CDD deformations of 2D IQFTs

A report on non-trivial behaviour in irrelevant deformations

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Work in progress with T. Fleury, M. Lencsés, G. Camilo and A. Zamolodchikov.

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Investigate this point more deeply by means of Wilson's interpretation of RG[1].

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    $\Longrightarrow$ $\exists \ell_*$ such that $\mathcal{A}_\ell \notin \Sigma$, $-\ell > \ell_*$;

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    $\implies \exists$ intrinsic UV scale $\Lambda_* = M e^{\ell_*}$, e.g. for QED is "Landau scale";
  - $\Sigma_{\ell_* = \infty}$ space of UV complete theories: can remove cutoff consistently.

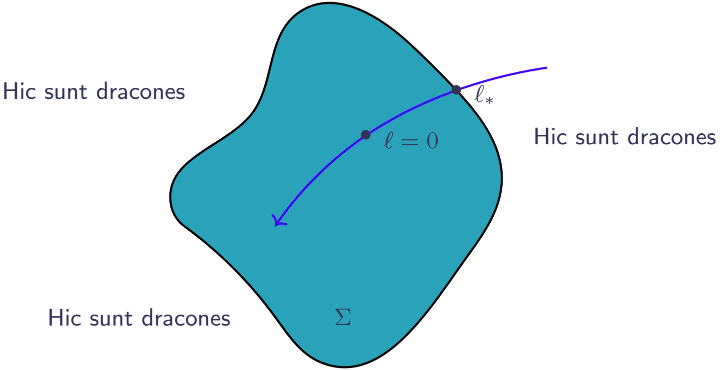

Figure: Pictorial representation of the space of quasi-local theories Σ, together with a flow. The arrow denotes the "forward RG time" direction and $-\ell_*$ the "critical RG time" before which the theory lies outside Σ.

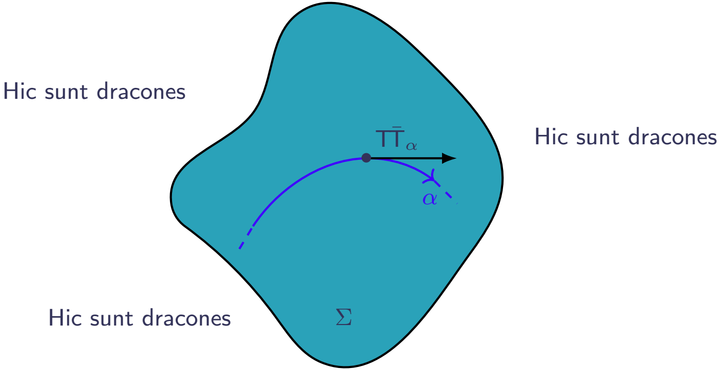

Figure: Pictorial representation of the $T\bar{T}$-flow

$$\frac{d}{d\alpha}\mathcal{A}_\alpha = -\int d^2x T\bar{T}_\alpha(x) \ ,$$

in the space of quasi-local theories Σ. At each point, the flow is tangent to the vector $T\bar{T}_\alpha(x)$. It is expected that $\ell_* = \infty$ although $\nexists$ UV fixed point.

The $T\bar{T}$ operator is defined as[2]

$$T\bar{T}(x) \doteq -\lim_{x\to x'} T(x,x') \;, \qquad T(x,x') = \frac{1}{2} e_{\mu\lambda} e_{\nu\rho} T^{\mu\nu}(x)\, T^{\lambda\rho}(x') \;.$$

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- singularities in collision limit are under control:

$$T(x,x') \simeq -\mathrm{T}\bar{\mathrm{T}}(x') + \delta(x-x') \, T^{\mu}_{\mu}(x') + \sum_{a} C^{a,\lambda}(x-x') \frac{\partial}{\partial x'^{\lambda}} O_{a}(x') \ ,$$

$$\Longrightarrow \left\langle T(x,x') \right\rangle = -\left\langle \mathrm{T}\bar{\mathrm{T}}(x) \right\rangle + \text{contact term} \ .$$

