2D gravity, matrix integrals and the minimal string

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IGST
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Plan (and based on)

- Part 1: Jackiw-Teitelboim gravity [Saad Shenker Stanford 19]
- Part 2: Generalization with defects [Maxfield, GJT 20] [Witten 20]
- Part 3: 2D gravity and the minimal string [SSS]
  [Mertens, GJT 20]
  [Usatyuk, Weng, GJT wip]

Jackiw-Teitelboim Gravity

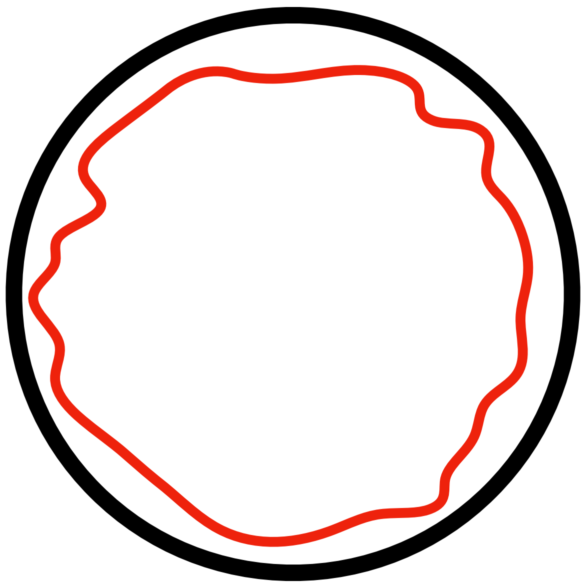
- Simple two dimensional theory of dilaton-gravity

$$I_{JT} = -\frac{S_0}{4\pi}\int R - \frac{1}{2}\int \phi(R+2) + I_{GHY}$$

- Asymptotically AdS_2 boundary conditions

$$\phi|_{\mathrm{bdy}} = \frac{\gamma}{\varepsilon}, \quad L|_{\mathrm{bdy}} = \frac{\beta}{\varepsilon}$$

$$\varepsilon \to 0$$



- Theory reduces to a boundary mode

- Broken conformal symmetry

[Almheiri, Polchinski 14] [Jensen 16]
[Maldacena,Stanford,Yang16] [Englesoy, Mertens Verlinde 16]…

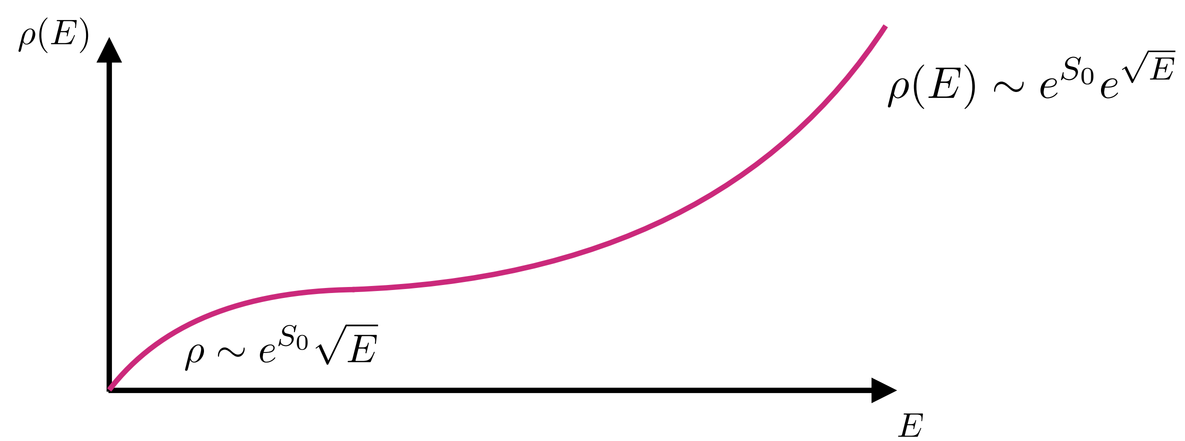
- Final answer for disk partition function $Z_{\text{disk}}(\beta) = e^{S_0}\sqrt{\frac{\gamma^3}{2\pi\beta^3}}e^{\frac{2\pi^2\gamma}{\beta}}$

  [Stanford, Witten 17]

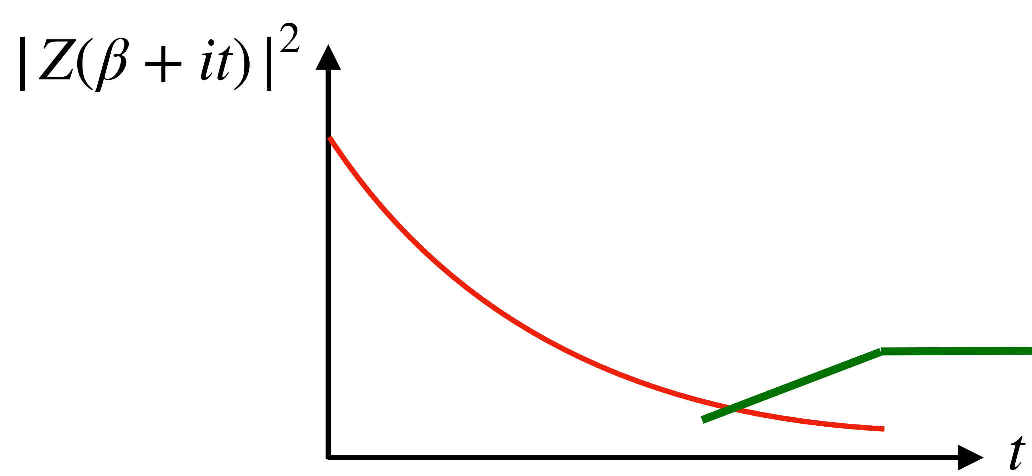
  Also [Altland, Bagrets, Kamenev 16] [Mertens, GJT, Verlinde 17] [Kitaev Suh 18] [Yang 18]

