Integrable Limits of Scattering Amplitudes

Georgios Papathanasiou

Integrability in Gauge and String Theory 2020
August 25, 2020

PRL 124, 161602 (2020) w/ Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,Verbeek
PRL 124, 161603 (2020) w/ Basso, Dixon

The Origin of the Six-Gluon Amplitude

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PRL 124, 161603 (2020) w/ Basso, Dixon
JHEP 08 (2019) 016 & 09 (2019) 061 w/ Caron-Huot,Dixon,Dulat,McLeod,Hippel
JHEP 02 (2016) 185 w/ Drummond

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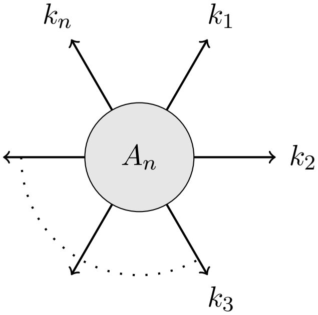

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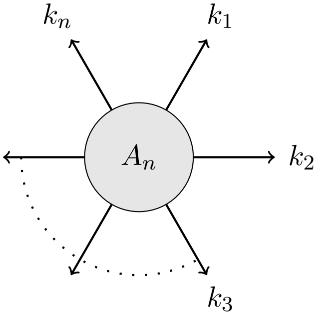

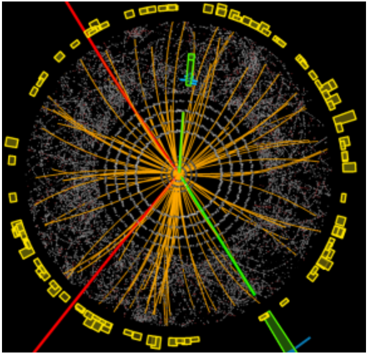

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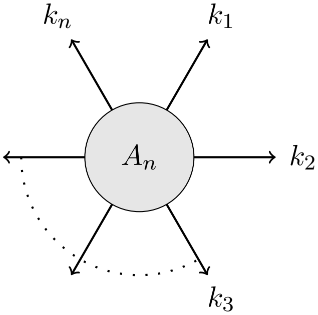

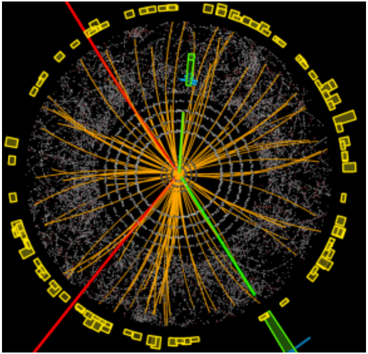

Scattering amplitudes $A = \langle \text{IN}|S|\text{OUT}\rangle$: $d\sigma \propto |A|^2$

- Computing efficiently necessary in practice
- Understanding beyond perturbation theory mathematically important [Millenium Prize]

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Celebrated example: The cusp anomalous dimension [Beisert,Eden,Staudacher]

$$\Gamma_{\text{cusp}} = 4g^2 \left[\frac{1}{1+\mathbb{K}}\right]_{11} = 4g^2 \left[1 - \mathbb{K} + \mathbb{K}^2 + \ldots\right]_{11} \quad \leftarrow \text{matrix component}$$

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt) J_j(2gt)}{e^t - 1}, \quad i,j = 1,2,\ldots \quad J_i(x) : \text{Bessel f}^{\text{n}}$$

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Can we hope for similar progress with amplitudes?

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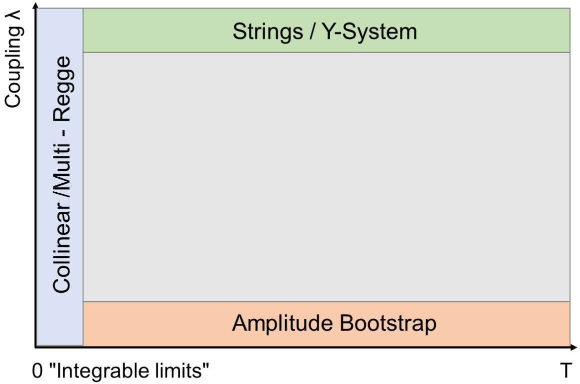

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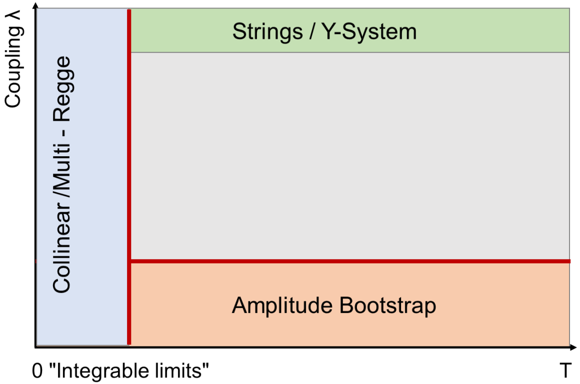

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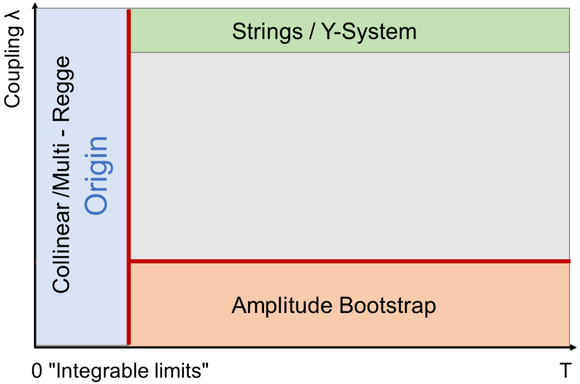

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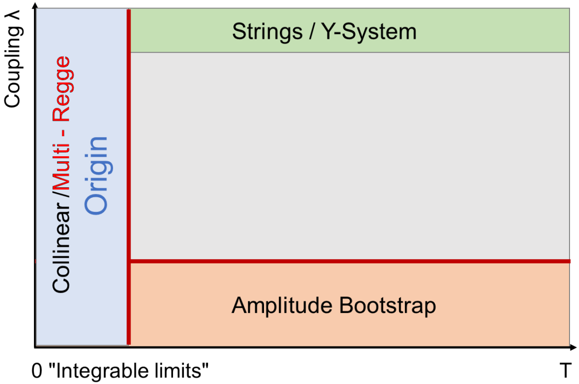