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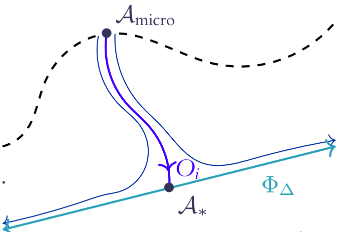
- the term "UV fragility" introduced[3] to denote this phenomenon;
- describe sub-leading critical behaviour;



$$F \underset{T\to T_c}{\sim} F_0 + a\,(T-T_c)^{2\nu} + a'\,(T-T_c)^{\xi} + \cdots$$

$$R_c^{-1} = M \underset{T\to T_c}{\sim} b\,(T-T_c)^{\nu} + b'\,(T-T_c)^{\eta} + \cdots$$

$T\bar{T}$ lowest $d(=4)$ irrelevant $\Rightarrow \xi = d_{T\bar{T}}\nu = 4\nu\ ,\ \eta = (d_{T\bar{T}}-1)\,\nu = 3\nu\ ,\ \frac{b'}{a'} = \frac{b}{a}\ .$

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- Finite size spectrum (cylinder) obeys Burgers equation[4]

$$\frac{\partial}{\partial\alpha}E_n(R,\alpha) + E_n(R,\alpha)\frac{\partial}{\partial R}E_n(R,\alpha) + \frac{1}{R}P_n(R)^2 = 0\,;$$

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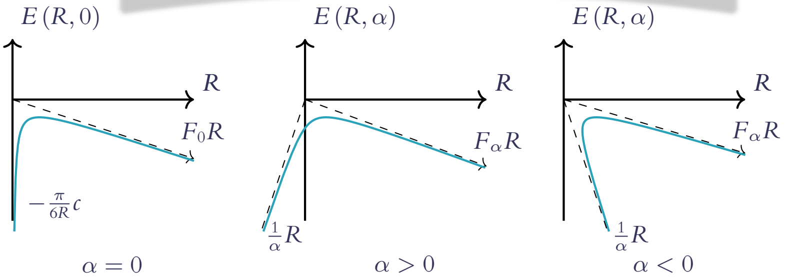

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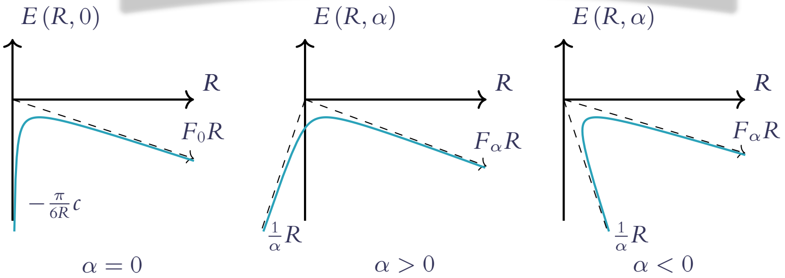

To derive, use $\langle T\bar{T}\rangle = -\det_{\mu\nu}\langle T^{\mu\nu}\rangle$ and standard identifications

$$\langle n|T^{xx}|n\rangle=-\frac{1}{R}E_n(R)\;,\quad \langle n|T^{xy}|n\rangle=\mathbb{i}\frac{1}{R}P_n(R)\;,\quad \langle n|T^{yy}|n\rangle=-\frac{d}{dR}E_n(R)\;.$$

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$$E\left(R,\alpha\right) = E\left(R - \alpha E\left(R,\alpha\right), 0\right) \ .$$

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- Hagedorn-type high energy spectrum[5]

$$\mathcal{N}(E) \sim e^{ER_H}$$

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- The $T\bar{T}$ deformation implies for S-matrix[6]

$$\frac{\delta S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)}{S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)} = \frac{\mathrm{i}}{2}\delta\alpha\left[\sum_{p_i<p_j}\vec{p}_i\wedge\vec{p}_j + \sum_{q_k<q_l}\vec{q}_k\wedge\vec{q}_l\right] .$$

