Geometrical 4-point functions in the 2d critical Q-state Potts model

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based on: 2002.09071 w/ Grans-Samuelsson, Jacobsen, Saleur

2005.07258 w/ Jacobsen, Saleur

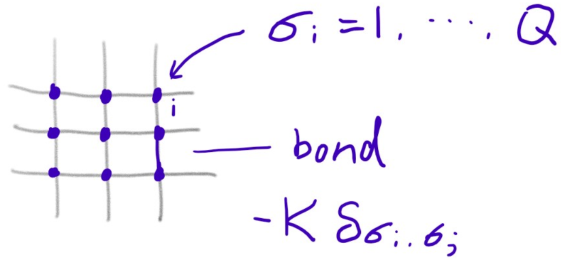
Potts, geometrical correlations, CFT

Potts $\leftrightarrow$ MM

interchiral conformal bootstrap

summary

Potts model and FK clusters

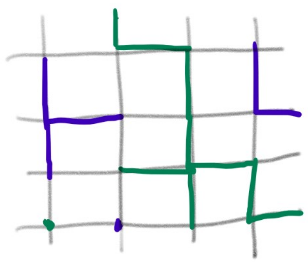


$$Z_{spin} = \sum_{\{\sigma\}} \prod_{ij} e^{K\,\delta_{\sigma_i \sigma_j}} \qquad Q \geq 2,\ Q \in \mathbb{Z}$$

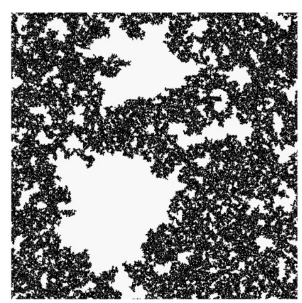
Ising

$v = e^K - 1$

$$Z_{FK} = \sum_{\text{diagrams}} v^{\#\text{ bonds}}\, Q^{\#\text{ clusters}} \qquad Q \in \mathbb{R}$$



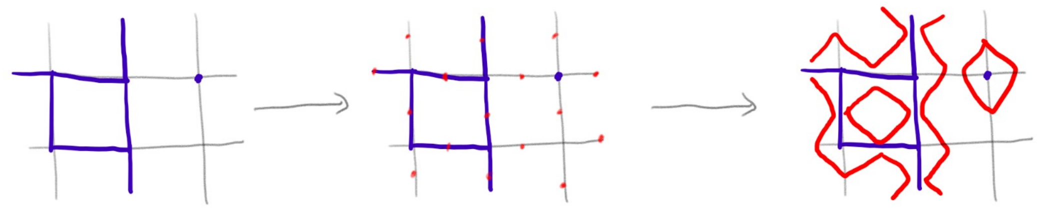
Fortuin-Kastelegn (FK) clusters



percolation : $Q \to 1$

2d square lattice, $v_c = \sqrt{Q}$ $\quad 0 \leq Q \leq 4$: continuous phase transition $\longrightarrow$ CFT

cluster $\Longleftrightarrow$ loops



$$Z_{FK} \Longrightarrow Z_{loop} \underset{\substack{\uparrow \\ \text{critical}}}{=} \sqrt{Q}^{\,\#\text{ sites}} \sum_{\substack{\text{loop} \\ \text{config}}} \sqrt{Q}^{\,\#\text{ loops}}$$

each loop carries weight $\sqrt{Q}$

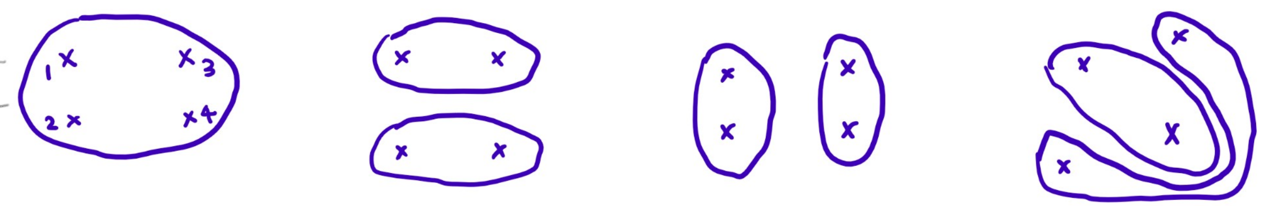
geometrical correlations

spin correlator $\langle \sigma \dots \sigma \rangle$ $\quad \sigma_\alpha = Q\,\delta_{\alpha,\sigma_i} - 1 \quad \alpha = 1, \dots, Q \quad Q \in \mathbb{Z}$

