Mapping between Witten and Lightcone SFTs

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The connection between covariant and lightcone approaches is an old question. (Giddings & D'Hoker; Kugo & Zwiebach; Baba, Ishibashi, & Murakami; ...)

We look for an explicit mapping between the field variables of Witten's open bosonic SFT and lightcone SFT, as formulated by Mandelstam, Kaku, and Kikkawa.

Upshot:

Lightcone SFT is an effective action derived from covariant SFT upon integrating out the unphysical modes of the string field.

Witten's SFT:

String field: $\Psi_W \in \mathcal{H}_{cov}$

Action: $$S_W = \frac{1}{2}\omega(\Psi_W, Q\Psi_W) + \frac{1}{3}\omega(\Psi_W, m_2^W(\Psi_W, \Psi_W))$$

Witten's open string star product

Lightcone SFT:

String field: $\Psi_{lc} \in \widehat{\mathcal{H}}_{lc} \subset \mathcal{H}_{lc}$

$\widehat{\mathcal{H}}_{lc}$ is defined by linear constraint $L_0^{\|}A = 0$.

No ghost or lightcone oscillators

Action:

$$S_{lc} = \frac{1}{2}\omega(\Psi_{lc}, c_0 L_0 \Psi_{lc}) + \frac{1}{3}\omega(\Psi_{lc}, m_2^{lc}(\Psi_{lc}, \Psi_{lc})) + \frac{1}{4}\omega(\Psi_{lc}, m_3^{lc}(\Psi_{lc}, \Psi_{lc}, \Psi_{lc}))$$

Consider a generic covariant open bosonic SFT with string field $\Psi \in \mathcal{H}_{cov}$. For the moment we will not be specific about the nature of the interactions, which we will discuss later.

To arrive at a corresponding lightcone SFT, we need to integrate out the unphysical modes (classically).

Unfortunately, this is not as simple as integrating out states with ghost and lightcone oscillators, since the BRST structures on $\mathcal{H}_{cov}$ and $\mathcal{H}_{lc}$ are not the same.

The states we need in the lightcone state space are:

$$\alpha^{i_1}_{-n_1}\ldots\alpha^{i_N}_{-n_N}c_1|k_\mu\rangle \in \widehat{\mathcal{H}}_{lc} \subset \mathcal{H}_{lc}$$

These correspond to the following states in the covariant state space:

$$\left(e^{-\frac{in_1}{2k_-}x^+}A^{i_1}_{-n_1}\right)\ldots\left(e^{-\frac{in_N}{2k_-}x^+}A^{i_N}_{-n_N}\right)c_1|k_\mu\rangle \in \mathcal{H}_{DDF} \subset \mathcal{H}_{cov}$$

DDF oscillators:

$$A^i_n = \oint \frac{d\xi}{2\pi i}e^{-\frac{in}{p_-}X^+(\xi)}i\sqrt{2}\partial X^i(\xi)$$

Procedure:

- Find a decomposition

$$\mathcal{H}_{cov} = \mathcal{H}_{DDF} \oplus \mathcal{H}_{else}$$

where the subspaces are BPZ orthogonal.

- Decompose the dynamical string field accordingly

$$\Psi = \Psi_{DDF} + \Psi_{else}$$

- Solve the equations of motion for Ψ_{else}:

$$Q\Psi_{else} = -(1 - \Pi_{DDF})m_2(\Psi_{DDF} + \Psi_{else}, \Psi_{DDF} + \Psi_{else}) + \dots$$

to determine Ψ_{else} as a function of Ψ_{DDF}.

- Plug this in to derive the effective action.

Problems:

- How do we characterize the complementary subspace $\mathcal{H}_{else}$?
- How do we fix a gauge to solve the equation of motion for Ψ_{else}? Unfortunately, Siegel gauge won't do because $\mathcal{H}_{else}$ contains "massless" states in the kernel of L_0, e.g. $c_0|0\rangle$.