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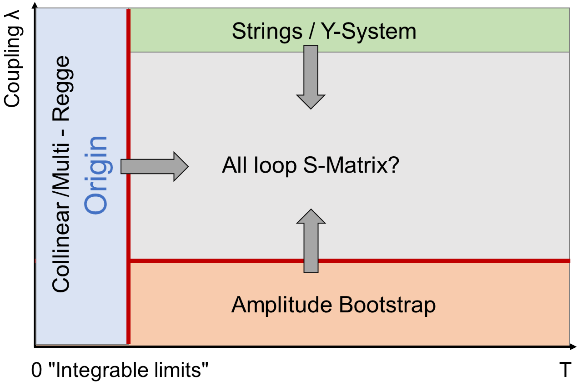

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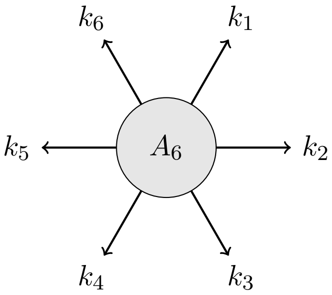
Outline

The Six-Gluon Maximally Helicity Violating (MHV) Amplitude

Simplest nontrivial case, $A_6(-,-,+,\ldots,+)$. Remarkable properties,



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- is dual to null hexagonal Wilson loop $W_{\hexagon}$
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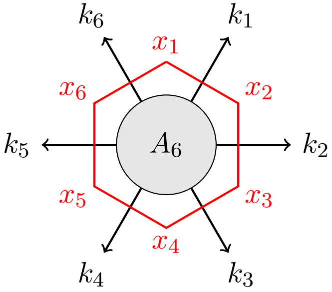

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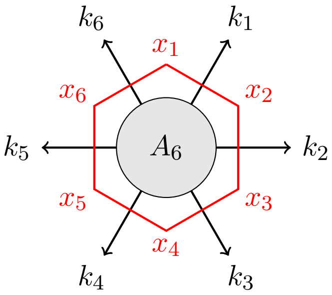

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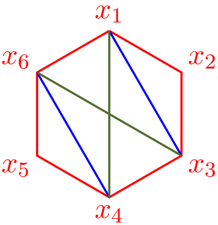

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- exhibits dual conformal symmetry under $x_i^\mu \to x_i^\mu / x_i^2$: After factoring out universal IR-divergent Bern-Dixon-Smirnov(-like) part, function of 3 conformal cross ratios u_i.

The Origin of the Six-Gluon Amplitude

$\mathcal{E}_6$ (and $\mathcal{E}_7$) computed most efficiently in general kinematics & at fixed order in the coupling via *Amplitude Bootstrap*.
[Recent Review: Caron-Huot,Dixon,Drummond,Dulat,Foster,Gürdoğan,Hippel,McLeod, GP]

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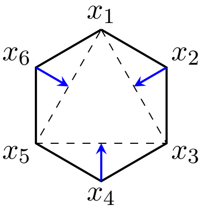
Natural to scan space of kinematics for all-loop patterns and simplifications. Here: Focus on limit when $u_i \to 0$: "origin"



$$u_1 = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2}\,, \quad x_{13}^2 = s_{12} \to 0 \quad \text{plus } i \to i+2$$

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In the origin limit $u_i \to 0$, from perturbative results up to 7 loops, observed that six-particle amplitude takes the form, [Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP]

$$\ln \mathcal{E}_6 = -\frac{\Gamma_{\text{oct}}}{24} \ln^2 (u_1 u_2 u_3) - \frac{\Gamma_{\text{hex}}}{24} \sum_{i=1}^{3} \ln^2 (\frac{u_i}{u_{i+1}}) + C_0 \,.$$

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Connection? How about $\Gamma_{\rm hex}, C_0$? Finite coupling?

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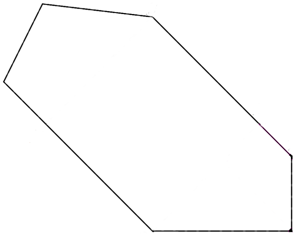

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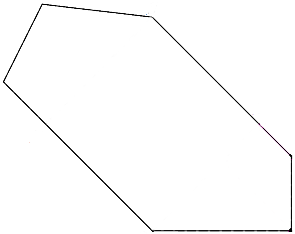

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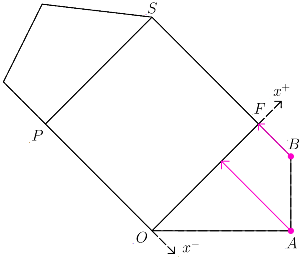

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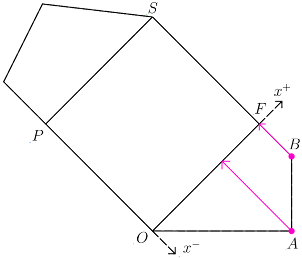
$$\mathcal{W}_6 \equiv \mathcal{E}_6 e^{\frac{1}{2}\Gamma_{\text{cusp}}(\sigma^2+\tau^2+\zeta_2)}$$



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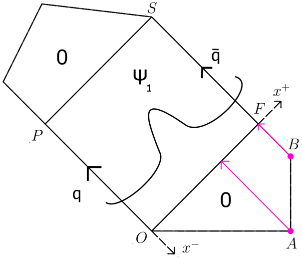

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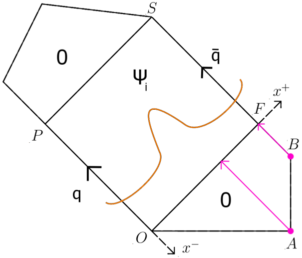

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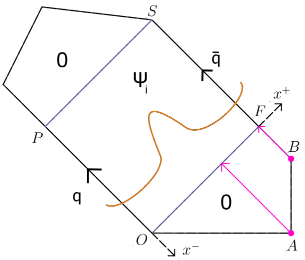

Wilson Loop 'Operator Product Expansion (OPE)'
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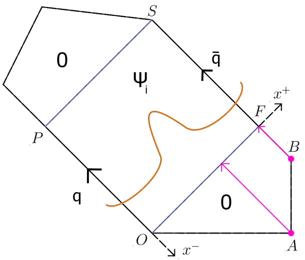

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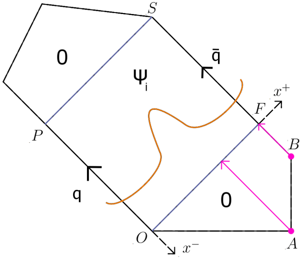

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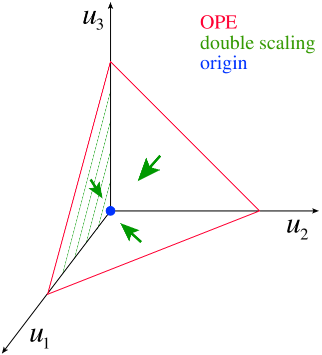
[Belitsky,Bonini,Bork,Caetano,Cordova,Drummond,Fioravanti,Hippel,Lam,Onishchenko, GP,Piscaglia,Rossi...