more general: geometrical correlations in FK clusters

eg. $P_{aabca}(i_1, i_2, i_3, i_4, i_5) \quad \underline{\underline{Q \in \mathbb{R}}}$

we focus on: $\sum$



probabilities $\quad P_{aaaa} \quad P_{abab} \quad P_{aabb} \quad P_{abba}$

✱ Goal: determine these most fundamental 4-pt functions in Potts CFT

Potts CFT

$0 \le Q \le 4$ parametrize $\sqrt{Q} = 2\cos\frac{\pi}{x+1}$ $x \in [1, \infty]$

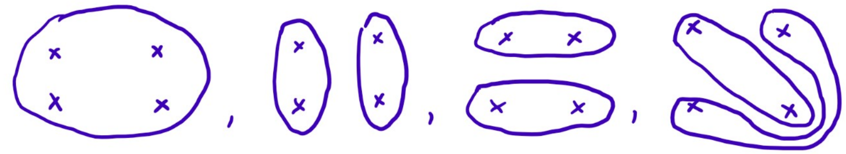
central charge $c = 1 - \frac{6}{x(x+1)} \in [-2, 1]$ non-unitary! (logarithmic)

use Kac parametrization $h_{r,s}$ can be fractions

non-degenerate!

order parameter $\Phi_{\frac{1}{2},0}$

$$\langle \Phi_{\frac{1}{2},0}(z_1,\bar{z}_1) \underbrace{\Phi_{\frac{1}{2},0}(z_2,\bar{z}_2)\, \Phi_{\frac{1}{2},0}(z_3,\bar{z}_3)}_{\text{bootstrap}}\, \Phi_{\frac{1}{2},0}(z_4,\bar{z}_4) \rangle \longrightarrow$$

spectrum of 

$$\langle \Phi_{\frac{1}{2},0}\,\Phi_{\frac{1}{2},0}\,\Phi_{\frac{1}{2},0}\,\Phi_{\frac{1}{2},0} \rangle = \sum_{\text{spectrum}} A_{h,\bar{h}}\, \mathcal{F}_h(z)\, \bar{\mathcal{F}}_{\bar{h}}(\bar{z})$$

amplitude : C^2_{OPE} $\left\{ \begin{array}{l} A_{aaaa} \\ A_{abab} \\ A_{aabb} \\ A_{abba} \end{array} \right.$

[Jacobsen, Saleur, 2018]

numerical + algebraic

$I =$)($e_i =$ ⌣⌢ $u =$ //⋯//

↓ generate

Affine-Temperley-Lieb (ATL) modules $\mathcal{W}$

$\mathcal{W} \supset$ a tower of conformal primaries + descendants

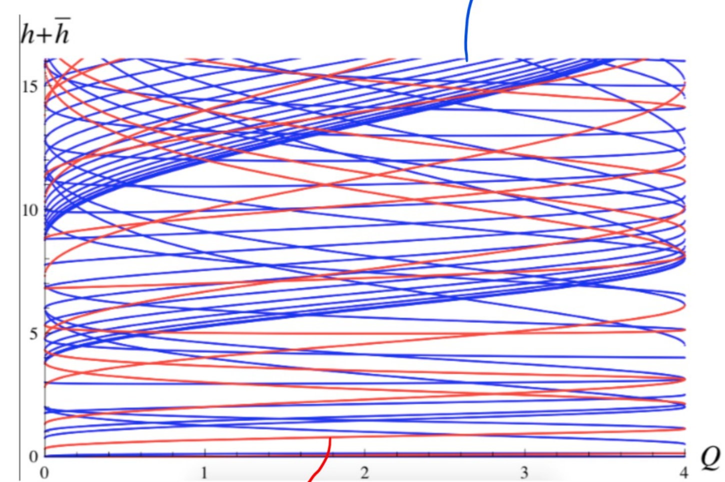
$\mathcal{W}_{j,\, e^{2i\pi\frac{p}{M}}}$: $(h_{r,s}, h_{r,-s})$ $\quad (r,s) = (\mathbb{Z} + \frac{p}{M},\ j \in 2\mathbb{Z})$

$\overline{\mathcal{W}}_{0,q^2}$: $(h_{r,s}, h_{r,s})$ $\quad (r,s) = (\mathbb{N}^*, 1)$

↑

degenerate!