Resolution: In the lightcone state space $\mathcal{H}_{lc}$ we know what to do. The decomposition simply corresponds to that between states which do or do not contain ghost and lightcone oscillators:

$$\mathcal{H}_{lc} = \widehat{\mathcal{H}}_{lc} \oplus \mathcal{H}_{L_0^{\|} \neq 0}$$

These are distinguished by whether or not the eigenvalue of $L_0^{\|}$ vanishes. Furthermore, we can find an operator $b_0^{\|}$ which satisfies

$$[Q_{lc}, b_0^{\|}] = L_0^{\|}$$

where Q_{lc} is the analogue of the BRST operator on the lightcone state space. The needed gauge fixing condition is $b_0^{\|} = 0$.

What we need is a similarity transformation between $\mathcal{H}_{lc}$ and $\mathcal{H}_{cov}$ such that

$$S \cdot Q_{lc} \equiv S^{-1} Q_{lc} S = Q$$

Then

$$\mathcal{H}_{DDF} = S^{-1} \widehat{\mathcal{H}}_{lc}$$
$$\mathcal{H}_{else} = S^{-1} \mathcal{H}_{L_0^{\|} \neq 0}$$

The needed gauge fixing condition is

$$(S \cdot b_0^{\|}) \Psi_{else} = 0$$

$\longrightarrow$ Aisaka and Kazama (2004)

WS variables from covariant point of view:

$$\underbrace{X^i,\ p_+,\ p_-,\ \alpha_n^+,\ \alpha_n^-}_{c=26},\ \underbrace{b,\ c}_{c=-26}$$

WS variables from lightcone point of view:

$$\underbrace{X^i}_{c=24},\ \underbrace{p_+,\ p_-}_{\substack{\text{lightcone}\\ \text{zero modes}}},\ \underbrace{\alpha_n^+,\ \alpha_n^-,\ b,\ c}_{\substack{c=0\ \text{cohomologically}\\ \text{trivial quartet}}}$$

WS Energy-momentum Tensors:

$T(z)$ = Total matter+ghost energy-momentum tensor with $c = 0$

$T^{\perp}(z)$ = Energy-momentum tensor of transverse free scalars with $c = 24$

$T^{\parallel}(z)$ = Energy momentum tensor of lightcone scalars at zero momentum and $\lambda = 1$ twisted b, c system with $c = 0$

$\widetilde{T}(z) = T^{\perp}(z) + T^{\parallel}(z) =$ Lightcone energy-momentum tensor with $c = 24$

Aisaka and Kazama's Result:

$$S \cdot Q_{lc} = Q$$

with

$$Q_{lc} = c_0 L_0 + \delta_{nonmin}, \qquad \delta_{nonmin} = \sqrt{2} p_- \sum_{n \neq 0} c_{-n} \alpha_n^-$$

cancels unphysical oscillators
by quartet mechanism

and

$$S = \exp \left[-\frac{1}{\sqrt{2} p_-} \sum_{n \neq 0} \frac{1}{n} \alpha_{-n}^+ \widetilde{L}_n \right]$$

The operator S looks like a funny conformal transformation. In fact we can show

$$S \cdot \phi(z) = f_S^{-1} \tilde{\circ}\, \phi(z)$$

where

$$f_S(\xi) = e^{\frac{i}{2p_-}x^+} \exp\left[-\frac{i}{p_-} X^+(\xi)\right]$$

$\phi(z)$ is an operator without contractions with X^+ and the conformal transformation is defined with respect to the lightcone energy-momentum tensor.

The conformal transformation is unusual since it explicitly depends on an operator of the worldsheet theory (namely, X^+).

One can show that the conformal transformation amounts to the following diffeomorphism of Lorenzian worldsheet coordinates (σ, τ):

$$\tau' = const. - \frac{1}{2p_-} X^+(\sigma, \tau)$$
$$\sigma' = -\frac{1}{2p_-} \int_0^\sigma ds \frac{d}{d\tau} X^+(s, \tau)$$

This is exactly the coordinate transformation which puts the string into lightcone gauge.