[6] S. Dubovsky, V. Gorbenko and M. Mirbabayi, JHEP 09 (2013) 045, arXiv: 1305.6939.
[7] A. Cavaglià, S.N., I. Szécsényi and R. Tateo, JHEP 1610 (2016), arXiv:1608.05534.
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$$S_{2\to 2}\left(\theta,\alpha\right) = e^{\mathrm{i}\alpha m^2\sinh(\theta)}S_{2\to 2}\left(\theta,0\right) .$$

Action flow via TBA/NLIE[7]. Gravitational phase shift[8] $\Delta t = -\alpha E$

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- $\alpha < 0$: healthy theory, no local observables (probably);

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$$\frac{\delta S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)}{S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)} = \frac{\mathrm{i}}{2}\delta\alpha\left[\sum_{p_i<p_j}\vec{p}_i\wedge\vec{p}_j + \sum_{q_k<q_l}\vec{q}_k\wedge\vec{q}_l\right] .$$

- In integrable case S matrix deformation can be taken as definition

$$S_{2\to2}\left(\theta,\alpha\right) = e^{\mathrm{i}\alpha m^2\sinh(\theta)}S_{2\to2}\left(\theta,0\right) .$$

Action flow via TBA/NLIE[7]. Gravitational phase shift[8] $\Delta t = -\alpha E$

- $\alpha < 0$: healthy theory, no local observables (probably);
- $\alpha > 0$: superluminal propagation, S-matrix well defined.

[6] S. Dubovsky, V. Gorbenko and M. Mirbabayi, JHEP 09 (2013) 045, arXiv: 1305.6939.
[7] A. Cavaglià, S.N., I. Szécsényi and R. Tateo, JHEP 1610 (2016), arXiv:1608.05534.
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- use TBA/NLIE to analyse finite-size spectra ($L = \log\left(1 + e^{-\varepsilon}\right)$, $r = mR$)

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I will present (partial) results for the special case of 2 R-CDDs

$$\Phi\left(\theta\right) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm1}\frac{1}{\cosh\left(\theta+\sigma\theta_0+\mathbb{i}\sigma'\gamma\right)} \ , \quad \begin{array}{c}\theta_0\in\mathbb{R}_{\geq0} \\ \gamma\in\left[0,\frac{\pi}{2}\right)\end{array} .$$

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$$f(\theta) = -\cosh\theta + \int_{\Theta} dt \Phi(\theta - t) f(t)\ , \quad \begin{cases} f(\theta) \geq 0 & , \quad \theta \in \Theta \\ f(\theta) < 0 & , \quad \theta \notin \Theta \end{cases}$$

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Here follow some plots for the 2 R-CDD case.

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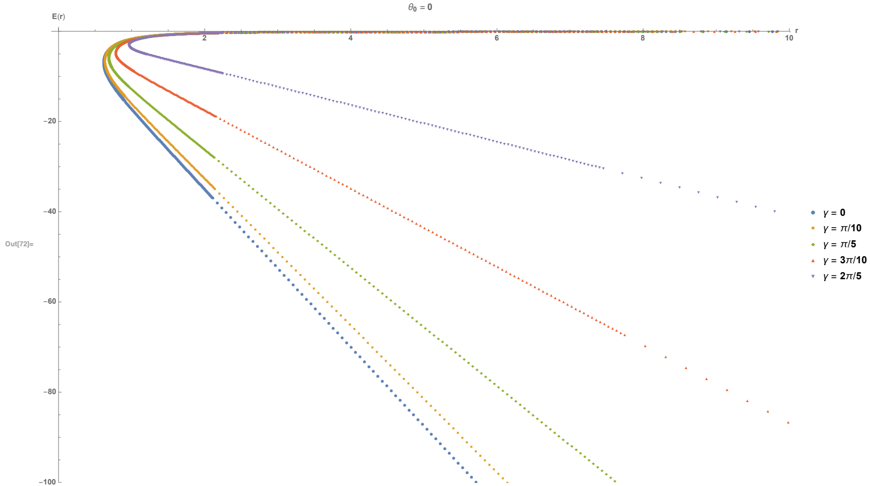

Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm} \text{sech}(\theta + \sigma\theta_0 + i\sigma'\gamma)$, $\theta_0 = 0$.