Potts spectrum



MM spectrum $(\mathbb{Z}+\frac{1}{2}, 2\mathbb{Z})$

Potts $\xleftrightarrow{\;?\;}$ MM

$$\langle \phi^D_{\frac{1}{2},0}\,\phi^N_{\frac{1}{2},0}\,\phi^D_{\frac{1}{2},0}\,\phi^N_{\frac{1}{2},0}\rangle \underset{\sim}{\propto} P_{aaaa} + \frac{2}{Q-2}P_{abab}$$

OPE : $(r,s)\in\left(\mathbb{Z}+\frac{1}{2}, 2\mathbb{Z}\right)$

Monte-Carlo

[Picco, Ribault, Santachiara, 2016 & 2019]

analytic continuation of type D MM in c

non-diagonal generalization c<1 Liouville

[Migliaccio, Ribault, 2018]

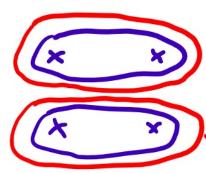
use RSOS lattice model $\xrightarrow{\text{continuum}}$ MM [Pasquier, 1987]

correlations $= \sum_{\text{diagrams}}$ clusters / loops [Kostov, 1989]

difference involve highly unlikely configs

$$\langle \phi^D_{\frac{1}{2},0}\,\phi^N_{\frac{1}{2},0}\,\phi^D_{\frac{1}{2},0}\,\phi^N_{\frac{1}{2},0}\rangle \propto P_{aaaa} + \tilde{P}_{abab}$$

"pseudo-probability"

$\sum$ 

loop weight $\neq$ Potts loop weight $\sqrt{Q}$

universal amplitude ratios

lattice numerical computation $\xrightarrow[\text{spectrum}]{\text{extract}}$ pseudo-probability $\tilde{P}$

$\tilde{P}$ involves $\sim$ same conformal fields as in P amplitude $\tilde{A}$

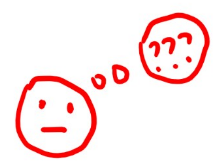
high precision numerical facts on the lattice : $(h_{r,s}, \bar{h}_{r,s})$

1. $\dfrac{\tilde{A}_{abab}(h_{r,s}, \bar{h}_{r,s})}{A_{abab}(h_{r,s}, \bar{h}_{r,s})} \xRightarrow{\text{depends on}} \mathcal{W}, Q$

2. $\dfrac{A_{abab}(h_{r,s}, \bar{h}_{r,s})}{A_{aaaa}(h_{r,s}, \bar{h}_{r,s})}, \dfrac{A_{aabb}(h_{r,s}, \bar{h}_{r,s})}{A_{aaaa}(h_{r,s}, \bar{h}_{r,s})} \xRightarrow{\text{depends on}} \mathcal{W}, Q$

} do not depend on lattice size! $\longrightarrow$ continuum

eg: $\dfrac{\tilde{A}_{abab}}{A_{abab}}(\mathcal{W}_{2,-1}) = \dfrac{2}{Q-2}$, $\dfrac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \dfrac{1}{1-Q}$, $\dfrac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \dfrac{2-Q}{2}$, . . .



interchiral conformal blocks

universal amplitude ratio :

$$P_{abab} \xrightarrow{\text{change loop weight}} \tilde{P}_{abab}$$

$$P_{aaaa} \xrightarrow[\text{Potts geometry}]{\text{different}} P_{abab}$$

$\Rightarrow$ only global amplitude $A(\mathcal{W})$ changes

relation among fields in the same $\mathcal{W}$ is rigid

$\Rightarrow$ interchiral conformal block $\mathbb{F}(\mathcal{W})$

$$P = \sum_{\mathcal{W}} A(\mathcal{W}) \mathbb{F}(\mathcal{W})$$

$$\tilde{P} = \sum_{\mathcal{W}} \tilde{A}(\mathcal{W}) \mathbb{F}(\mathcal{W})$$

LCFT: interchiral algebra $\supset \mathrm{Vir} \otimes \overline{\mathrm{Vir}}$

[Gainutdinov, Read, Saleur, 2012]