We can compute

$$S \cdot \alpha_n^i = e^{\frac{in}{2p_-}x^+} A_n^i$$

$$S \cdot b_0^{\|} = b_0 + \frac{i}{p_-} \oint \frac{d\xi}{2\pi i} \frac{b(\xi)}{\partial X^+(\xi)}$$

$$S \cdot L_0^{\|} = \widetilde{L}_0 - 1 + \frac{p_-}{i} \oint \frac{d\xi}{2\pi i} \frac{T^{\perp}(\xi) - 2\{X^+(\xi), \xi\}}{\partial X^+(\xi)}$$

We can now solve the equations of motion for Ψ_{else} in the gauge $(S \cdot b_0^{||})\Psi_{else} = 0$.

$$\Psi_{else} = -\frac{(S \cdot b_0)(1 - \Pi_{DDF})}{S \cdot L_0^{||}} \Big[m_2(\Psi_{DDF} + \Psi_{else}, \Psi_{DDF} + \Psi_{else})$$

$$+ m_3(\Psi_{DDF} + \Psi_{else}, \Psi_{DDF} + \Psi_{else}, \Psi_{DDF} + \Psi_{else})$$

$$+ \ldots \Big]$$

The operator

$$\frac{(S \cdot b_0)(1 - \Pi_{DDF})}{S \cdot L_0^{||}}$$

is a kind of "propagator."

We can plug this equation into itself so that Ψ_{else} appears only on the right hand side. Up to cubic order in Ψ_{DDF}, we obtain

$$\begin{aligned}\Psi_{else} = -\frac{(S\cdot b_0)(1-\Pi_{DDF})}{S\cdot L_0^{\|}}\Big[& m_2(\Psi_{DDF},\Psi_{DDF}) \\ & -m_2\left(\Psi_{DDF},\frac{(S\cdot b_0)(1-\Pi_{DDF})}{S\cdot L_0^{\|}}m_2(\Psi_{DDF},\Psi_{DDF})\right) \\ & -m_2\left(\frac{(S\cdot b_0)(1-\Pi_{DDF})}{S\cdot L_0^{\|}}m_2(\Psi_{DDF},\Psi_{DDF}),\Psi_{DDF}\right) \\ & +m_3(\Psi_{DDF},\Psi_{DDF},\Psi_{DDF}) \quad + \quad \ldots\Big]\end{aligned}$$

Next we plug the solution for Ψ_{else} back into the action. This gives an effective action expressed in terms of the field Ψ_{DDF}.

Ψ_{DDF} is equivalent to the lightcone string field $\Psi_{lc} \in \widehat{\mathcal{H}}_{lc}$ through the similarity transformation

$$\Psi_{DDF} = S^{-1}\Psi_{lc}$$

All in all, after integrating out the unphysical oscillators we obtain an action (up to quartic order)

$$
\begin{aligned}
S = & \frac{1}{2}\omega(\Psi_{lc}, c_0 L_0 \Psi_{lc}) \\
& + \frac{1}{3}\omega(\Psi_{lc}, S m_2(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc})) \\
& + \frac{1}{4}\underbrace{\omega\left(\Psi_{lc}, S m_2\left(S^{-1}\Psi_{lc}, \frac{(S\cdot b_0)(1-\Pi_{DDF})}{S\cdot L_0^{\parallel}} m_2(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc})\right)\right)}_{\text{effective vertex}} \\
& + \frac{1}{4}\underbrace{\omega\left(\Psi_{lc}, S m_2\left(\frac{(S\cdot b_0)(1-\Pi_{DDF})}{S\cdot L_0^{\parallel}} m_2(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc}), S^{-1}\Psi_{lc}\right)\right)}_{\text{effective vertex}} \\
& + \frac{1}{4}\omega(\Psi_{lc}, S m_3(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc})) + \ldots
\end{aligned}
$$

The procedure we are following may be formalized as a so-called homotopy transfer. (Markl)

We have a consistent interacting theory of a lightcone string field with the correct kinetic term.

However, if we start with generic open string vertices in the covariant theory we will not obtain the lightcone interactions of Mandelstam and Kaku and Kikkawa.

