2- JT gravity and near-AdS_2

Today we will study JT gravity. In particular, we will see that:

* JT gravity reduces to a boundary theory that is called "Schwarzian".
* The thermodynamics yield a linear-in-temp specific heat.
* If we have time, correlation functions (and 4-pt out-of-time-ordered correlators)

2.1 JT (Euclidean) action

$$S_{JT} = \underbrace{-\frac{\phi_0}{16\pi G_N}\left[\int_M d^2x \sqrt{h}\, R_h + 2\int_{\partial M} K\right]}_{\text{topological and gives the extremal entropy}}$$

$$- \frac{1}{16\pi G_N}\left[\int_M d^2x \sqrt{h}\, \phi\,(R_h+2) + 2\int_{\partial M} \phi_b K\right]$$

$\Rightarrow$ The theory is in 2 dimensions. It has a <u>metric</u> h and a scalar field ϕ, that we call the <u>dilaton.</u>

$\boxed{\text{IMPORTANT NOTE: } \phi_0 \gg \phi}$

Given an action, we can obtain the equations of motion for the metric and the dilaton:

$$\frac{\delta}{\delta\phi}: \quad R_h + 2 = 0 \Rightarrow \boxed{R_h = -2} \rightarrow \text{locally } AdS_2$$

$$\frac{\delta}{\delta h}: \quad \underbrace{\frac{1}{8\pi G}\left(\nabla_\mu\nabla_\nu\phi - h_{\mu\nu}\nabla^2\phi + h_{\mu\nu}\phi\right)}_{T^{\phi}_{\mu\nu}} = 0$$

$$T^{\phi}_{\mu\nu} = 0$$

Now it is time to find solutions:

- We will use Poincaré coordinates for the metric

$$\begin{cases} ds^2 = \dfrac{dt^2 + dz^2}{z^2} \\[2ex] \phi = \cancel{\dfrac{\tilde{\phi}}{z}} \quad \dfrac{\alpha + \gamma t + \delta (t^2 + z^2)}{z} \end{cases}$$

And so it looks that we arrived to a problem. We started saying that $\phi \ll \phi_0$ but it seems that as $z \to 0$, then $\phi \to \infty$.

What is the solution? We cannot go all the way up to the boundary. We will need to cut-off our spacetime $\Rightarrow$ $\boxed{\text{near - } AdS_2 \text{ spacetimes}}$

For this we will need to first impose boundary (Dirichlet) conditions:

$$\boxed{\phi|_{bdy} = \phi_b = \frac{\tilde{\phi}}{\varepsilon}, \qquad h|_{bdy} = \frac{\tilde{h}}{\varepsilon^2}}$$

How do we fix the length of the boundary? First, we will parametrise the curve by $\mathcal{C} = \{ t(u), z(u) \}$.

a boundary time "u"

$\rightsquigarrow$ For simplicity just put $\tilde{h} = 1 \Rightarrow \dfrac{1}{\varepsilon^2} = \dfrac{t'(u)^2 + z'(u)^2}{z(u)^2}$

$$\Rightarrow z(u) = \varepsilon \sqrt{t'(u)^2 + z'(u)^2} = \varepsilon\, t'(u) + \mathcal{O}(\varepsilon^3)$$

$\rightsquigarrow$ so now all the system just depends on the variable $t(u)$.

COMMENT ON SYMMETRIES: • All curves have the same "topological" action $\Rightarrow$ there is full reparametrization sym.

• However, not all reps. change the $nAdS_2$ spacetime! In particular,

$t(u) \to \dfrac{a\,(t(u)) + b}{c\, t(u) + d}$ $\quad ad - bc = 1$, leave the spacetime invariant $\Rightarrow$ $SL(2,\mathbb{R})$ transf do not change the curve $\Rightarrow$ Picking a curve SPONTANEOUSLY BREAKS

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But what is the action for this $t(u)$? We evaluate the JT action "on-shell".

$$S_{JT} = -\frac{1}{16\pi G_N}\underbrace{\int_M d^2x\,\sqrt{h}\,\phi\,(R+2)}_{} + 2\int_{\partial M} d^2x\,\phi_b\,K$$

EOM imposes $R+2=0 \rightsquigarrow \phi$ is a constraint.