- Density of states:

$$\boxed{\rho_{\text{JT}}(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(2\pi\sqrt{2\gamma E}\right)}$$



- Problem: The spectrum is continuous! SFF decays in time forever, matter correlators, etc…



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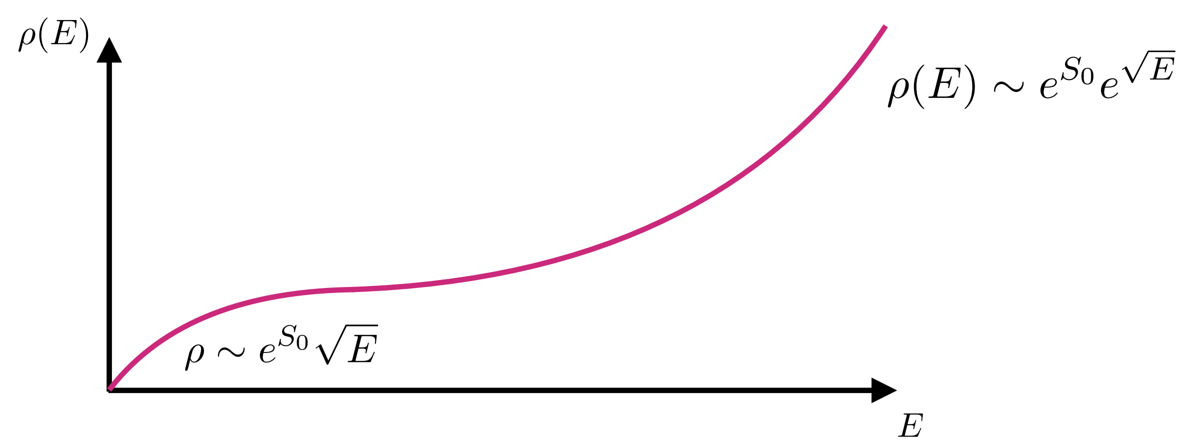


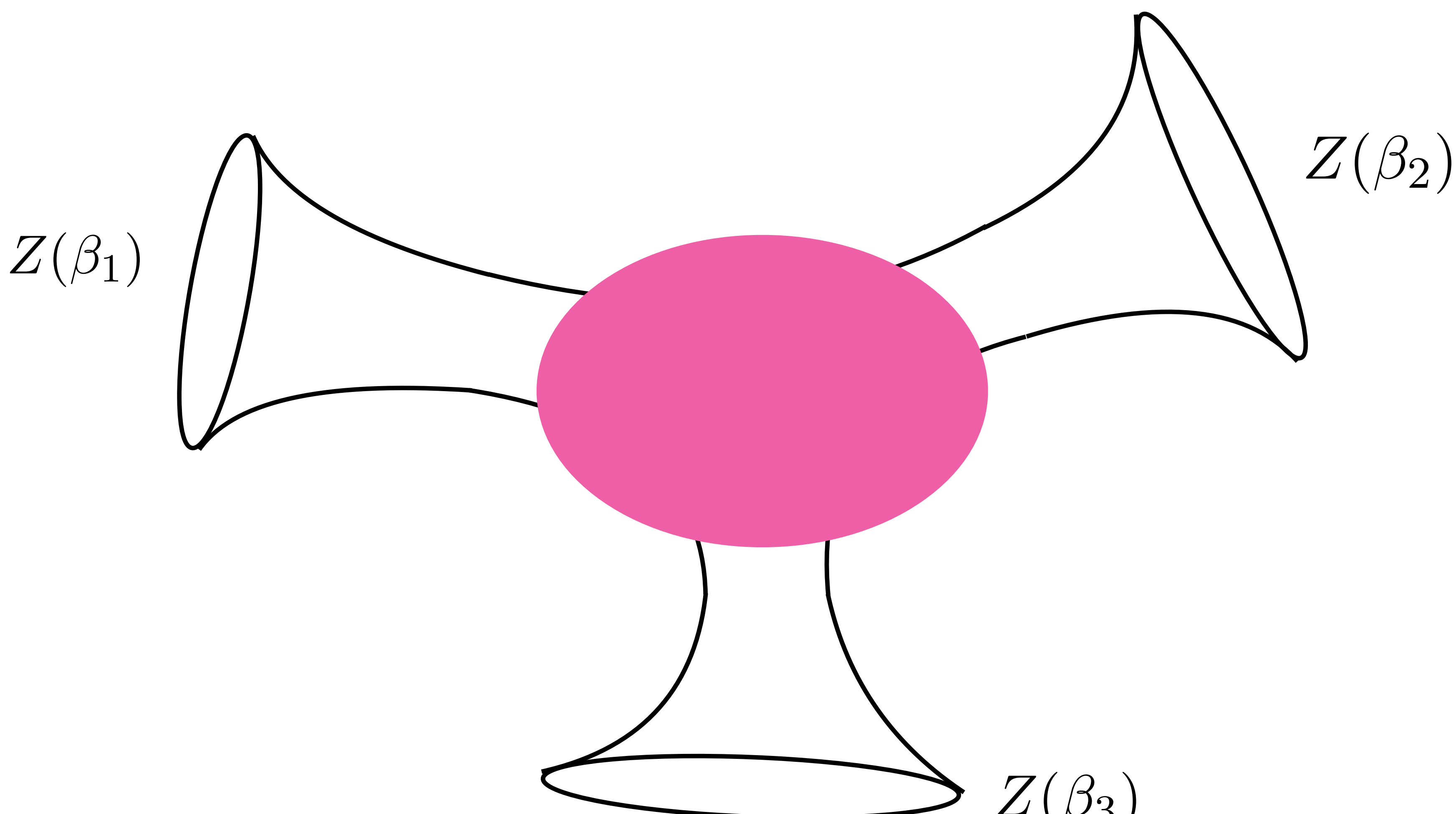
- Problem: The spectrum is continuous! SFF decays in time forever, matter correlators, etc…