A Path to Originality



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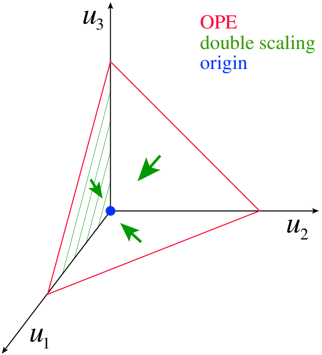

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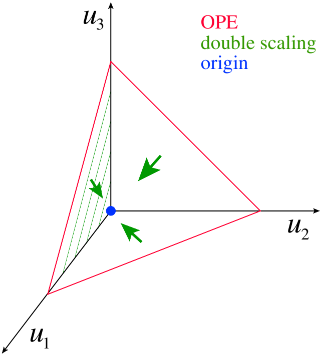

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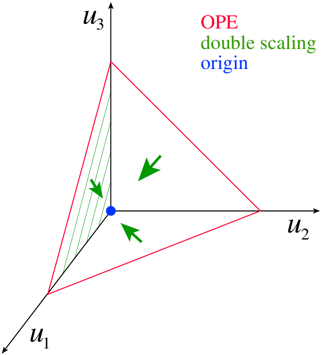
- However part of double scaling limit:



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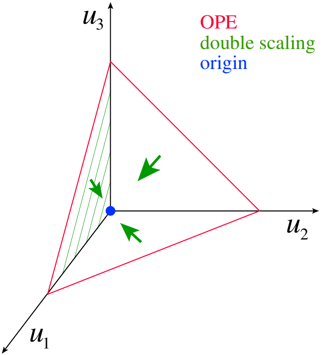

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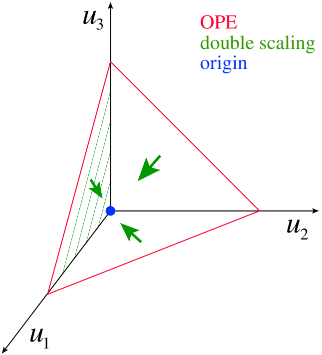

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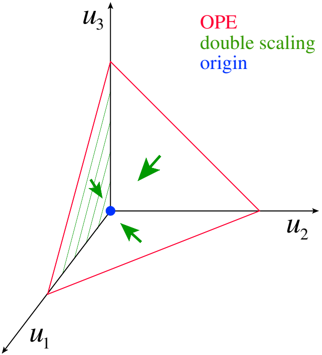

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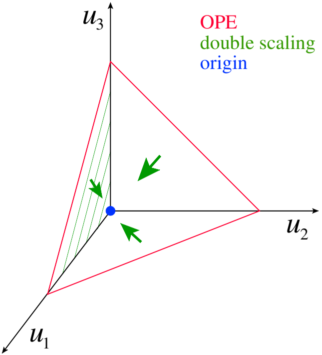

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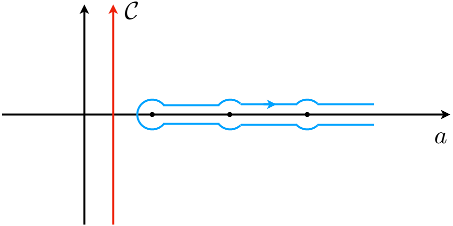
Sommerfeld-Watson Transform

Similar to Regge theory, where it amounts to analytic continuation in spin,

$$\sum_{a\geqslant 1}(-1)^a f(a) \to \int_{+\infty-i\epsilon}^{+\infty+i\epsilon} \frac{i f(a) da}{2\sin(\pi a)}\,,$$

provided $f(z)$ decays faster than $1/z$ as $z \to \infty$.



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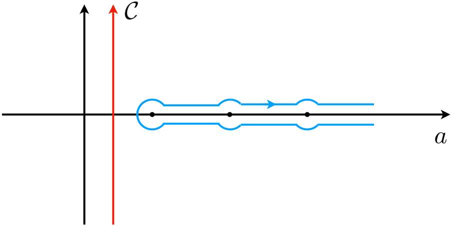

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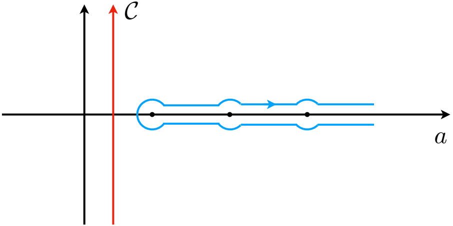

Finally, closing contour around $a=0$ on the left-hand side yields all nonvanishing terms at origin at finite coupling!

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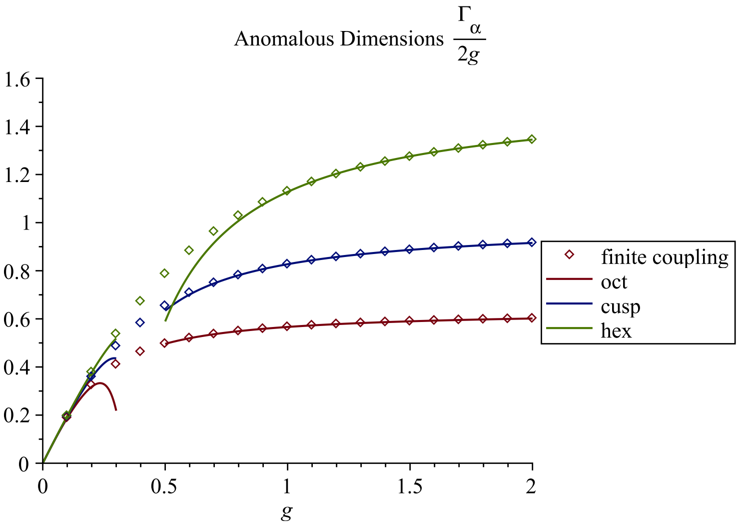
Comparison: Finite-coupling numerics & weak/strong coupling analytics



$$\Gamma_{\text{oct}} = \Gamma_0\,, \quad \Gamma_{\text{cusp}} = \Gamma_{\pi/4}\,, \quad \Gamma_{\text{hex}} = \Gamma_{\pi/3}$$