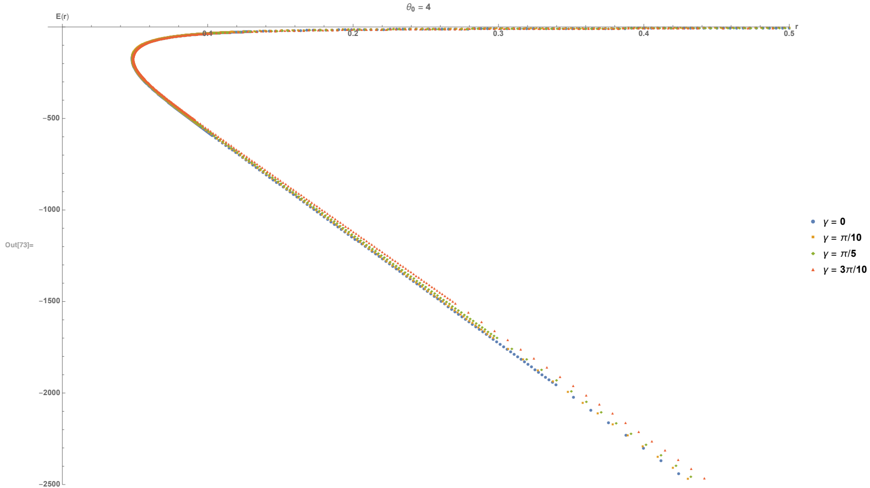

Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi} \sum_{\sigma,\sigma'=\pm} \operatorname{sech}(\theta + \sigma\theta_0 + \mathrm{i}\sigma'\gamma)$, $\theta_0 = 4$.

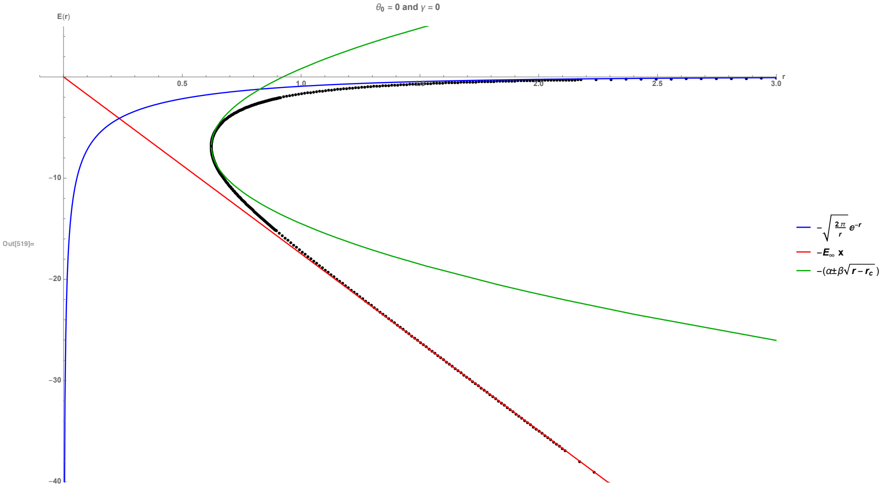

Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm}\operatorname{sech}(\theta + \sigma\theta_0 + \mathrm{i}\sigma'\gamma)$, $\theta_0 = 0$, $\gamma = 0$.
Parameters are $E_\infty = 17.475179499(1)$, $\alpha = 6.8407(8)$, $\beta = 12.4505(9)$, $r_c = 0.6215(7)$.