- Saad-Shenker-Stanford: This can be solved by allowing to sum over topologies

Sum over topologies

- In 2D topologies are classified by genus. We will also include the possibility of having any number of boundaries

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\ldots,\beta_n)$$



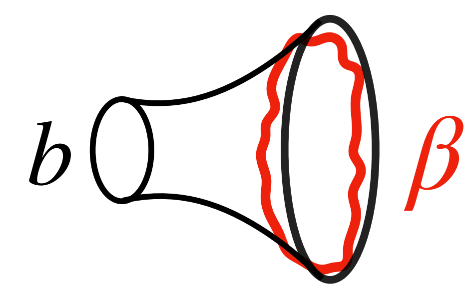
Sum over topologies

- Convenient to organize the path integral in the following way

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\dots,\beta_n)$$

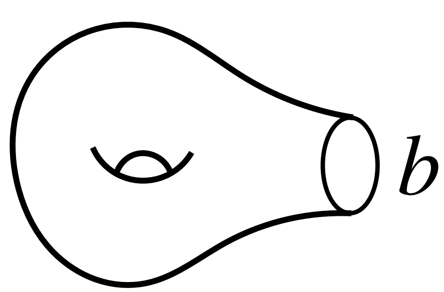
- Ingredients:

  - ☑ Trumpet: $Z_{\text{trumpet}}(\beta,b) = \sqrt{\dfrac{\gamma}{2\pi\beta}}\, e^{-\frac{\gamma}{2}\frac{b^2}{\beta}}$ $\rightarrow$



  - ☑ Hyperbolic surfaces with geodesic boundaries:

$$V_{g,n}(b_1,\dots,b_n) \quad \rightarrow$$

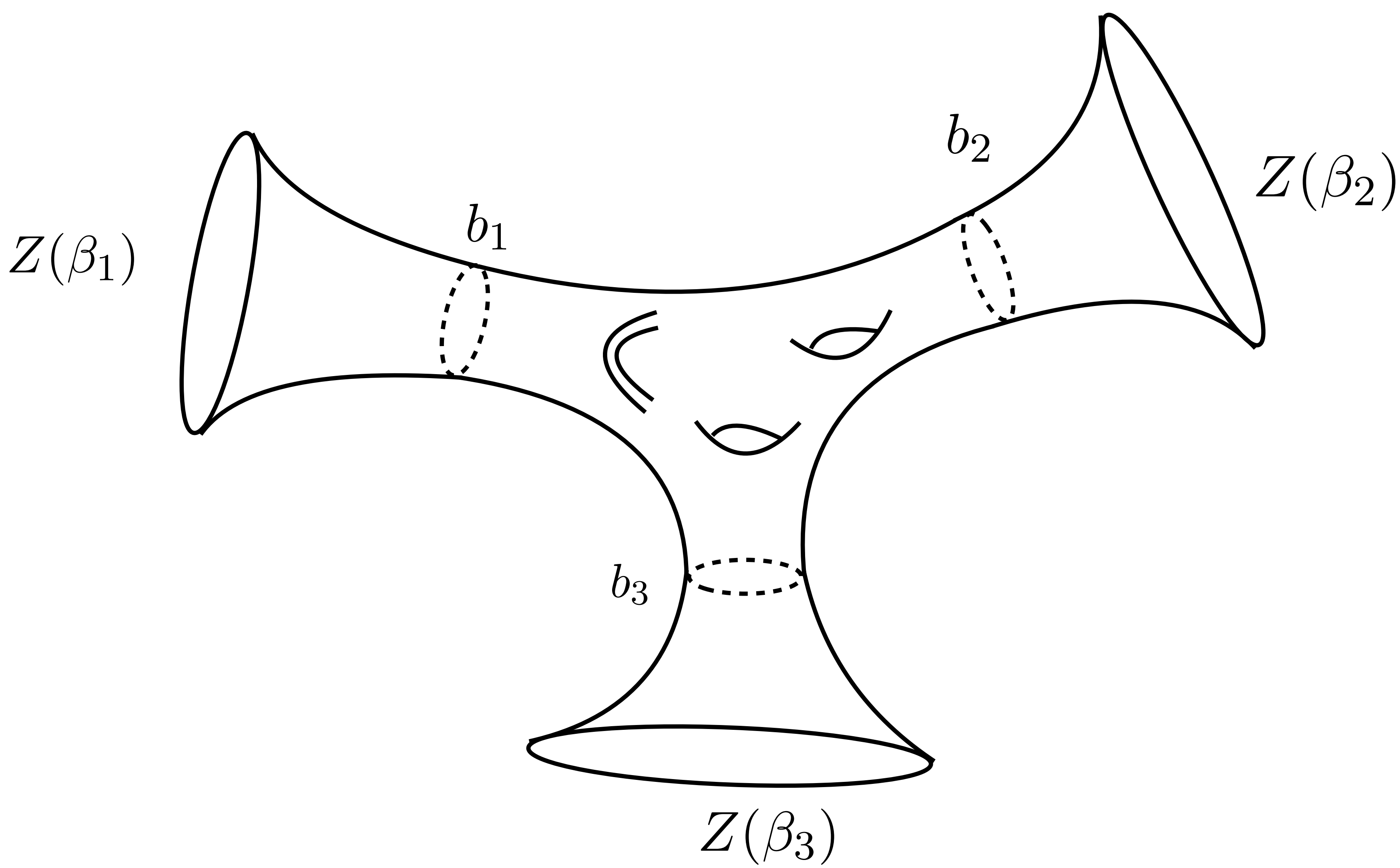


$\Rightarrow$ Weil-Petersson volumes, computed using Mirzakhani recursion

Sum over topologies

- Final answer obtained by gluing:

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int_0^\infty b_1 db_1 Z_{\text{trumpet}}(\beta_1, b_1)\cdots\int_0^\infty b_n db_n Z_{\text{trumpet}}(\beta_n, b_n) V_{g,n}(b_1,\cdots,b_n)$$



JT gravity and matrices

- SSS figured out a systematic way to sum over topologies when doing the path integral of JT gravity.