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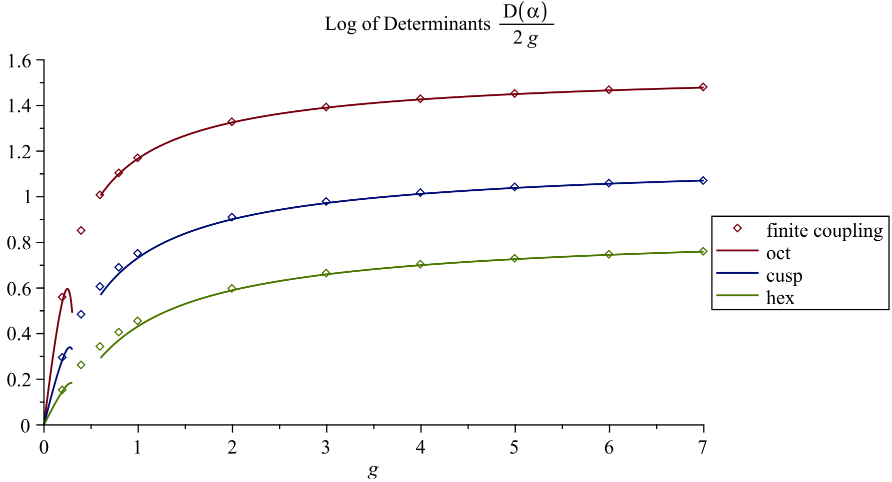

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Strong coupling: Expansion & Comparison with String Theory

Expanded Γ_α to four orders in $1/g$, and C_0 to two. For example,

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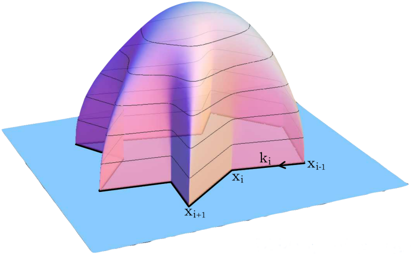

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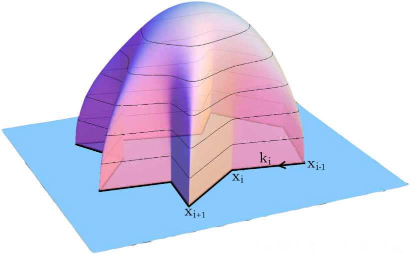

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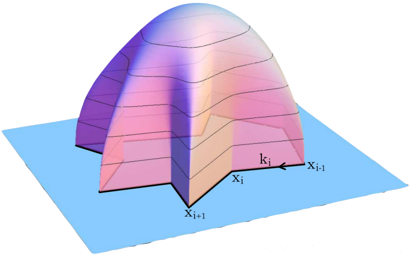

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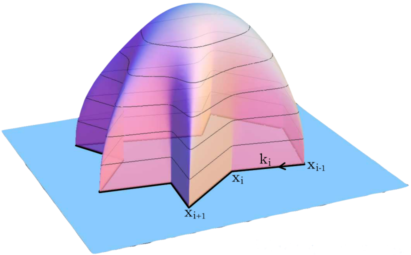

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Can also confirm Γ_{hex}. [Ito,Satoh,Suzuki]

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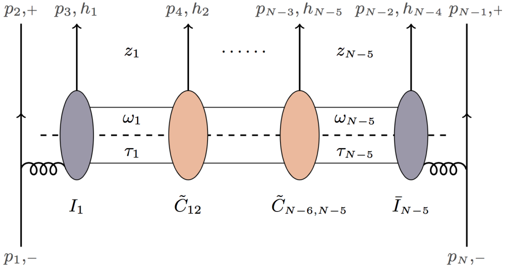

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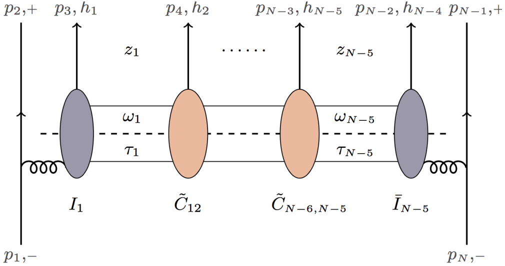

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All-order amplitudes in MRL at *any multiplicity*!

[Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek;PRL 124, 161602 (2020)]

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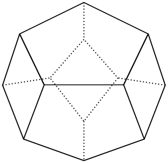

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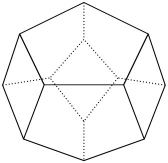

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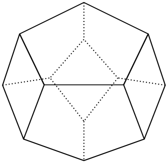

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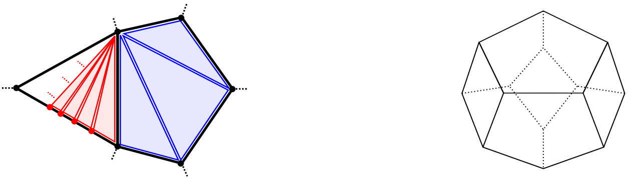

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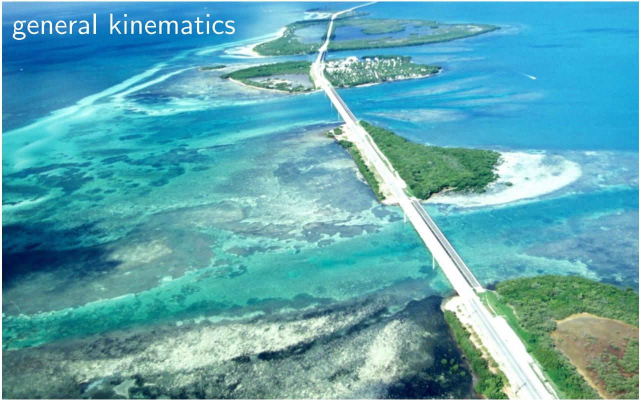

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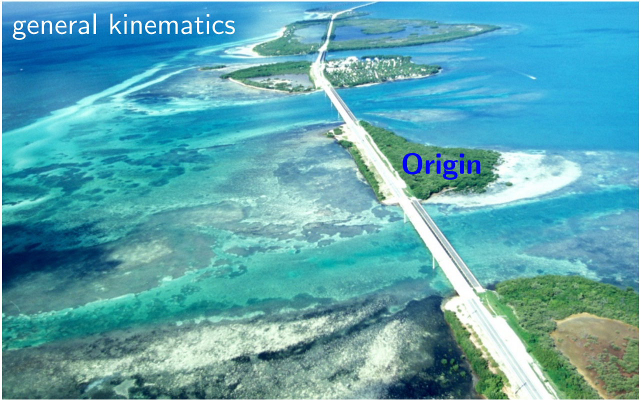