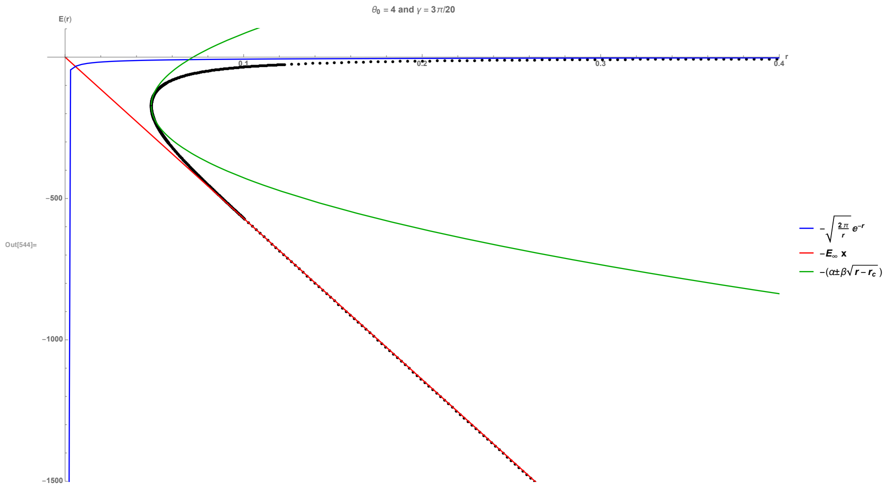

Figure: $E(r)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm} \operatorname{sech}(\theta + \sigma\theta_0 + \mathrm{i}\sigma'\gamma)$, $\theta_0 = 4$, $\gamma = 3\pi/20$. $E_\infty = 5700.693492914\cdots$, $\alpha = 171.7260(6)$, $\beta = 1121.4579(4)$, $r_c = 0.0482(8)$.

Out[644]=

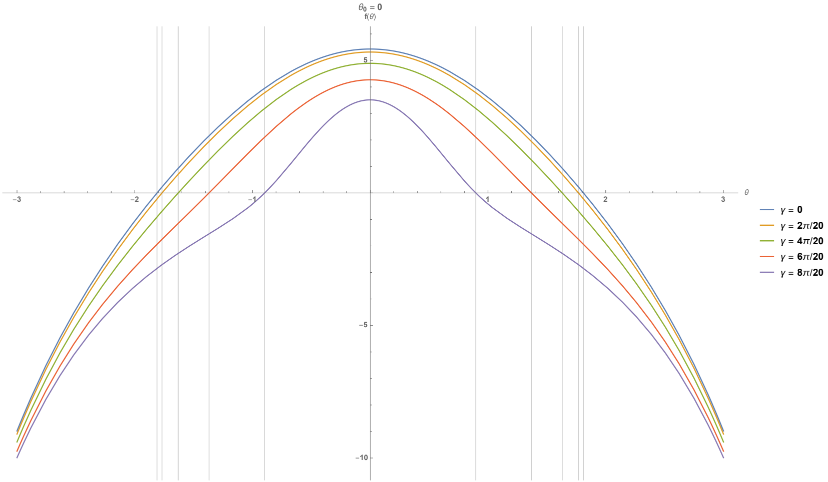

Figure: $f(\theta)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm} \operatorname{sech}(\theta + \sigma\theta_0 + \mathbb{i}\sigma'\gamma)$, $\theta_0 = 0$.
Grey vertical lines are the edges of $\Theta = [-B, B]$.