- The theory is equivalent, in a holographic sense, to a **double scaled** matrix integral

$$\langle Z(\beta_1) \cdots Z(\beta_n) \rangle = \int dH \; \mu(H) \; \mathrm{Tr}(e^{-\beta_1 H}) \cdots \mathrm{Tr}(e^{-\beta_n H})$$

$\uparrow$

Operator that "creates" a boundary

- New insight: the observable dual to JT gravity is a boundary thermal partition function, averaged over hamiltonians for some measure

$$\mu(H) \propto e^{-L \, \mathrm{Tr} \, V(H)}$$

$$\langle Z(\beta_1)\ldots Z(\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\ldots,\beta_n)$$

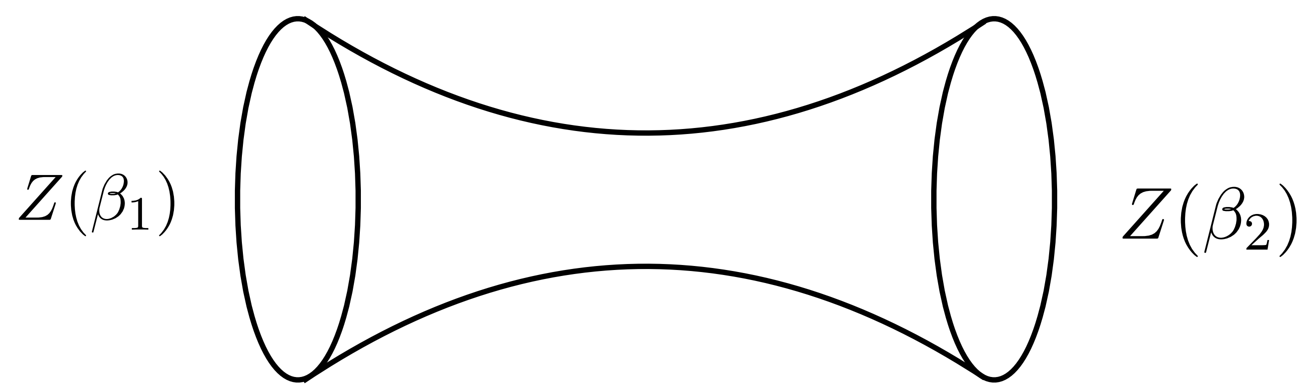
- **JT gravity:** Computed in terms of WP volumes. They satisfy a recursion of their own found by Mirzakhani.
- **Matrix Integral:** Computed in terms of the topological recursion of matrix models with $\rho_{\text{disk}} = \rho_{JT}(E)$

Eynard and Orantin proved that both recursions are identical (up to an integral transform)

SSS: This implies that pure JT gravity is holographically dual to a matrix integral, interpreted as an average over Hamiltonians

Factorization vs Disorder

- The spacetime wormholes in the Euclidean path integral break factorization of non-interacting boundaries



$$\langle Z(\beta_1) Z(\beta_2)\rangle \neq \langle Z(\beta_1)\rangle \langle Z(\beta_2)\rangle$$

- Explained by ensemble average over Hamiltonian of boundary theory. How to fix in string theory?

[Coleman]
[Giddings Strominger]

Holography: [Maldacena Maoz] [Arkani-Hamed Orgera Polchinski] …

JT gravity with a gas of defects

- Motivations for doing this:

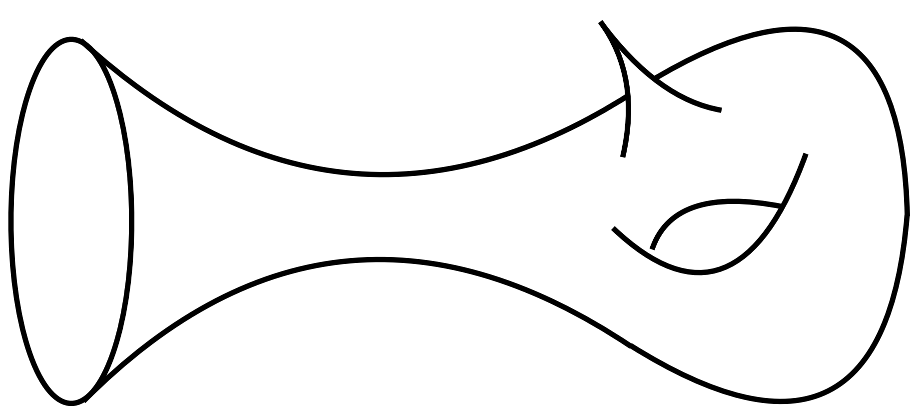
    1. Generalize the dual matrix integral to general dilaton gravity theories

    2. Application to 3D gravity [Maxfield, GJT 20] [Witten 20]

- Repeat the same procedure but allow the presence of dynamical defects. Sum over any number of them and any position.

    - Defect fugacity: λ

    - Deficit angle: $\theta = 2\pi(1-\alpha)$



2D dilaton-gravity

- A defect is equivalent to inserting $\lambda \int \sqrt{g} e^{-2\pi(1-\alpha)\phi}$ in the JT path integral.