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> Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

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Evade Feynman diagrams by exploiting analytic structure

QFT Property	Computation

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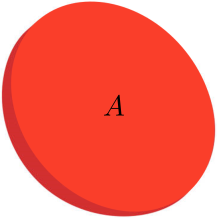
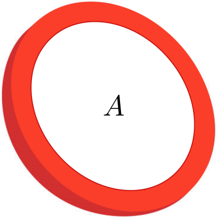


QFT Property	Computation
Physical Branch Cuts	$A_6^{(L)}, L = 3, 4$
[Gaiotto,Maldacena, Sever,Vieira]	[Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]

Higher Loops and Legs: The Amplitude Bootstrap

Evade Feynman diagrams by exploiting analytic structure



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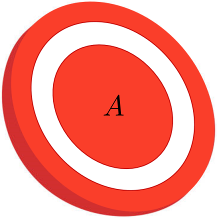

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See also recent $S(A_7) \to A_7$ work by [Dixon,Liu]

Weak coupling expansion of Γ_α

	$L=1$	$L=2$	$L=3$	$L=4$
Γ_{oct}	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$
Γ_{cusp}	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$
Γ_{hex}	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$

$$
\begin{aligned}
\frac{\Gamma_\alpha}{4g^2} &= 1 - 4c^2\zeta_2 g^2 + 8c^2(3+5c^2)\zeta_4 g^4 \\
&\quad - 8c^2\left[(25+42c^2+35c^4)\zeta_6 + 4s^2\,\zeta_3^2\right]g^6 + \dots\,, \\
D(\alpha) &= 4c^2\zeta_2 g^2 - 4c^2(3+5c^2)\zeta_4 g^4 \\
&\quad + \frac{8}{3}c^2\left[(30+63c^2+35c^4)\zeta_6 + 12s^2\,\zeta_3^2\right]g^6 + \dots\,,
\end{aligned}
$$

$$
\Gamma_{\text{oct}} = \Gamma_0\,, \qquad \Gamma_{\text{cusp}} = \Gamma_{\pi/4}\,, \qquad \Gamma_{\text{hex}} = \Gamma_{\pi/3}
$$

Strong coupling expansion of Γ_α

Letting $a = \alpha/\pi$, find

$$\Gamma_\alpha = \frac{8ag}{\sin(2\pi a)}\left[1 - \frac{s_1}{2\sqrt{\lambda}} - \frac{as_2}{4\lambda} - \frac{a(s_1 s_2 + as_3)}{8(\sqrt{\lambda})^3} + \ldots\right],$$

where

$$s_{k+1} = \{\psi_k(1) - \psi_k(\tfrac{1}{2} + a)\} + (-1)^k\{\psi_k(1) - \psi_k(\tfrac{1}{2} - a)\},$$

and $\psi_k(z) = \partial_z^{k+1} \ln \Gamma(z)$ the polygamma function.

Secretly Gaussian integral

Origin=OPE integrand in modified integration contour. Can recast as infinite-dimensional integral,

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i F(\vec{\xi}\,)\, e^{-\vec{\xi}\cdot M\cdot\vec{\xi}}\,,$$

where $M = (1+\mathbb{K})\cdot\mathbb{Q}$ and $F(\xi,\phi,\tau,\sigma)$ complicated Fredholm determinant. Remarkably, observe that perturbatively

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i\, e^{-\vec{\xi}\cdot(M+\delta M)\cdot\vec{\xi}}\,,$$

becomes Gaussian but with modified kernel⇒ evaluate explicitly!

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Construction of a Crossing-Simmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories.

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The High-Energy or (Multi-)Regge Limit

Implications for (Quantum) Gravity

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- Recently, essential in resolving disputes in binary black hole dynamics [Bern,Ita,Parra-Martinez,Ruf]

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Phenomenological Applications and Beyond

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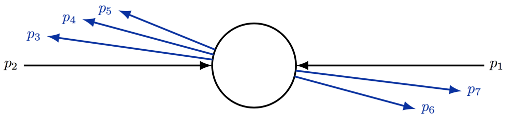
- Beauty: First instance of integrability in gauge theory! [Lipatov][Faddeev,Korchemsky]

Multi-Regge Kinematics (MRK)

For $1 + 2 \to 3 + \ldots + n$ scattering, defined (in COM frame) as

$$p_3^+ \gg p_4^+ \gg \cdots \gg p_n^+ \,,$$
$$p_i^+ \equiv p_i^0 + p_i^4 \,,$$



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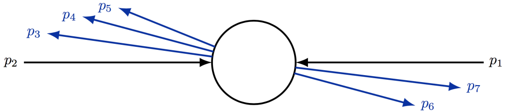

with 2D transverse plane components held fixed,

$$\mathbf{p}_{i+1} \equiv p_{i+1}^1 + i p_{i+1}^2 \equiv \mathbf{x}_i - \mathbf{x}_{i-1} \,, \; i = 1, \ldots n-2.$$

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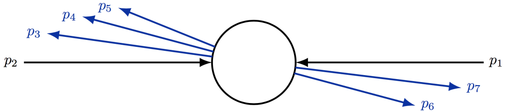

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MSYM: Dual conformal symmetry $\to SL(2,\mathbb{C})$ (plus small)!

$$z_i \equiv \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} \,, \quad \tau_i \equiv \sqrt{u_{1i+3} u_{ni+2}} \to 0 \,, \quad i = 1 \ldots n-5 \,.$$

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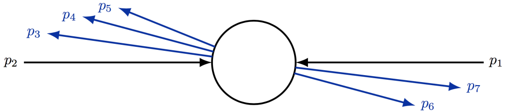

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Euclidean region: $\mathcal{R}_n \to 1$. Here: analytically continue (2-Reggeon region)

$$u_{2,n-1} \to e^{-2\pi i} u_{2,n-1} \,.$$

The six-particle amplitude in the multi-Regge limit

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BFKL approach: Dispersion integral

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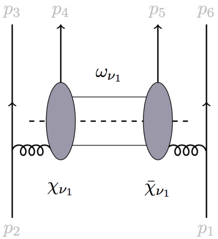

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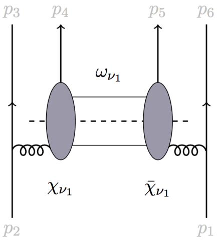