✲ extract A_{Potts} from A_{MM} — analytical continuation

analytically known ✓

$c<1$ Liouville : A^L

$$\sum_{\mathcal{W}} (A_{aaaa} + \tilde{A}_{abab}) \mathbb{F} = P_{aaaa} + \tilde{P}_{abab} = \text{MM 4-pt function}$$

$$\sum_{\mathcal{W}} (A_{aaaa} + A_{abab}) \mathbb{F} = P_{aaaa} + P_{abab}$$

$$A_{aaaa}(\mathcal{W}_{0,-1}) = A^L(\mathcal{W}_{0,-1}), \quad A_{abab}(\mathcal{W}_{2,-1}) = \frac{Q-2}{2} A^L(\mathcal{W}_{2,-1}), \quad A_{aaaa}(\mathcal{W}_{4,-1}) = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} A^L(\mathcal{W}_{4,-1})$$

interchiral conformal bootstrap

crossing equations:

$$P^{(s)}_{aaaa} = P^{(t)}_{aaaa}$$

t-channel

spectrum, A_{aaaa}

$$P^{(s)}_{abab} = P^{(t)}_{abab}$$

$$P^{(s)}_{aabb} = P^{(t)}_{abba}$$

$$P^{(s)}_{abba} = P^{(t)}_{aabb}$$

eg: $$\sum_{\mathcal{W} \in \mathrm{Spec}\, P_{aabb}} A_{aabb}(\mathcal{W}) \mathbb{F}^{(s)}(\mathcal{W}) = \sum_{\mathcal{W} \in \mathrm{Spec}\, P_{abba}} A_{abba}(\mathcal{W}) \mathbb{F}^{(t)}(\mathcal{W})$$

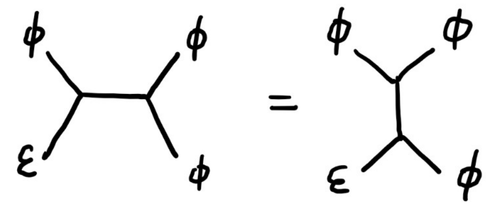
$\Rightarrow$ **solve for** $A_{aaaa}(\mathcal{W})$, $A_{abab}(\mathcal{W})$, $A_{aabb}(\mathcal{W})$, $A_{abba}(\mathcal{W})$

determine 

degeneracy $\longrightarrow \mathbb{F}(\mathcal{W})$

to construct $\mathbb{F}(\mathcal{W})$: degeneracy of ε : $(h_{2,1}, h_{2,1}) \in \mathcal{W}_{0,q^2}$

☆ $\varepsilon \times \phi_{r,s} \sim \phi_{r+1,s} + \phi_{r-1,s}$



$\Rightarrow$ recursion of C_{OPE}

$c<1$ Liouville theory : $(h_{2,1}, h_{2,1})$, $(h_{1,2}, h_{1,2})$ degeneracy $\Rightarrow \dfrac{C_{r+1,s}}{C_{r-1,s}}, \dfrac{C_{r,s+1}}{C_{r,s-1}} \Rightarrow$ full analytic solution

[Zamolodchikov ×2, 1995]

[Teschner, 1995]

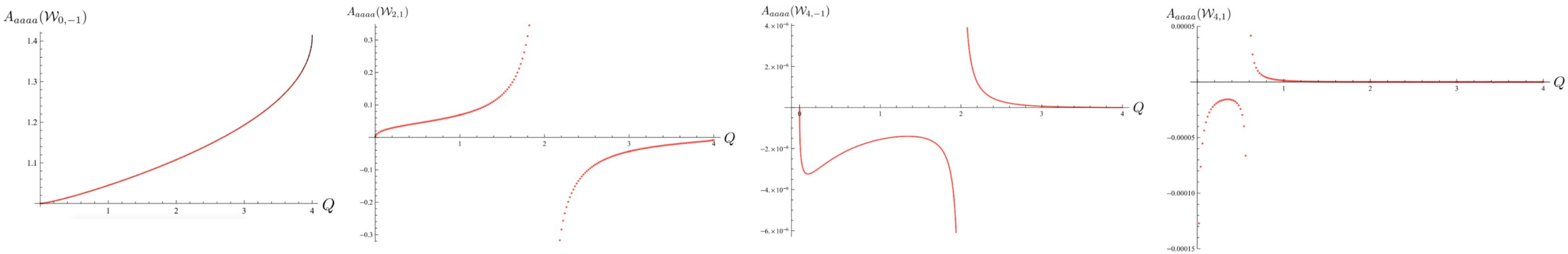
$(h_{1,2}, h_{1,2})$ not degenerate in Potts !!!