[Mertens, GJT 19] [Witten 20]

- Then JT gravity with a gas of defects is equivalent to the following modification of the action

$$I = -\frac{1}{2} \int d^2x \sqrt{g}(\phi R + U(\phi))$$

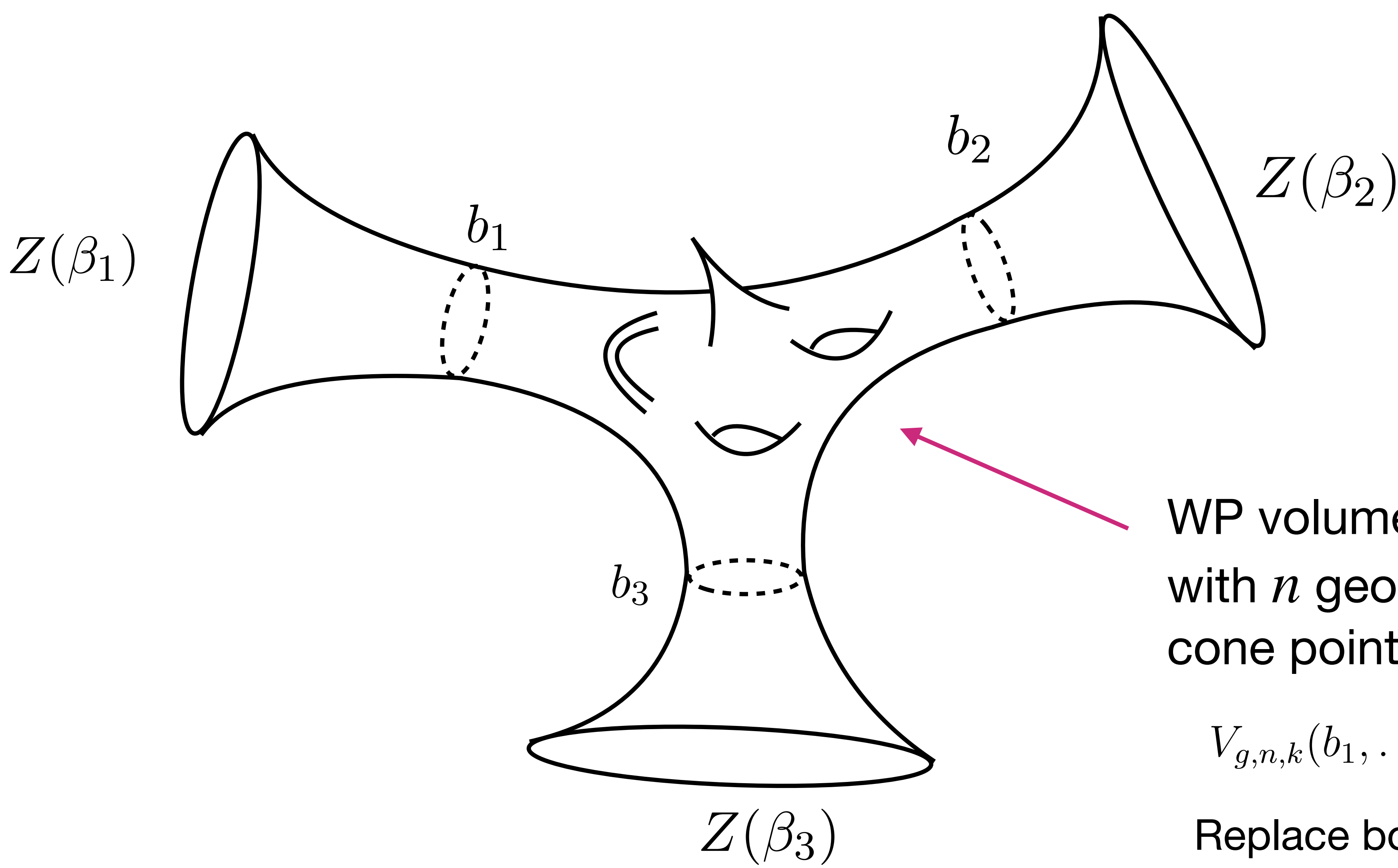
With potential

$$\boxed{U(\phi) = 2\phi + 2 \sum_i \lambda_i : e^{-2\pi(1-\alpha_i)\phi} :}$$

- This covers a large class of two-derivative pure dilaton-gravity.

Cut and glue v2

For defecit angles that satisfy $\alpha < 1/2$ there is always a geodesic homologous to the holographic boundary. Therefore we can still use trumpets to glue. For example



WP volume over hyperbolic surfaces with n geodesic boundaries and k cone points

$$V_{g,n,k}(b_1,\ldots,b_n;\alpha_1,\ldots,\alpha_k) = \mathrm{Vol}\left(\mathcal{M}_{n,g,k}\right)$$

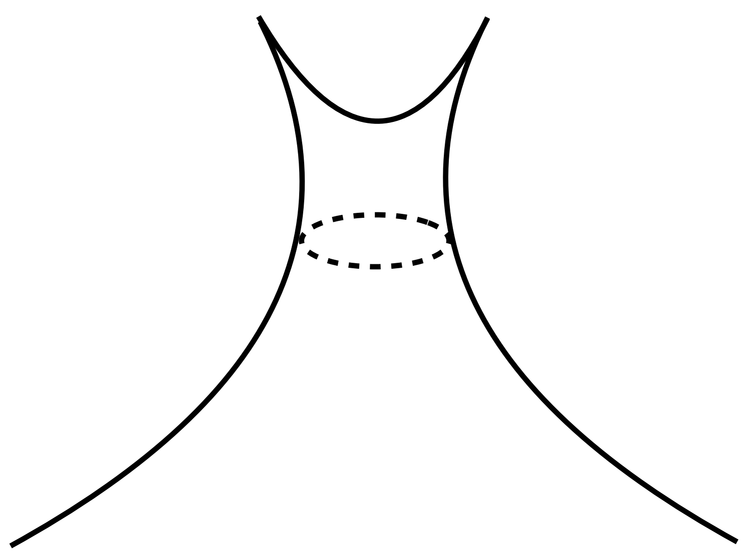
Replace borders by defects $b \to 2\pi i\alpha$

