Entanglement entropy in QM and QFT
1/5 - Entanglement in QM

Pablo Bueno

It from Qubit
Simons Collaboration on Quantum Fields, Gravity and Information

Instituto Balseiro

— **II School of Holography and Entanglement Entropy** —
December, 2020

Overview of the lectures

- Monday 23rd, 10:30h-11:30h - **Entanglement in QM**: basics of QM for discrete systems, subsystems, Schmidt decomposition, entanglement entropy, the first law of EE, additional measures and inequalities.
- Monday 23rd, 13h-14h - **Entanglement in QFT I**: aspects of quantum fields and algebras, the Reeh-Schlieder theorem, EE in QFT.
- Tuesday 24th, 13h-14h - **Entanglement in QFT II**: EE for free fields, monotonicity theorems, quantum Bekenstein bound.
- Wednesday 25th, 13h-14h - **The "extensive mutual information" (EMI) model**: general structure of EE and universal terms, explicit calculations for the EMI model.
- Saturday 28th, 13h-14h - **Holographic entanglement entropy**: holographic principle and AdS/CFT, Ryu-Takayanagi prescription, corrections to the RT formula, some explicit calculations, gravity from entanglement.

Outline

Some References

- The issue of entanglement in QM, including the Schmidt decomposition appears discussed in many lecture notes that can be found online. To mention a few http://www.hartmanhep.net/topics2015/18-entanglement-intro.pdf, https://arxiv.org/pdf/1801.10352.pdf, http://users.cms.caltech.edu/~vidick/teaching/120_qcrypto/LN_Week2.pdf.
- The first law of EE was introduced in https://arxiv.org/pdf/1305.3182.pdf and https://arxiv.org/pdf/1305.3291.pdf.
- The notions of relative entropy and mutual information are quantum versions of extremely standard notions in classical statistics/information theory. They appear discussed in lots of places. A standard reference is Nielsen and Chuang's "Quantum Computation and Quantum Information" book. This is not an extremely advanced book but it contains a lot of stuff on other topics like quantum computation, algorithms, quantum noise, error-correction, etc.
- The Rényi entropies discussed here are quantum versions of the notions introduced by the person who gives them name in the 60's (https://projecteuclid.org/download/pdf_1/euclid.bsmsp/1200512181) in the context of classical information theory.

Basics of QM for discrete systems

Some basics of Quantum Mechanics

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Quantum mechanical system $\Leftrightarrow$ Hilbert space $\mathcal{H}$. A state $|\psi\rangle \in \mathcal{H}$ can be written in some basis $|i\rangle$ as

$$|\psi\rangle = \sum_i c_i \, |i\rangle \ , \quad \sum_i |c_i|^2 = 1 \, .$$

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More generally, states $\Leftrightarrow$ density matrices. Given $|\psi\rangle$, we can associate

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Mixed states are those which cannot be described by vectors (only by density matrices), *e.g.*,

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Pure states can be both described by vectors and by density matrices (like $|\psi\rangle \Leftrightarrow \rho$ above)

Some basics of Quantum Mechanics

In QM, we are usually interested in expectation values of observables $\mathcal{O}$:

$$\langle\psi|\mathcal{O}|\psi\rangle \Leftrightarrow \mathrm{Tr}(\rho\mathcal{O})$$

where "Tr" is the trace of the corresponding operator (for matrices, this is just the sum of the elements in the diagonal):

$$\mathrm{Tr}(\mathcal{O}) \equiv \sum_i \langle i|\, \mathcal{O}\, |i\rangle$$

Some basics of Quantum Mechanics

For example, consider a single qubit system (two-level discrete system). Any pure state can be written as

$$|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle \equiv \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}, \quad |c_0|^2 + |c_1|^2 = 1\,.$$

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The density matrix associated reads

$$\rho = |c_0|^2 |0\rangle\langle 0| + c_0 c_1^* |0\rangle\langle 1| + c_1 c_0^* |1\rangle\langle 0| + |c_1|^2 |1\rangle\langle 1| \equiv \begin{bmatrix} |c_0|^2 & c_0 c_1^* \\ c_1 c_0^* & |c_1|^2 \end{bmatrix}.$$

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Expectation value of $\mathcal{O} \equiv |0\rangle\langle 0|$?

$$\langle\psi|\mathcal{O}|\psi\rangle = (c_0^*\,\langle 0| + c_1^*\,\langle 1|)\,|0\rangle\langle 0|\,(c_0\,|0\rangle + c_1\,|1\rangle) = |c_0|^2$$

$$\mathrm{Tr}(\rho\mathcal{O}) = \langle 0|\,\rho\,|0\rangle\langle 0|0\rangle + \langle 1|\,\rho\,|0\rangle\langle 0|1\rangle = \langle 0|\,\rho\,|0\rangle = |c_0|^2$$

Subsystems, entanglement and Schmidt decomposition

QM of subsystems

Imagine now the system is made of two subsystems A and B. Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We can write a general state $|\psi\rangle \in \mathcal{H}$ as

$$|\psi\rangle = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B ,$$

where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

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$$\mathrm{Tr}_{AB}(\mathcal{O}) \equiv \sum_{ij} \langle i|_A \langle j|_B \, \mathcal{O} \, |i\rangle_A |j\rangle_B \ .$$

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The subindex "AB" indicates that we are tracing over both A and B, but we can also trace over each subsystem:

$$\mathrm{Tr}_A(\mathcal{O}) \equiv \sum_i \langle i|_A \mathcal{O} |i\rangle_A \ , \quad \mathrm{Tr}_B(\mathcal{O}) \equiv \sum_j \langle j|_B \mathcal{O} |j\rangle_B$$

Entangled states

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If $|\psi\rangle$ cannot be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \Rightarrow |\psi\rangle$ is an entangled state.