$\Rightarrow$ then the action becomes only a boundary action and we need to compute the extrinsic curvature of the bdy curve.

$$\Rightarrow \boxed{S_{eff} = -\frac{1}{8\pi G_N}\int_{\partial M}\phi_b\,K = -\frac{1}{8\pi G_N}\int\frac{du}{\varepsilon}\,\frac{\tilde{\phi}}{\varepsilon}\,K}$$

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* what is K? We need a tangent and a normalised normal vector $\Rightarrow$

$$\boxed{K = -h_{ab}\,\frac{T^a T^c \nabla_c n^b}{h_{ab}T^a T^b}}$$

$\zeta = (t(u), z(u))$

(Matt Headrick's "Compendium of useful formulas")

In our case, $T^a = (t', z')$

$$n^a = \frac{z}{\sqrt{t'^2+z'^2}}(-z', t')$$

$$\Rightarrow K = \frac{t'(t'^2+z'^2+z'z'') - z\,z'\,t''}{(t'^2+z'^2)^{3/2}}$$

Now we can plug $z(t)$ in K and evaluate what happens for small ε.

The result is: $K = 1 + \underbrace{\frac{2t't''' - 3t''^2}{2t'^2}}_{Sch(t, u)\,\to\,\text{Schwarzian derivative}}\varepsilon^2 + O(\varepsilon^4)$

So we arrived to our first important result

$$\boxed{S_{Sch} = -\frac{1}{8\pi G_N}\int du\, \tilde{\Phi}(u)\, S(t(u),u)}$$

NOTE: $S(t(u),u) = S\left(\frac{a\,t(u)+b}{c\,t(u)+d}, u\right)$

$\Rightarrow$ So this action breaks the repar. symmetry explicitly.

We want to find solutions $\Rightarrow$ EOMS: $\frac{\left[S(t(u),u)\right]'}{t'} = 0$

So we want $t(u)$ with constant Schwarzian. $\rightsquigarrow$ $t(u)=u$ works. ($Sch = 0$)

We can also find finite temperature solutions by going from Poincaré to Rindler coordinates. $t(u) = \tan\left(\frac{\tau(u)}{2}\right)$

$$\Rightarrow S(t,u) = S(\tau,u) + \frac{1}{2}\tau'(u)^2$$

It is easy to check that $\tau(u) = \frac{2\pi}{\beta}u$ is a solution to our action.

2.2 Thermodynamics (Alejandra's lectures)

* On-shell action gives the partition function of the bdy theory $Z(\beta) = e^{-S_{grav}}$

Now it is very easy to evaluate

$$S_{grav} = S_{Sch} = -\frac{1}{8\pi G_N}\int_0^\beta du\, \tilde{\Phi}(u)\left(S(\tau,u) + \frac{1}{2}\tau'(u)^2\right)$$

We need to evaluate for $\tau(u) = \frac{2\pi}{\beta}u$

$$\Rightarrow S_{Sch} = -\frac{\tilde{\Phi}}{8\pi G_N}\int_0^\beta du\left(\frac{2\pi}{\beta}\right)^2 \cdot 1 = -\underbrace{\frac{(2\pi)^2\tilde{\Phi}}{8\pi G_N}}_{c}\cdot\frac{1}{\beta}$$

If we include the topological term: $S_{grav} = -S_0 - \frac{c}{\beta}$

Now we identify $-\beta F = \log Z$

$$\Rightarrow S_{th} = (1 - \beta \partial_\beta) \log Z = S_0 + 4\pi^2 \frac{C}{\beta}$$

$$C(\beta) = \beta^2 \partial_\beta^2 \log Z = 4\pi^2 \frac{C}{\beta} \left(\text{linear in temperature} \right)$$

$\boxed{\text{2.3 SUMMARY}}$

- We studied JT dilaton-gravity theory.
- We found it becomes a theory of a "reparametrization mode" of the boundary curve.
- The action is the boundary Schwarzian action
- We found saddle-points at zero and finite temperature
- We used them to compute the thermodynamics finding a linear specific heat.

*What I couldn't do: Schwarzian action leads to maximal chaos $\to \lambda_L = \frac{2\pi}{\beta}$.

* What's next? See how this features also emerge in the SYK model.