We will obtain novel and complicated nonpolynomial effective action.

This is the case, for example, if we apply this procedure to Wiitten's SFT.

There is an exception:

If the vertices of the covariant theory are defined in the same way as in lightcone SFT (Kugo & Zwiebach), integrating out the unphysical modes does not alter the vertices.

In particular,

$$\omega(\Psi_{lc}, Sm_2^{lc}(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc})) = \omega(\Psi_{lc}, m_2^{lc}(\Psi_{lc}, \Psi_{lc}))$$

$$\omega(\Psi_{lc}, Sm_3^{lc}(S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc}, S^{-1}\Psi_{lc})) = \underbrace{\omega(\Psi_{lc}, m_3^{lc}(\Psi_{lc}, \Psi_{lc}, \Psi_{lc}))}_{\substack{\text{effective}\\ \text{vertices}}} = 0$$

The Mandelstam/Kaku-Kikkawa vertices are the unique geometrical vertices with this property.

Therefore, the conventional lightcone SFT is obtained after integrating out the unphysical modes of a "covariant" SFT with lightcone-style interactions, as considered by Kugo and Zwiebach.

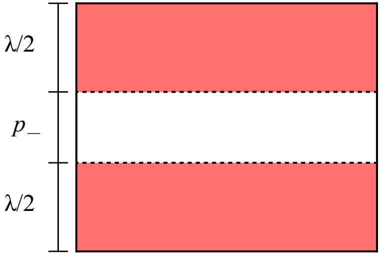
The Kugo-Zwiebach theory is not actually covariant since the conformal maps defining the vertices depend on the p_- momentum of the states which are interacting.

What we wanted is to get the conventional lightcone SFT from Witten's SFT. So we need an additional step to relate Witten's theory to the Kugo-Zwiebach theory.

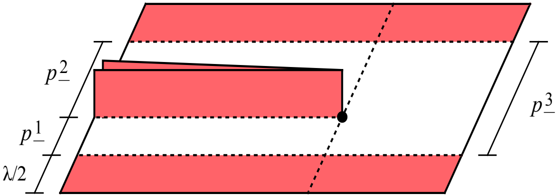
Since the Witten and Kugo-Zwiebach theories have the same kinetic terms, this merely requires finding a 1-parameter family of vertices which interpolate between them.

A way to do this was proposed by Kaku (1987).

The idea is to consider a lightcone string carrying "string Chan-Paton indices."



We may define vertices through the lightcone interaction together with contraction of the Chan-Paton indices.



This defines string products m_2^λ and m_3^λ.

The Kugo-Zwiebach theory is obtained in the limit $\lambda \to 0$, and the Witten theory in the limit $\lambda \to \infty$.

Working on the tensor algebra, the object

$$\mathbf{m}^{\lambda} = \mathbf{Q} + \mathbf{m_2}^{\lambda} + \mathbf{m_3}^{\lambda}$$

is nilpotent and defines an A_∞ structure.

To relate the Witten and Kugo-Zwiebach theories, we need a coderivation $\boldsymbol{\mu}^{\lambda}$ satisfying

$$\frac{d}{d\lambda}\mathbf{m}^{\lambda} = [\mathbf{m}^{\lambda}, \boldsymbol{\mu}^{\lambda}]$$

$\boldsymbol{\mu}^{\lambda}$ is essentially given by $\mathbf{m}^{\lambda}$ with additional b-ghost insertions.

With this the relation between the dynamical fields of the Kugo-Zwiebach and Witten theories is given by

$$\Psi_W = \pi_1 \mathbf{G} \frac{1}{1 - \Psi_{KZ}}$$

where **G** is a cohomomorphism given by the path ordered exponential

$$\mathbf{G} = \mathcal{P} \exp \left[\int_0^\infty d\lambda \boldsymbol{\mu}^\lambda \right]$$

Integrating out the unphysical modes of the Kugo-Zwiebach theory then relates the fields of Witten and lightcone SFTs.

Thank you!