Chern–Simons Origin of Superstring Integrability

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4d Chern-Simons

Long-suspected connection CS theory $\leftrightarrow$ YBE realised for

$$S_{4d} = \int \omega \wedge \left[ A \wedge A + \frac{2}{3} A \wedge A \wedge A \right]$$

gauge group $G$

$\omega$ a holomorphic one-form $\omega = dz$  

$$A = A_w dw + A_{\bar{w}} d\bar{w} + A_z d\bar{z} + A_{\bar{z}} dz$$

$\Sigma$: Riemann surface ($\mathbb{R}^2$ today)

$C$: complex curve ($\mathbb{C}^*$ today)
$S_{4d} = \frac{1}{\hbar} \int \omega \wedge \left[ A_x dA + \frac{2}{3} A \wedge A \wedge A \right]$

$\Sigma^{(0,1) \times C^2}$

\[ \text{Eoms:} \quad F_{\omega \omega} = 0 \quad \text{flat ball on } \Sigma \]

\[ F_{\omega \bar{z}} = F_{\bar{z} \omega} = 0 \quad \text{holomorphic on } C \]

Mixed topological/holomorphic theory

All counterterms vanish by Eoms IR-free theory

quantization possible
4d $\mathcal{CS}$ has no local observables (like 3d $\mathcal{CS}$)

Observables are open Wilson lines $\Sigma \times \{z_i\}$

$\sim$

Topological in $\Sigma$: so $\sim \Rightarrow$ YBE ✓

Lines don't cross in 4d.

The free theory: gauge fields vanish at infinity so

Holonomy: good observable (no trace)
Interactions in 4d CS

\[ 1 + \hbar r(z_{1}, z_{2}) \]

\( r \) is classical \( r \)-matrix of hom. \( G \) spin chain!
Costello-Yamazaki '19: $A, \omega$ poles + zeros in $\mathbb{C}$  
200 of integrable $\sigma$-models from $S^4$! 

Let $L_{\omega, \sigma} = \langle A_{\omega, \sigma} \rangle$ 

Like CY '19 we set $G = PSU(2,2|4)^4$

$$\omega = \prod_{k=1}^{3} \frac{(z - q_k)(\bar{z} - \bar{q}_k)}{(z - p_i)^2} \, \, dz$$

Quantum 4d CS well-defined \[ PSU(1) \]

$$A_{\omega} \sim \frac{1}{z - q_k}, \quad A_{\bar{\omega}} \sim \frac{1}{z - \bar{q}_k}$$

$$A_{\omega}, A_{\bar{\omega}}, A_{\xi} \sim (z - p_i)$$

gauge $\mathbb{Z}_4$ sym \[ \{ A^i \rightarrow SL(A^{i+1}) \} \]

$$z \rightarrow e^{\pi i / 2} \, z$$

get $\sigma$-model on \[ (2, 2|4) \, \cdot \, SO \, \times \, SO(5) \]

Pure spinor $q_{\, \omega} \Rightarrow 0$ Metsaev-Tseytlin $q_{\, \nu}, \tilde{q}_{\, \nu} \Rightarrow 0$
Pure spinor $q_i \to 0$

Metsaev-Tseytlin $q_{1,2,3} \tilde{q}_8 \to 0$

$q_i \to \infty$

$\tilde{q}_1 \tilde{q}_2 \tilde{q}_3 \to \infty$

\[
\begin{align*}
A_w &\sim \frac{1}{z^2} \quad A_w \sim z^2 \\
A_w &\sim \frac{1}{z} \quad A_w \sim z
\end{align*}
\]

$z \sim 0 \\
\sim \infty$
4d CS theory and diffeomorphisms

On manifold without boundary
4d CS is diffeomorphism invariant

With bdry/singular fields not necessarily!

$$\delta_{\nu} A_{a} \sim \partial_{\nu} v^b A_b$$

Near $z = 0$

$$A_{\bar{w}} \sim \frac{1}{z} \quad A_{w} \sim \frac{1}{z^2}$$

$\delta_{\nu}$ mixes $A_{\bar{w}}$ and $A_{w}$ so not compatible with
Beltrami-Chern-Simons Theory

Diff invariance adds new field $\beta$ coupling to de Rham differential.

\[ d \rightarrow d + d\beta \partial_{\beta} + d\bar{z} \partial_{\bar{z}} \]

\[ S_{4d} \rightarrow S_{4d} + S_{\beta} \]

Cartan's formula for $d$

\[ d_{\nu} = [d, e_{\nu}] \]

$S_{\beta}$: BRST-exact $\Rightarrow$ field redefinition removes $S_{\beta}$

\[ A_{\nu} \rightarrow A_{\nu} - \beta_{\nu} A_{\nu} \]

\[ S_{4d} + S_{\beta} \rightarrow S_{4d} \]

$A_{\nu}, A_{\bar{\nu}}$ poles $\Rightarrow$ cannot do redefinition $\beta$ a bona fide field

\[ S_{\text{BCS}} = S_{4d} + S_{\beta} \]

