Today we will study JT gravity. In particular, we will see that:

* JT gravity reduces to a boundary theory that is called "Schwarzian".
* The thermodynamics yield a linear-in-temp specific heat.
* If we have time correlation functions (and 4-pt out-of-time-ordered correlators).

(Euclidean)

2.1 JT Vacuum

\[ S_{\text{JT}} = -\frac{\phi_0}{16\pi G} \left[ \int \frac{d^2x}{\sqrt{h}} R_h + 2 \int \frac{dM}{M} \right] \rightarrow \text{topological and gives the extremal entropy} \]

\[ -\frac{1}{16\pi G} \left[ \int \frac{d^2x}{\sqrt{h}} \phi \left( R_h + 2 \right) + \int \frac{dM}{M} \phi_0 K \right] \]

⇒ The theory is in 2 dimensions. It has a metric \( h \) and a scalar field \( \phi \), that we call the dilaton.

**Important note:** \( \phi_0 \gg \phi \)

Given an action, we can obtain the equations of motion for the metric and the dilaton:

\[ \frac{\delta}{\delta \phi} : R_h + 2 = 0 \Rightarrow R_h = -2 \quad \text{locally } \text{AdS}_2 \]

\[ \frac{\delta}{\delta h} : \frac{1}{16\pi G} \left( \nabla_\mu \nabla_\nu \phi - \frac{h}{16\pi G} \nabla^2 \phi + \frac{h}{4 \sqrt{h}} \phi \right) = 0 \]

\[ \nabla_\mu \phi = 0 \]
Now it is time to find solutions:

\[ ds^2 = \frac{dt^2 + dz^2}{z^2} \]

\[ \phi = \frac{c}{z^2} \left( x + y t + \phi_0 (t^2 + z^2) \right) \]

And so it looks that we arrived to a problem. We started saying that \( \phi < \phi_0 \) but it seems that as \( z \to 0 \), then \( \phi \to \infty \).

What is the solution? We cannot go all the way up to the boundary. We will need to cut our spacetime \( \text{Mecr - AdS}_2 \) spacetimes

For this we will need to first impose boundary (Dirichlet)

\[ \phi \bigg|_{\text{bdy}} = \phi_0 = \frac{\phi_0}{z^2}, \quad h \bigg|_{\text{bdy}} = \frac{h}{z^2} \]

How do we fix the length of the boundary? First, we will parameterise the curve by \( z = \{ x(t), z(t) \} \).

A boundary time \( u \)

\[ \Rightarrow x(u) = E \sqrt{t'(u)^2 + z'(u)^2} = E t'(u) + O(e^3) \]

so now all the system just depends on the variable \( t(u) \).

**Comment on SYMMETRIES:** All curves have the same "topological" action \( S_{\text{top}} \). However, not all res. change the \( \text{AdS}_2 \) spacetime. In particular

\[ (u) = a(t(u)) t(u) + c \]

\( \text{AdS}_2 \) leave the spacetime invariant \( \Rightarrow \text{SL}(2,1) \).
But what is the action for this $t(u)$? We evaluate the 3D action “on-shell”:

$$S_{ST} = \frac{1}{16\pi G} \int d^2 x \sqrt{\gamma} \phi (R^2 + 2) + 2 \int d^2 x \sqrt{\gamma} \phi \kappa$$

Each term $R^2 + 2 = 0 \Rightarrow \phi$ is a constraint.

Then the action becomes only a boundary action and we need to compute the extrinsic curvature of the boundary.

$$\Rightarrow S_{eff} = \frac{1}{16\pi G} \int \partial u \partial \phi \kappa = \frac{1}{16\pi G} \int \frac{\partial u}{\partial E} \frac{\partial \phi}{\partial E} \kappa$$

* What is $K$? We need a tangent and a normalised normal vector $\mathbf{\kappa}$.

$$\mathbf{\kappa} = (t(u), z(u))$$

In our case, $\mathbf{T}^a = (t', z')$

$$\mathbf{n}^a = \frac{z'}{\sqrt{t'^2 + z'^2}}$$

$$\Rightarrow K = \frac{t' (t'^2 + z'^2 + z'^2) - 2z' t'}{(t'^2 + z'^2)^{3/2}}$$

Now we can plug $Z(t)$ into $K$ and evaluate what happens for small $\epsilon$.

The result is: $K = 1 + 2t' t^{''''} - 3t'^2 t^{''} + O(\epsilon^4)$
So we wanted to find first important result:
\[ S_{\text{sch}} = -\frac{1}{8\pi\hbar} \int du \, \tilde{\phi}(u) \, S(\tau(u), u) \]

Note: \( S(\tau(u), u) = S(\tau(\alpha(u), \eta), \alpha(u), \eta) \),
\[ \Rightarrow \text{so this action leads to the usual symmetric explicitly} \]

We want to find solutions \( \exists \) so:\
\[ \left[ S(\tau(u), u) \right]' = 0 \]

\[ \tau(u) = \text{constant} \], \( \Rightarrow \tau(u) = u \)

We want \( \tau(u) \) with constant Schwinger. \( \Rightarrow \tau(u) = u \)

\( \text{waves. } \) \( \text{Sch = 0} \)

We can also find finite temperature solutions by going from Pruuner to Rindler coordinates:
\[ \tau(u) = \tan \left( \frac{\tau(u)}{2} \right) \]

\[ \Rightarrow S(\tau(u), u) = S(\tau, u) + \frac{1}{2} \tau^2(\tau(u))^2 \]

It is easy to check that \( \tau(u) = \frac{2\pi}{\beta} u \) is a solution to our action.

2.2 Thermodynamics

(Alejandro's lectures)

* On-shell action gives the partition function of the body theory:
\( Z(\beta) = e^{-S_{\text{grav}}} \)

Now it is very easy to evaluate:
\[ S_{\text{grav}} = S_{\text{ch}} = -\frac{1}{8\pi\hbar} \int du \, \tilde{\phi}(u) \left( S(\tau(u), u) + \frac{1}{2} \tau(u)^2 \right) \]

We need to evaluate for \( \tau(u) = \frac{2\pi}{\beta} u \):
\[ \Rightarrow S_{\text{ch}} = -\frac{\tilde{\phi}}{8\pi\hbar} \int du \left( \frac{2\pi}{\beta} \right)^2 1 = -\frac{(2\pi)^2}{8\pi\hbar} \tilde{\phi} \cdot \frac{1}{\beta} \]

If we exclude the topological term: \( S_{\text{grav}} = -S_0 - \frac{c}{\beta} \)
Now we identify $-\beta F = \log Z$

$\Rightarrow S_{th} = (1 - \beta^2 \partial \beta) \log Z = S_0 + \frac{4\pi^2}{\beta} \frac{C}{B}$

$C(\beta) = \beta^2 \partial^2 \beta \log Z = \frac{4\pi^2}{\beta} \frac{C}{B}$ (linear in temperature)

### 2.3 Summary

- We studied JT chiral gravity theory.
- We found it becomes a theory of a "renormalization mode" of the boundary curve.
- The action is the boundary Schwarzman cell.
- We found saddle points at zero and finite temperature.
- We used them to compute the thermodynamics, finding a linear specific heat.

*What I couldn't do*: Schwarzman action leads to maximal chaos $\lambda_L = \frac{2\pi}{\beta}$

*What's next?* See how this features also emerge in the SYK model.