[Tan Wong Zhang] [Do Norbury]

Cut and glue v2

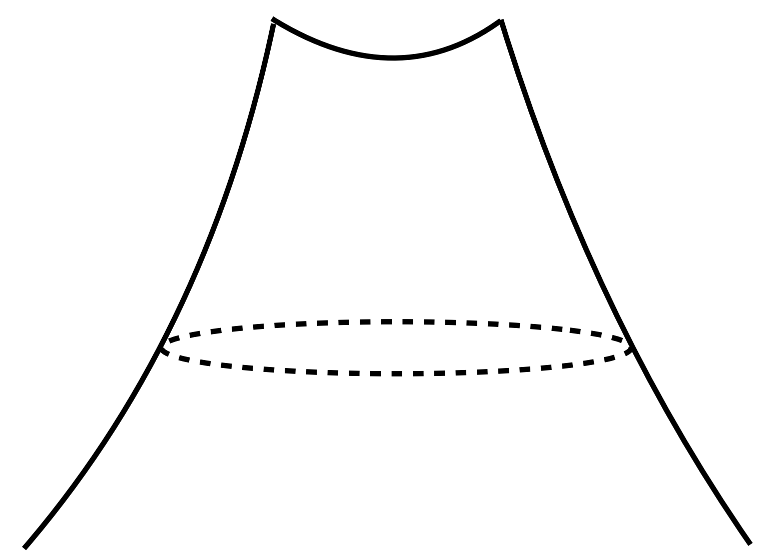
The fact that we restrict to $\alpha < 1/2$ is important. Consider for example the following two situations

$\alpha < 1/2$



Geodesic

$1/2 < \alpha < 1$



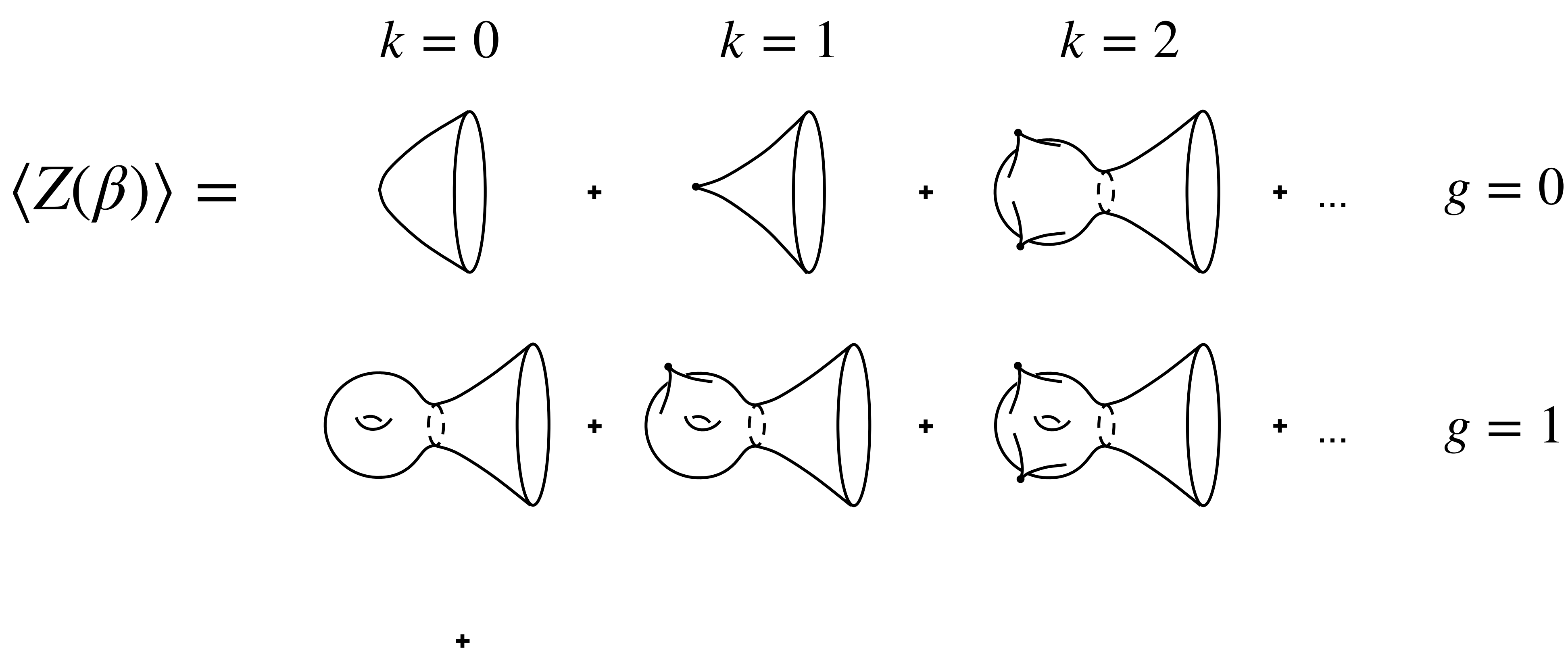
No geodesic!

SSS recipe cannot be applied. We will use a different method later

Now the calculation becomes the same as in JT gravity but with a double expansion. We can also generalize to several flavors of defects:

$$\left\langle Z(\beta_1)\cdots Z(\beta_n)\right\rangle_C = \sum_{g,k_1,k_2,\ldots=0}^{\infty} e^{-(2g+n-2)S_0}\left(\prod_i \frac{\lambda_i^{k_i}}{k_i!}\right) Z_{g,n,k}(\beta_1,\ldots,\beta_n;\alpha_1,\ldots,\alpha_k)$$

For example, in the case of the single boundary:



JT gravity with defects

- Main questions: 1) Can we perform the sum over defects explicitly to get new d.o.s? And 2) Is the theory dual to a matrix integral?

The answer to both questions is **yes**!