Easy check for $S_{\text{BCS}} = 0$ diff $\checkmark$

\[ S_{\beta} S_{\text{BCS}} = 0 \Rightarrow A_{\nu}^{(2)} A_{\nu}^{(2)} \bigg|_{z=0} = 0 \]

\[ A_{\nu}^{(2)} A_{\bar{\nu}}^{(2)} \bigg|_{\bar{z}=0} = 0 \]

VIRASORO!
Kappa Symmetry

\( S_{8CS} \) gauge invariant

\[ \delta_k S_{8CS} = 0 \]

away from boundary

Usually gauge variation \( \kappa = 0 \) on \( \partial \mathcal{D} \) to eliminate \( \partial \mathcal{D} \) terms in \( \delta_k S_{8CS} \)

Instead, we allow

\[ \kappa \sim \frac{3^{(3)}}{2} + \text{regular} \]

as expected for K-symmetry

So expect \( \delta_k S_{8CS} \neq 0 \)

\( z = 0 \)

\[ A^w \sim z^3 + \cdots, \quad A^i, A^0, \quad w \sim z^3 \]

Explicitly near

\[ \delta_k S_{4d} = \int \frac{d^3 z}{(2\pi)^3} \left[ A_w \partial_\kappa \left( A_w, \kappa \right) - A_w \partial_\kappa \left( A_w, \kappa \right) \right] \]

\[ \sim \int \frac{d^3 z}{(2\pi)^3} \left[ \frac{A_w}{2} \partial_\kappa \left( \frac{1}{2^2} \right) \left[ A_w, \kappa \right]^{(3)} + \frac{A_w}{2} \partial_\kappa \left( \frac{1}{2^2} \right) \left[ A_w, \kappa \right]^{(3)} \right] \]

\[ + \frac{A_w}{2} \partial_\kappa \left( \frac{1}{2^2} \right) \left[ A_w, \kappa \right]^{(3)} + \frac{A_w}{2} \partial_\kappa \left( \frac{1}{2^2} \right) \left[ A_w, \kappa \right]^{(3)} \]

\[ = \int \frac{d^2 \theta}{(2\pi)^2} A_w^{(3)} \left[ A_w^{(2)} \right]^{(3)} + 2 \delta^2 \left( \kappa \right) A_w^{(2)} \left[ A_w^{(2)} \right]^{(3)} \]

\[ z = 0 \]

\[ z = \infty \]

\[ \delta_k S_{4d} = \frac{1}{2} \int \left[ A^{(1)} A^{(2)} \right]^{(3)} \]
\( \delta_x S_{\text{4d}} \neq 0 \) localized at \( z = 0, \infty \)

\( \delta_x S_{\text{4d}} \) familiar from Arutyunov + Frolov!

it is \( \kappa \)-variation of matter fields in G S action

For some G, famous Fierz identities \( \Rightarrow \)
can cancel \( \delta_x S_{\text{4d}} \) by \( \delta_x (\text{metric}) \)

For BCS this is

\[ \delta^a_{\Sigma} = A^a_w \kappa^a \kappa^b A^b_w \]

\[ \delta_x \beta^w = \frac{\delta_k \kappa^{k \Sigma}}{2} + \kappa^{i \Sigma} [\kappa (3)] A^{(3)} A^i_{\Sigma \Sigma} \]

giving

\[ \delta_x S_{\text{BCS}} = 0 \]
Quantizing BCS theory

Vacuum config has to reduce to BMN geodesic

\[ J_w = J_{\bar{w}} = \kappa (T_E - T_J) \]

where \( T_E, T_J \in \text{psu}(2,2|4) \)

with the MT boundary conditions

\[ a^{BMN}_{w} = (w + \bar{w}) \right_\partial \left. f \right|_{(T_E - T_J)} \]

\[ a^{BMN}_{\bar{w}} = (f + \frac{1}{2z}) (T_E - T_J) \]

\[ a^{BMN}_{w} = (f + \bar{w}^2) (T_E - T_J) \]

where \( f(z, \bar{z}) \) is a simple fn w/ suitable asymptotics

\[ (f, \bar{f}) \sim 0, \quad f \sim (-1)^z, \quad z = \rho \bar{z} \]

The BRST operator w/ bkd \( a^{BMN} \) is

\[ Q_{BRST} = \partial_b + a^{BMN} \]

This breaks

\[ \text{psu}(2,2|4) \rightarrow \text{psu}(2|2) \]
and is starting point for quantization
Conclusions

Beltrami-Chern-Simons theory is a new formulation of Green-Schwarz superstring with integrability manifest works in plane-wave & flat space.

Other kids?

Quantize BCS theory as Beltrami's ghosts, $e = \Theta$-symmetry

TFT(ω,ω)

Church b-c ghosts from BCS anti-chiral b-c ghosts

Derive holographic integrability