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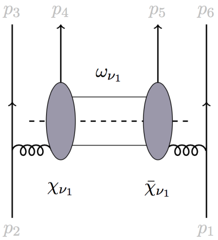

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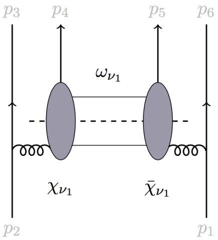

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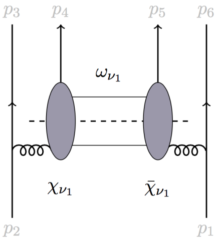

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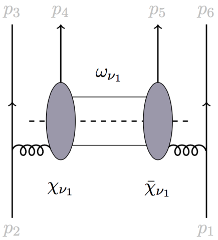

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- Kinematic dependence in z_i and $L_1 = \log \tau_1 + i\pi$

Direct computation very challenging beyond first few orders. However, progress via collinear limit.

From Collinear to Multi-Regge Limit for $n = 6$ particles

Coll. Kinematics: $T \to 0$ in convenient choice of kin.variables F, T, S.

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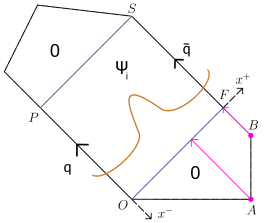
[Alday,Gaiotto,Maldacena,Sever,Vieira]

$$\mathcal{W}_6^{\text{coll}} = \sum_{a=1}^{\infty} (FT)^a \int \frac{du}{2\pi} \mu_a(u) S^{ip_a(u)} T^{\gamma_a(u)} =$$



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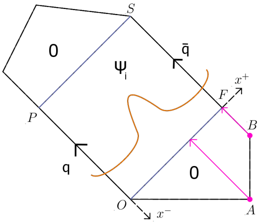

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Compare with MRK: $$\frac{\mathcal{W}_6}{2\pi i} = \sum_{n_1} \left(\frac{z_1}{z_1^*}\right)^{\frac{n_1}{2}} \int \frac{d\nu_1}{2\pi} \tilde{\Phi}_1 |z_1|^{2i\nu_1} e^{-L_1\omega_1}$$

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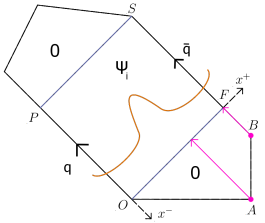
[Alday,Gaiotto,Maldacena,Sever,Vieira]

$$\mathcal{W}_6^{\text{coll}} = \sum_{a=1}^{\infty} (FT)^a \int \frac{du}{2\pi}\, \mu_a(u) S^{ip_a(u)} T^{\gamma_a(u)} =$$



where γ_a, p_a, μ_a excitation energy, momentum and integration measure known at finite coupling [Basso,Sever,Vieira]

Compare with MRK: $$\frac{\mathcal{W}_6}{2\pi i} = \sum_{n_1} \left(\frac{z_1}{z_1^*}\right)^{\frac{n_1}{2}} \int \frac{d\nu_1}{2\pi} \tilde{\Phi}_1 |z_1|^{2i\nu_1} e^{-L_1\omega_1}$$

Intriguing resemblance. . .

From Collinear to Multi-Regge Limit for $n = 6$ particles

Beyond resemblance

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For example,

$$\omega(u,n) = -4g(\mathbb{M}\cdot\kappa)_1\,, \quad \mathbb{M} \equiv (1+\mathbb{K})^{-1}\,,$$

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}\,, \quad i,j=1,2,\ldots \quad J_i(x): \text{Bessel f}^{\text{n}}$$

$$\kappa_j(u,n) = -\int_0^\infty \frac{dt}{t}\frac{J_j(2gt)}{e^t-1}\left(\frac{1}{2}\left[e^{\frac{t(1+(-)^j)}{2}} - (-)^j e^{\frac{t(1-(-)^j)}{2}}\right]\cos(ut)e^{-\frac{nt}{2}} - J_0(2gt)\right).$$

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For $n = 7$ at LLA, can generalize $n = 6$ BFKL approach.
[Bartels,Kormilitzin, Lipatov, Prygarin]

$$\mathcal{W}_7^{\mathrm{MRL}} = \prod_{i=1}^{2} \sum_{n_i} \left(\frac{z_i}{z_i^*}\right)^{\frac{n_i}{2}} \int \frac{d\nu_i}{2\pi} \times$$

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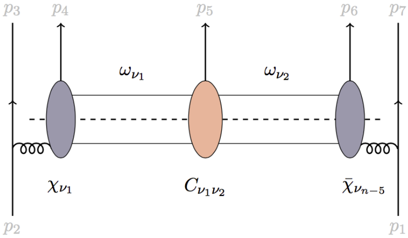

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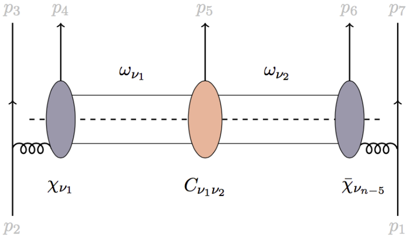

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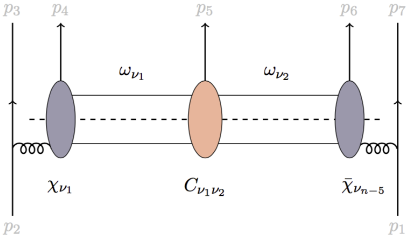

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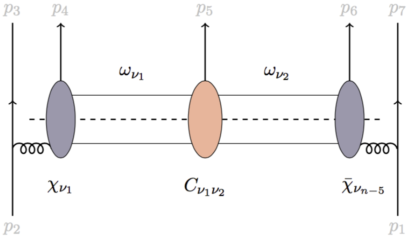

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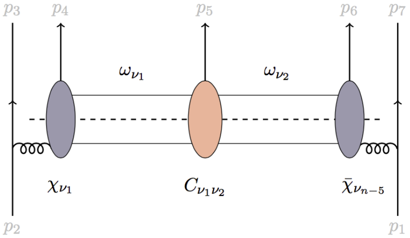

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Beyond LLA however, contains unphysical divergences! For $n_i = \nu_i = 0$, double pole $\Phi_i \to \nu_i^{-2}$ pinching integration contour.