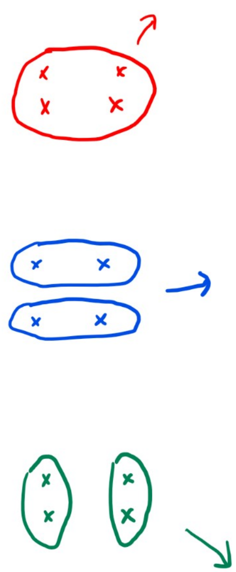
non-diagonal generalization ✓ [Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2017] $\Rightarrow A^2$

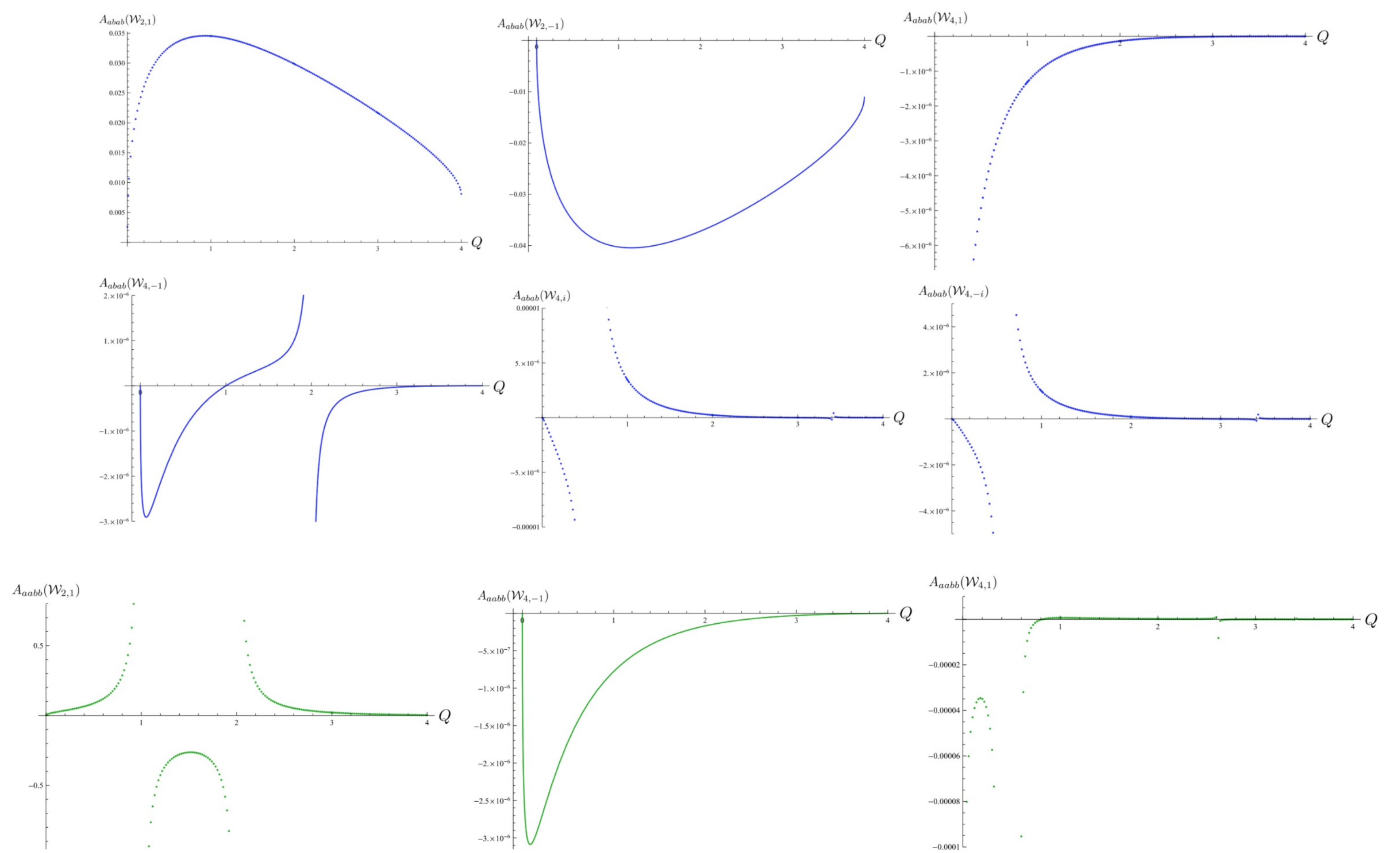
for $\langle \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \rangle$: degeneracy $\longrightarrow \dfrac{A_{r+1,s}}{A_{r,s}} =$ relation of fields in the same $\mathcal{W}$

