inconsistent, since $L \sim e^{-\varepsilon} \ll 1$ or $L \sim -\varepsilon \ll r$;
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       2.3.1 $\Phi$ is integrable, then $(\Phi \ast L) \sim 1$ inconsistent;
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3. $\varepsilon \sim (\Phi \ast L)$ only possible if $\exists \Theta \subset \mathbb{R}$ s.t. $\varepsilon(\theta) < 0, \theta \in \Theta$.

Two subcases
3.1 $\varepsilon(\theta| r) = -h(\theta| r) - rf(\theta) + \cdots$ two relations
   $$h(\theta| r) = \int_{\Theta} dt \Phi(\theta - t) h(t| r),$$
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$$\varepsilon \sim Ar^{-\gamma} \cosh \theta \implies (\Phi \ast L) \sim 2r^\gamma \int_0^\infty dT \log \left[ 1 + e^{-T} \right]$$

hence $\gamma = 1$ and the equation is balanced.
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CDD deformations

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It is only possible if $\Phi$ is has not a definite sign or it is everywhere positive and $|\Phi|_1 = \int_{-\infty}^{\infty} dt \Phi(t) > 1$.

If $\Phi$ is everywhere positive, let $\theta_M$ s.t. $f(\theta_M) = \max_{\theta \in \Theta} f(\theta)$, then

$$f(\theta_M) \leq -\cosh \theta_M + f(\theta_M) \int_\Theta dt \Phi(t) < -1 + f(\theta_M) |\Phi|_1,$$

which is implies $|\Phi|_1 > 1 + 1/f(\theta_M)$. 
We need a way to handle singular points such as branch points.
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Use methods of numerical analysis in bifurcation theory for dynamical systems\(^9\).

\(^9\)E. Allowger and K. Georg, Numerical Continuation Methods, C.S.U.
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Use external parameter \(\varsigma\) to parametrize solutions of TBA as pairs

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(Some of) the questions are:

1. Is the singular point a square root branch cut (as in $T \bar{T}$)?
2. Are there more than two branches?
3. Can we exclude more complicated behaviours (e.g. bifurcation points)?
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Here follow some plots for the 2 R-CDD case.

\(^9\)E. Allowger and K. Georg, Numerical Continuation Methods, C.S.U.
Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma, \sigma'} \pm \text{sech} (\theta + \sigma \theta_0 + i \sigma' \gamma)$, $\theta_0 = 0$. 
Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma, \sigma'} \pm \text{sech}(\theta + \sigma\theta_0 + i\sigma'\gamma)$, $\theta_0 = 4$. 
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Parameters are $E_\infty = 17.475179499(1)$, $\alpha = 6.8407(8)$, $\beta = 12.4505(9)$, $r_c = 0.6215(7)$.
Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma, \sigma'} \pm \text{sech} (\theta + \sigma \theta_0 + i\sigma' \gamma)$, $\theta_0 = 4$, $\gamma = 3\pi/20$. $E_\infty = 5700.693492914 \ldots$, $\alpha = 171.7260(6)$, $\beta = 1121.4579(4)$, $r_c = 0.0482(8)$. 
Figure: $f(\theta)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma, \sigma'} \text{sech}(\theta + \sigma \theta_0 + i \sigma' \gamma)$, $\theta_0 = 0$.
Grey vertical lines are the edges of $\Theta = [-B, B]$. 

$\theta_0 = 0$

$\gamma = 0$

$\gamma = \frac{2\pi}{20}$

$\gamma = \frac{4\pi}{20}$

$\gamma = \frac{6\pi}{20}$

$\gamma = \frac{8\pi}{20}$
Figure: $f(\theta)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma, \sigma'} \pm \text{sech}(\theta + \sigma \theta_0 + i\sigma' \gamma)$, $\theta_0 = 5$.
Grey vertical lines are the edges of $\Theta = [-B, B]$. 
Figure: Edge $B$ of $\Theta = [-B, B]$ as function of $\theta_0$. Solid line is $\theta_0 + 1$. 
Figure: $g(\theta|r)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma,\sigma'} \pm \sech(\theta + \sigma \theta_0 + i\sigma' \gamma)$, $\theta_0 = 4$ and $\gamma = 3\pi/20$. 
Conclusions & outlook

what is left to do?

1. is the singular point a square root branch cut (as in $\bar{T}T$)? YES!

There are still many questions left to answer, amongst which:

• understand the behaviour of $g(\theta|r) \Rightarrow$ sub-leading behaviour of $E(r)$ on the second branch. Analytics suggest $\sim 1/r$ while numerics show $\sim r^\gamma e^{-\chi r}$;

• obtain a better control on the dependence of $B$ on the parameters. E. g. the limit $\gamma \to \pi/2$ is described by a finite difference equation $Y(\theta) = e^{-r \cosh \theta (1 + Y(\theta + \theta_0)) (1 + Y(\theta - \theta_0))}$, $Y(\theta) = e^{-\varepsilon(\theta)}$;

• similar analysis for other models (e. g. 3-R-CDD, E-CDDs, Elliptic sG).

Is the square root behaviour a universal feature?

• Turn attention to more rich models, with bound states;

• relate the existence of turning points to the properties of $S$-matrix.

(Very) long-term goal.

Conclusions & outlook

1. is the singular point a square root branch cut (as in $\bar{T}T$)? YES!
2. are there more than two branches? NO!

There are still many questions left to answer, amongst which:

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1. is the singular point a square root branch cut (as in $\tilde{T\tilde{T}}$)? YES!
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There are still many questions left to answer, amongst which:

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\[ ^{10} \text{G. Mussardo and S. Penati, arXiv: hep-th/9907039.} \]
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Conclusions & outlook

**what is left to do?**

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Thank you!