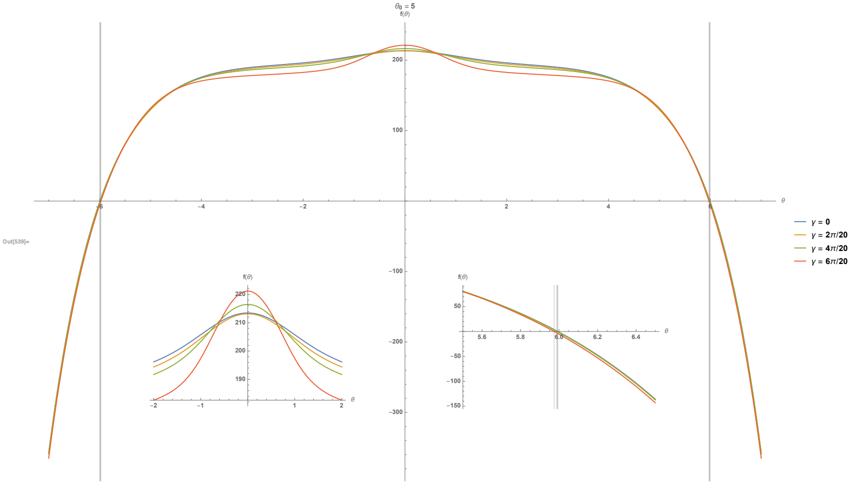

Figure: $f(\theta)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm} \operatorname{sech}(\theta + \sigma\theta_0 + \mathrm{i}\sigma'\gamma)$, $\theta_0 = 5$. Grey vertical lines are the edges of $\Theta = [-B, B]$.

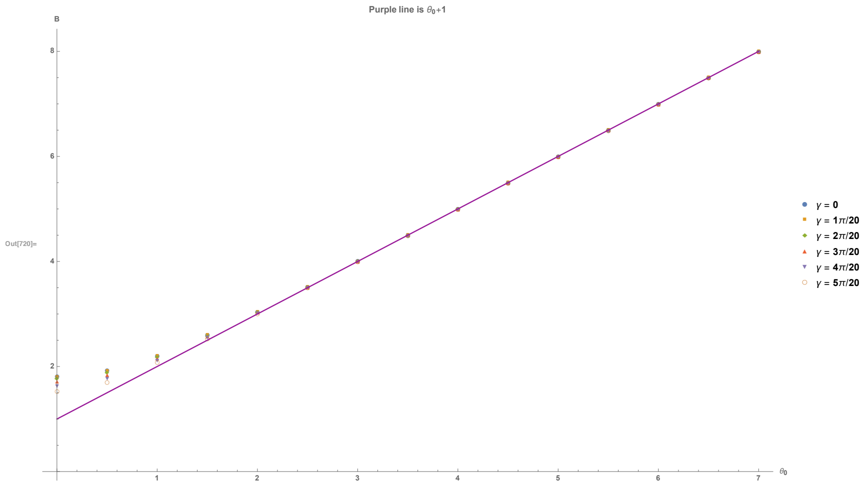

Figure: Edge B of $\Theta = [-B, B]$ as function of θ_0. Solid line is $\theta_0 + 1$.

Out[657]=

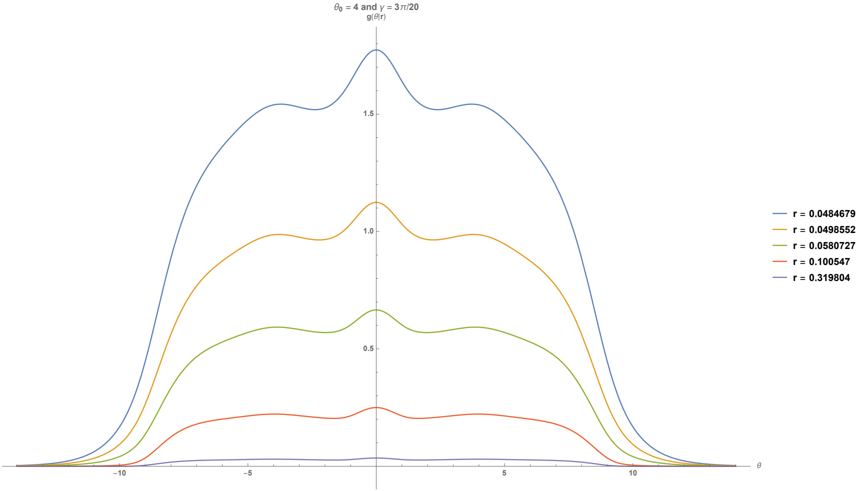

Figure: $g(\theta|r)$ for $\Phi(\theta) = \frac{1}{2\pi}\sum_{\sigma,\sigma'=\pm} \mathrm{sech}\,(\theta + \sigma\theta_0 + \mathrm{i}\sigma'\gamma)$, $\theta_0 = 4$ and $\gamma = 3\pi/20$.

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[10] G. Mussardo and S. Penati, arXiv: hep-th/9907039.

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Thank you!