$$\mathbb{F}(\mathcal{W}) = \sum_{\text{fields} \in \mathcal{W}} \text{recursion} \times \text{Virasoro block}$$

results







lattice ✓

cancel poles ✓

"renormalized" Liouville recursions

non-diagonal $c<1$ Liouville : $\underbrace{\text{degenerate } (h_{1,2}, h_{1,2})}_{\text{not in Potts !!!}} \Rightarrow \left.\begin{array}{c} \dfrac{A^L(\mathcal{W}_{j+1,\ })}{A^L(\mathcal{W}_{j-1,\ })} \\ + \\ \text{degenerate } (h_{2,1}, h_{2,1}) \end{array}\right\} \to \text{analytic}$

in Potts : a "renormalized" version of Liouville recursion

$$\left.\begin{array}{l} \dfrac{A_{aaaa}(\mathcal{W}_{4,-1})}{A_{aaaa}(\mathcal{W}_{0,-1})} = \dfrac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} \dfrac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{0,-1})} \\[2ex] \dfrac{A_{abab}(\mathcal{W}_{4,-1})}{A_{abab}(\mathcal{W}_{2,-1})} = \dfrac{(Q-1)(Q-4)(Q^2-4Q+2)}{2Q(Q-3)^2} \dfrac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{2,-1})} \\[2ex] \dfrac{A_{aaaa}(\mathcal{W}_{4,1})}{A_{aaaa}(\mathcal{W}_{2,1})} = \dfrac{(Q-2)^2}{(Q-1)^2(Q^2-4Q+2)} \dfrac{A^L(\mathcal{W}_{4,1})}{A^L(\mathcal{W}_{2,1})} \end{array}\right\} \Rightarrow \text{analytically solvable?!}$$

dressed with a rational function of Q — ???

summary

☑ uncovered Potts $\longleftrightarrow$ MM

☑ determined  numerically semi-analytically IF

[*] logCFT at generic c [Grans-Samuelsson, Liu, YH, Jacobsen, Saleur, 2020]

See also [Nivesvivat, Ribault, 2020] & [Gorbenko, Zan, 2020] on $O(n)$ LCFT

[?] rational c : percolation ($Q \to 1$), Ising clusters ($Q \to 2$)

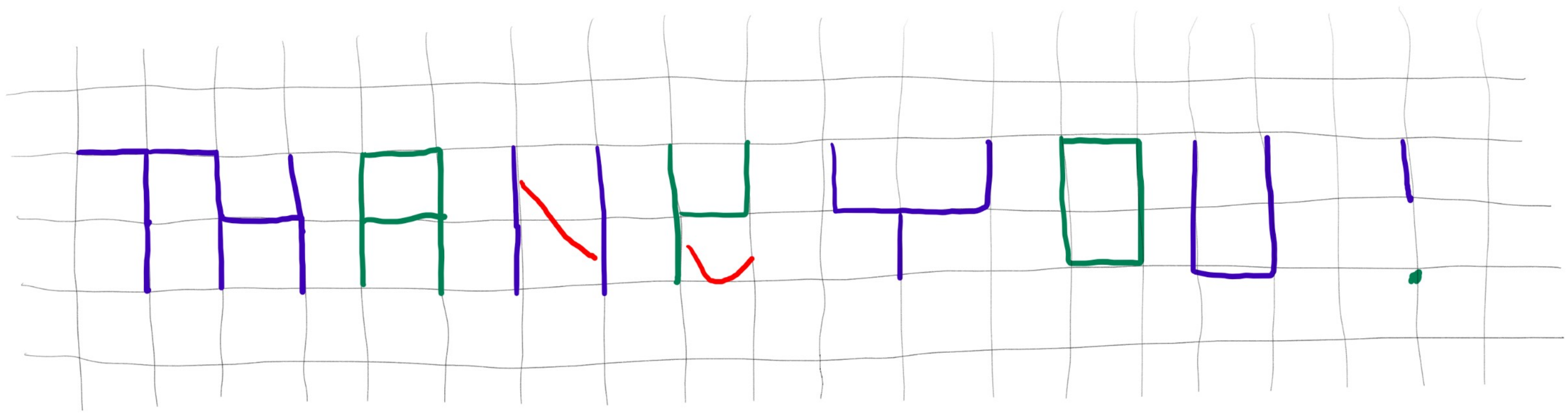
[?] universal amplitude ratios

"renormalized" Liouville recursions

$\Longrightarrow$ full analytical solution ?

$\downarrow$

symmetry aspects


THANK YOU!