- Before, it is instructive to consider the following question. Can we define a theory where we include a finite number of defects? For example, only one.

$$\rho(E) \sim e^{S_0} \left[\sqrt{E} + \underbrace{\frac{\lambda}{\sqrt{E}}}_{\text{1 defect}} - \underbrace{\frac{\lambda^2}{2E^{3/2}}}_{\text{2 defect}} + \dots \right]$$

No, we **have** to sum over defects

$$\rho(E) \sim e^{S_0} \sqrt{E - E_0}$$

(Including a single defect is analogous to the Maloney Witten partition function in 3D, and its ill-defined for similar reasons)

Genus zero WP volumes

- To compute the genus zero d.o.s. we need to sum over defects. This is done with the following formula for genus zero WP volumes

[Mertens, GJT 20] [Budd wip]

$$\boxed{V_{0,n}(b_1,\ldots,b_n) = \frac{1}{2}\left(-\frac{\partial}{\partial x}\right)^{n-3}\left[J_0\left(b_1\sqrt{u_{\mathrm{JT}}(x)}\right)\cdots J_0\left(b_n\sqrt{u_{\mathrm{JT}}(x)}\right)u'_{\mathrm{JT}}(x)\right]\Big|_{x\to 0}}$$

Where

$$\frac{\sqrt{u_{\mathrm{JT}}}}{2\pi} I_1\left(2\pi\sqrt{u_{\mathrm{JT}}}\right) = x$$

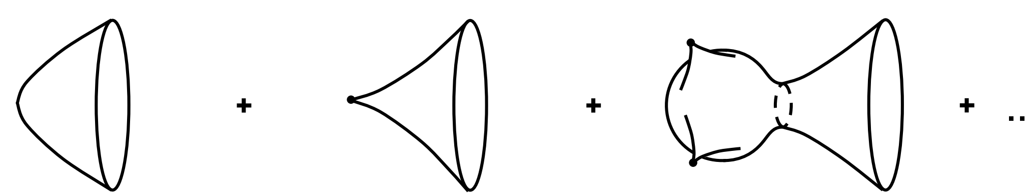
[Zograf 98]

Replace borders by defects $b \to 2\pi i\alpha$

Exact density of states

- Using the previous formula for WP volumes we can compute the disk d.o.s. as

$$\langle \rho(E) \rangle_{g=0} = \frac{e^{S_0}}{4\pi} \int_{E_0}^{E} \frac{du}{\sqrt{E-u}} \left(I_0\left(2\pi\sqrt{u}\right) + \sum_i \lambda_i \frac{2\pi\alpha_i}{\sqrt{u}} I_1\left(2\pi\alpha_i\sqrt{u}\right) \right)$$

$=$ 

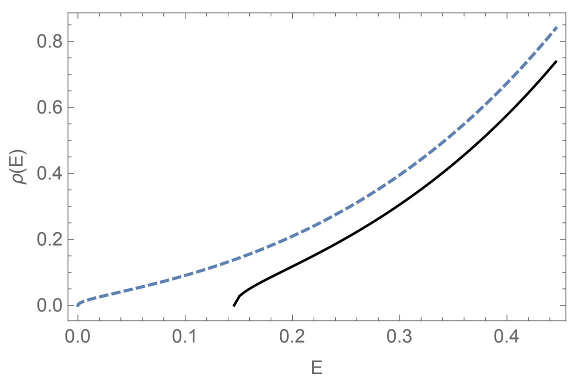
The new edge of the spectrum depends implicitly on the fugacity through

$$\boxed{\sqrt{E_0} I_1\left(2\pi\sqrt{E_0}\right) + 2\pi \sum_i \lambda_i I_0\left(2\pi\alpha_i\sqrt{E_0}\right) = 0}$$

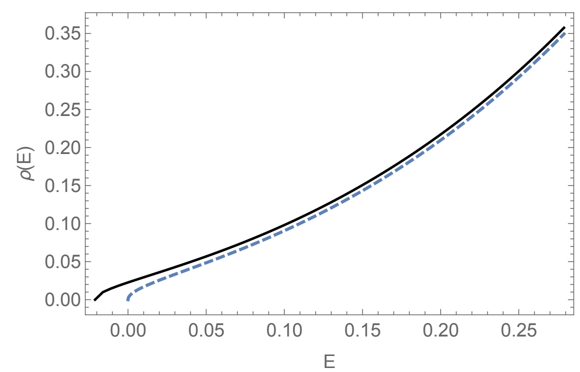
- We can check this matches the previous perturbative calculation.

Exact density of states

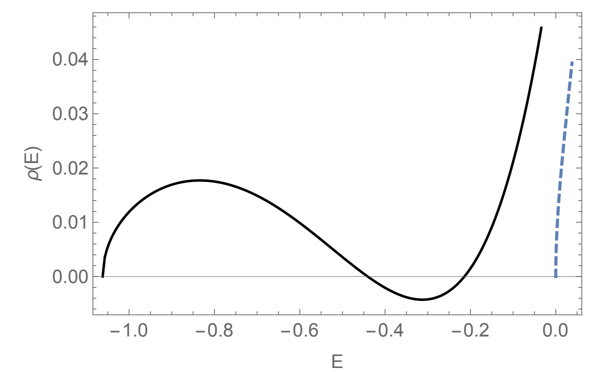
- Some numerical calculation of the density of states:



$\lambda < 0$



$0 < \lambda < \lambda_c$



$\lambda_c < \lambda$

- The theory is perfectly fine for $\lambda < 0$. For $\lambda_c < \lambda$ the density of states can become negative! This critical value is finite.

- The interpretation and fate of the model beyond the critical fugacity is an open question.