The seven-particle amplitude in MRK beyond the leading logarithm

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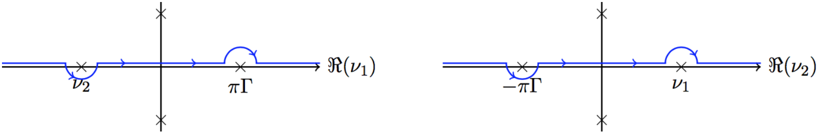

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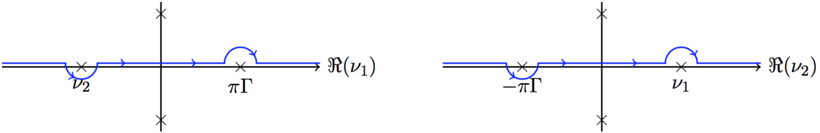

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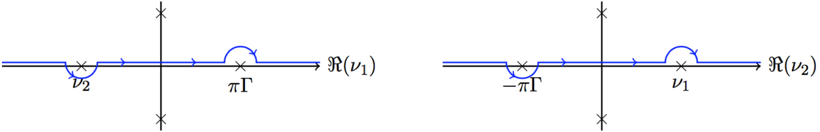

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$$\omega_0(\pm\pi\Gamma) = 0,\quad \text{Res}_{\nu=\pm\pi\Gamma}\left(\tilde{\Phi}(\nu,0)\right) = \pm\frac{1}{2\pi},$$
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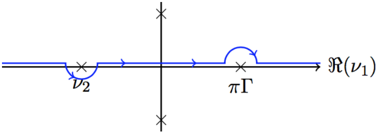

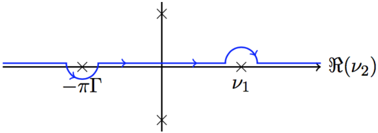

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For earlier work on $n=6$, see also [Caron-Huot]

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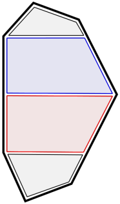

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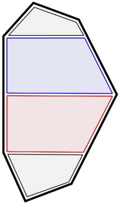

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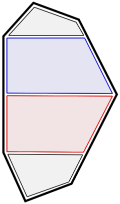

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7-gluon amplitude ($\tilde{C}_{12}$) in MRL *to all loops*!

Central Emission Block to All Loops

[Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek;PRL 124, 161602 (2020)]

$$\tilde{C}_{12} = \frac{\tilde{C}_{12}^{(0)}}{g^2} Z_{12} h_1 \check{h}_2 \exp\left(f_{12} - f_{\tilde{1}\tilde{2}} - i f_{\tilde{1}2} + i f_{1\tilde{2}} - A\right)$$

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$$f_{rs} = 4\kappa(u_r, n_r) \cdot \mathbb{Q} \cdot \mathbb{M} \cdot \kappa(u_s, n_s)\,,\ \text{with}\ \ \mathbb{Q}_{ij} = \delta_{ij}(-1)^{i+1} i\,,$$

whereas the infinite-dimensional matrix $\mathbb{M}$ and vector κ were defined earlier (with similar definitions for $\kappa \to \tilde{\kappa}$, $f_{rs} \to f_{\tilde{r}s}$)

Amplitudes at Any Multiplicity in the Multi-Regge Limit

Our regularization works for any number of gluons n, yielding

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as a well-defined $(n-5)$-fold *Fourier-Mellin (FM) transform* $\mathcal{F}$.

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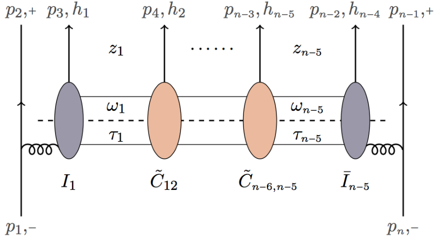
Integrand factorization:

$$A_n = \prod_{i=1}^{n-5} e^{-L_i \omega_i} \tilde{\Phi}_i \times I_1 \prod_{r=1}^{n-6} \tilde{C}_{rr+1} \bar{I}_{n-5} =$$



Amplitudes at Any Multiplicity in the Multi-Regge Limit

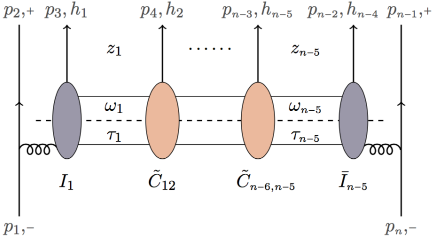
Our regularization works for any number of gluons n, yielding

$$\mathcal{W}_n^{\text{MRL}} = \prod_{i=1}^{n-5} \sum_{n_i=\infty}^{\infty} \left(\frac{z_i}{z_i^*}\right)^{\frac{n_i}{2}} \int \frac{d\nu_i}{2\pi} |z_i|^{2i\nu_i} A_n \equiv \mathcal{F}[A_n]$$

as a well-defined $(n-5)$-fold *Fourier-Mellin (FM) transform* $\mathcal{F}$.

Integrand factorization:

$$A_n = \prod_{i=1}^{n-5} e^{-L_i \omega_i} \tilde{\Phi}_i \times I_1 \prod_{r=1}^{n-6} \tilde{C}_{rr+1} \bar{I}_{n-5} =$$



All-order amplitudes in MRL at *any multiplicity*!

[Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek; PRL 124 (2020)]

Application: Integral Evaluation & Maximal Transcendentality

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Single-valued combinations of multiple polylogarithms (SVMPLs),

$$G(a_1,\ldots,a_k;z) \equiv \int_0^z \frac{dt_1}{t_1 - a_1} G(a_2,\ldots,a_k;t_1)\,, \quad G(;z) = 1\,,$$

k : weight or transcendentality,

with their complex conjugates, such that all branch cuts cancel. [F.Brown]

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> An L-loop gluon amplitude in multi-Regge kinematics in planar MSYM has uniform weight $2L$, for any helicity configuration and any number of legs.

Dual Conformal Invariance

- In reality DCI broken by divergences, (IR in massless $\mathcal{N} = 4$/UV in cusped WL). Breaking controlled by conformal Ward identity. [Drummond,Henn,Korchemsky,Sokatchev]

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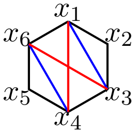

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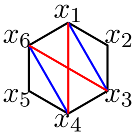
e.g. $n = 6$: $\quad u_1 = \dfrac{x_{46}^2\, x_{31}^2}{x_{36}^2\, x_{41}^2} =$  + cyclic

- In general, # of independent u_i: $m = 4n - n - 15 = 3n - 15$

Six-particle BDS(-like) Ansatz

[Bern,Dixon,Smirnov; Alday,Maldacena]

$$A_6^{\text{BDS-like}} = \exp\left[\sum_{L=1}^{\infty}(g^2)^L\Big(f^{(L)}(\epsilon)\hat{M}_6(L\epsilon) + C^{(L)}\Big)\right],$$

where

$$f(\epsilon) \;=\; \sum_{L=1}^{\infty}(g^2)^L f^{(L)}(\epsilon) \;=\; \frac{1}{4}\Gamma_{\text{cusp}} + \mathcal{O}(\epsilon).$$

and

$$\hat{M}_6(\epsilon) = (4\pi e^{-\gamma_E})^{\epsilon}\sum_{i=1}^{6}\Bigg[-\frac{1}{\epsilon^2}\left(1 + \epsilon\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) + \frac{\epsilon^2}{2}\ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right) + \frac{1}{2}\ln^2\left(\frac{s_{i,i+1}}{s_{i+1,i+2}}\right) - \frac{1}{4}\ln^2\left(\frac{s_{i,i+1}}{s_{i+3,i+4}}\right) + \frac{3}{2}\zeta_2\Bigg] + \mathcal{O}(\epsilon)\,,$$

Relation to original, BDS ansatz:

$$A_6^{\text{BDS}} = A_6^{\text{BDS-like}}\, e^{\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}_6^{(1)}}\,, \qquad \mathcal{E}_6^{(1)} = \sum_{i=1}^{3}\text{Li}_2\left(1 - \frac{1}{u_i}\right)$$

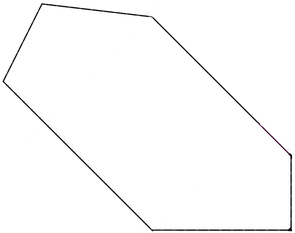
Special Conformal Ward Identity

[Drummond,Henn,Korchemsky,Sokatchev]

$$\begin{aligned}\mathbb{K}^\nu \ln W_n &= \sum_{i=1}^{n} (2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \ln W_n \\ &= -\sum_{l\geq 1} g^{2l} \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{l\epsilon} + \Gamma^{(l)} \right) \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \mu^2 \right)^{l\epsilon} x_i^\nu + O(\epsilon),\end{aligned}$$

Γ: collinear anomalous dimension

The Collinear Limit



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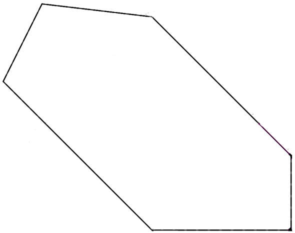

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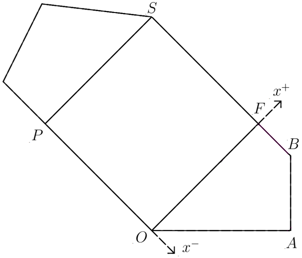
- Form null square (OPSF), by connecting two non-intersecting edges.



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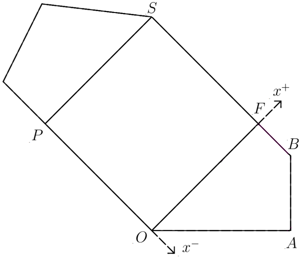

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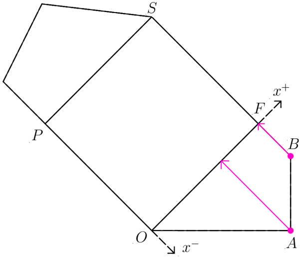

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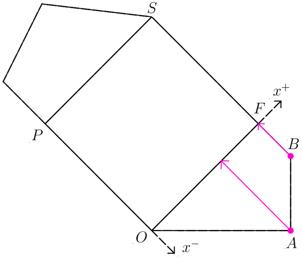
$$u_2 = \frac{1}{e^{2\tau}+1}\,, \quad u_1 = e^{2\tau+2\sigma} u_2 u_3\,,$$

$$u_3 = \frac{1}{1+e^{2\sigma}+2e^{\sigma-\tau}\cosh\varphi+e^{-2\tau}}\,.$$



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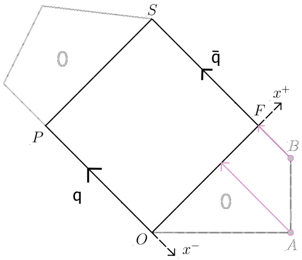

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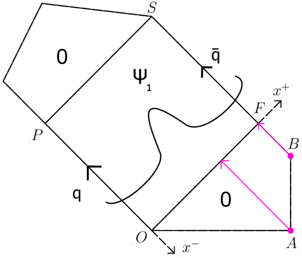

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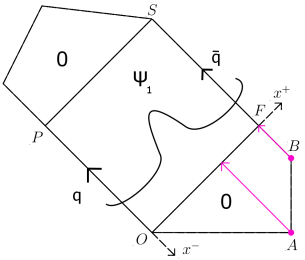

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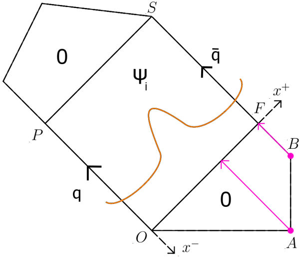

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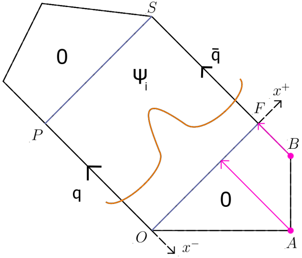

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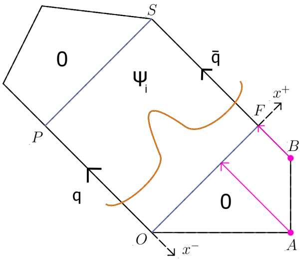

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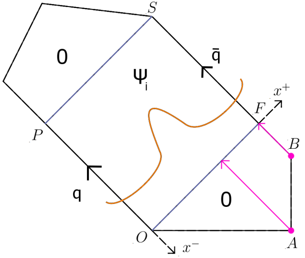

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MSYM: ψ_i mapped to excitations of integrable spin chain ⇒ exact $E, \mathcal{P}$
[Basso,Sever,Vieira]

Single-particle Gluon Contribution to Six-Particle Wilson Loop/Amplitude

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1. Evaluated for $a = 1, 2$, proved space of functions they span to all loops [GP'13] [GP'14]
2. Perturbatively resummed $\forall a$. [Drummond,GP]