- This can actually be generalized to any number of boundaries with g=0

$$\boxed{\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_{g=0} = \frac{e^{(2-n)S_0}}{2\pi^{n/2}} \frac{\sqrt{\beta_1\cdots\beta_n}}{\beta_1+\ldots+\beta_n} \left(\frac{\partial}{\partial x}\right)^{n-2} e^{-u(x)(\beta_1+\ldots+\beta_n)}\Big|_{x=0}}$$

With "string equation": $$\frac{\sqrt{u(x)}}{2\pi} I_1\left(2\pi\sqrt{u(x)}\right) + \lambda\, I_0\left(2\pi\alpha\sqrt{u(x)}\right) = x,$$

- This is the answer for a hermitian matrix integral in the double scaling limit!

[Ambjorn, Jurkiewicz, Makeenko]

[Moore Seiberg Staudacher]

- Useful to work with the "string equation" instead of the disk d.o.s

$$\langle Z(\beta)\rangle_{g=0} = \frac{e^{S_0}}{\sqrt{4\pi\beta}} \int_0^\infty dx e^{-\beta u(x)}$$

[Brezin, Kazakov] [Douglas Shenker]
[Gross Migdal]

[Banks Douglas Seiberg Shenker]

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\dots,\beta_n)$$

- **JT gravity + defects:** Computed in terms of WP volumes and includes a sum over defects

- **Matrix Integral:** Computed in terms of the topological recursion of matrix models applied to the new density of states $\rho_{\text{disk}} = \rho_{JT+def}(E)$

Eynard and Orantin “deformation theorem” guarantees both procedures agree!

JT gravity with a gas of defects (or pure dilaton-gravity) is holographically dual to a matrix integral, interpreted as an average over Hamiltonians

Minimal string theory

- The world-sheet description is the (p, q) minimal model coupled to Liouville gravity and ghosts

- We focus on the series $(2,p)$ with $p = 3,5,\ldots$ Liouville coupling is $b = \sqrt{2/p}$. Dual to one-matrix integral

[Kazakov] [Staudacher]

- In the limit $p \to \infty$ the theory becomes JT gravity. New twist: interpret the matrix as a dual random Hamiltonian.

[SSS 19]

[Mertens GJT 20]

The minimal string theory

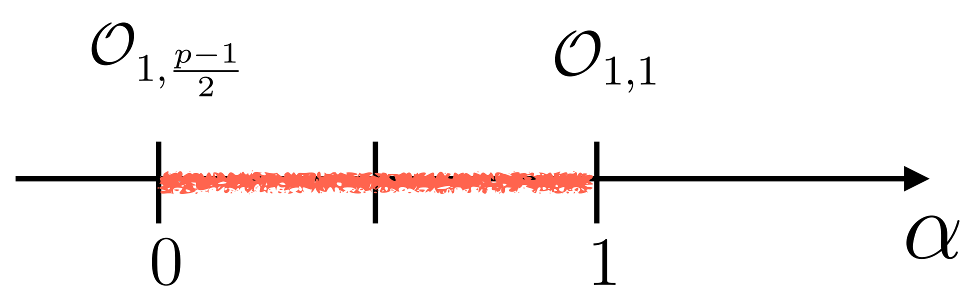
- The density of states of the minimal string for $p \to \infty$ matches JT gravity

$$\rho_p(E) = \frac{1}{4\pi^2}\sinh\left(\frac{p}{2}\cosh^{-1}\left(1+\frac{8\pi^2}{p^2}E\right)\right) \quad \to \quad \rho_{\rm JT}(E) = \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right)$$

- Consider the following deformation of the minimal string

$$I = I_{(2,p)} + \sum_n \tau_n \int e^{2\alpha_n \phi}\mathcal{O}_{1,n}$$

For MM ops: $n = p\dfrac{1-\alpha}{2}$



- We can check the deformed minimal string "string equation" reduces for $p \to \infty$, to the JT + defects "string equation"!

[Moore Seiberg Staudacher 91] [Belavin Zamolodchikov 08]

JT gravity with generic defects

[Usatyuk, Weng, GJT wip]

[Budd wip]

- The minimal string result is valid for any defect angle. Using Belavin-Zamolodchikov string equation, it gives a prediction in the JT limit

$$\boxed{\frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_{L=1}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \left(\frac{2\pi(1-L(1-\alpha))}{\sqrt{u}} \right)^{L-1} I_{L-1}\left(2\pi(1-L(1-\alpha))\sqrt{u}\right) = x}$$

Gives disk d.o.s $\rho_{\lambda,\alpha}(E)$ with $0 < \alpha < 1$

- This is outside the SSS framework. No geodesics, no trumpets.

- Non-trivial check: gives back JT when $\alpha \to 1$

- Non-analytic behavior in the angle

- Summary:
  - Pure JT gravity is dual, in a holographic sense, to an ensemble of Hamiltonians
  - The same is true for a two-parameter deformation of JT gravity, which is equivalent to more general pure dilaton-gravity theories
  - This duality can be understood as the $p \to \infty$ limit of the $(2,p)$ minimal string. This is useful to get new JT gravity results too!
- Some open questions:
  - Similar dualities for theories with matter?
  - Matrix integral emerge from triangulation of hyperbolic surfaces?

    [Kazakov Staudacher Wynter]
  - Relation between minimal strings and double-scaled SYK?

    [Berkooz Isachenkov Narovlansky Torrents]

- Applications of quantum effects in disk:
  - Large quantum effects at low temperature in near extremal black holes. Reliably captured by JT gravity.
  - Under general assumptions, Schwarzian sector inside 2D CFT

    [Ghosh Maxfield GJT 19]
  - Quantum effects near extremality in higher dimensions?

    [Iliesiu GJT 20] …
- Applications of non-perturbative effects:
  - Near extremality, sum over Seifert manifolds in 3D is related to sum over defects. Non-perturbative shifts of extremality bound

    [Maxfield GJT 20]
  - 2D CFT Ensemble dual to pure 3D gravity?

    [Cotler Jensen 20] …

Thank you for your attention!!