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Entangled states

Example: two qubits

$$|\psi_1\rangle \equiv \frac{1}{\sqrt{2}}\left[|01\rangle + |00\rangle\right] = |0\rangle_A \otimes \frac{1}{\sqrt{2}}\left[|1\rangle_B + |0\rangle_B\right] \quad \text{(not entangled)}$$

$$|\psi_2\rangle \equiv \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right] \quad \text{(entangled, } |\psi_2\rangle \neq |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B\text{)}$$

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In the second case, the state of each subsystem cannot be fully described without the other. The corresponding density matrices read

$$\rho_1 \equiv |\psi_1\rangle\langle\psi_1| = \frac{1}{2}\left[|01\rangle\langle 01| + |00\rangle\langle 01| + |01\rangle\langle 00| + |00\rangle\langle 00|\right],$$

$$\rho_2 \equiv |\psi_2\rangle\langle\psi_2| = \frac{1}{2}\left[|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right].$$

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$$\rho_A \equiv \mathrm{Tr}_B\, \rho_{AB} = \sum_j \langle j|_B\, \rho_{AB}\, |j\rangle_B \ ,$$

where ρ_{AB} is the full state (*e.g.*, ρ_1 or ρ_2 in our examples).

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In our examples:

$$\rho_{A,1} = \mathrm{Tr}_B\, \rho_1 = |0\rangle\langle 0| \ , \quad \text{(pure)}$$

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Taking partial traces we loose information.

Schmidt decomposition

Let's have an even closer look at this...

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Let's have an even closer look at this...

Schmidt decomposition: it is always possible to write any pure state

$$|\psi\rangle = \sum_{i,j} c_{ij} |i\rangle_A |j\rangle_B \ ,$$

in a different basis of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_{\alpha=1}^{r} \sigma_\alpha |v_\alpha\rangle_A |w_\alpha\rangle_B \ , \quad \text{(single sum!)}$$

with

$$\sigma_\alpha > 0 \quad \forall\alpha \, , \quad \sum_{\alpha=1}^{r} \sigma_\alpha^2 = 1 \, .$$

State will be separable if $r = 1$, entangled otherwise.

Schmidt decomposition

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From the Schmidt decomposition it follows that

$$\rho_A = \sum_{\alpha=1}^{r} \sigma_\alpha^2 \left|v_\alpha\right\rangle \left\langle v_\alpha\right| \,, \quad \rho_B = \sum_{\alpha=1}^{r} \sigma_\alpha^2 \left|w_\alpha\right\rangle \left\langle w_\alpha\right| \,.$$

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We will use the σ_α to define notions of how entangled A and B are...

- Maximal entanglement if

$$\sigma_\alpha = \frac{1}{\sqrt{r}} \quad \forall \alpha$$

- No entanglement at all if

$$\sigma_1 = 1 \quad \text{and} \quad \sigma_\alpha = 0 \quad \forall \alpha \neq 1$$

Schmidt decomposition

$$|\psi\rangle = \sum_{\alpha=1}^{r} \sigma_\alpha \, |v_\alpha\rangle_A \otimes |w_\alpha\rangle_B$$

Examples:

$$|\psi_1\rangle \equiv \frac{1}{\sqrt{2}} \left[|01\rangle + |00\rangle \right] = 1 \cdot |0\rangle \otimes \left[\frac{1}{\sqrt{2}} \left[|1\rangle + |0\rangle \right] \right] ,$$

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$$|\psi_3\rangle \equiv \frac{1}{2}\left[|01\rangle + |00\rangle + |10\rangle + |11\rangle\right] = 1 \cdot \left[\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]\right] \otimes \left[\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]\right] ,$$

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$$|\psi_4\rangle \equiv \frac{1}{\sqrt{3}}\left[|00\rangle + |01\rangle + |11\rangle\right] = \sqrt{\frac{1}{6}\left[3+\sqrt{5}\right]} |v_1\rangle \otimes |w_1\rangle - \sqrt{\frac{1}{6}\left[3-\sqrt{5}\right]} |v_2\rangle \otimes |w_2\rangle$$

where

$$|v_1\rangle \equiv \frac{\left[\alpha |0\rangle + |1\rangle\right]}{\sqrt{1+\alpha^2}} , \quad |v_2\rangle \equiv \frac{\left[\beta |0\rangle + |1\rangle\right]}{\sqrt{1+\beta^2}} , \quad |w_1\rangle \equiv \frac{\left[|0\rangle + \alpha |1\rangle\right]}{\sqrt{1+\alpha^2}} , \quad |w_2\rangle \equiv \frac{\left[|0\rangle + \beta |1\rangle\right]}{\sqrt{1+\beta^2}} ,$$

and $\alpha \equiv -1 + \frac{1}{2}(3+\sqrt{5})$, $\beta \equiv -1 + \frac{1}{2}(3-\sqrt{5})$.

Entanglement entropy

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Let λ_a be the eigenvalues of ρ, then: $S(\rho) = -\sum_a \lambda_a \log \lambda_a$.

- $S(\rho) \geq 0$ for any state.
- It vanishes for pure states: $S(\rho) = 0$ if ρ is pure.

Entanglement entropy

Now, given a system composed of two subsystems A and B in some pure state ρ_{AB}, the **entanglement entropy** of A with respect to B is defined as the Von Neumann entropy of ρ_A:

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$$S_{\text{EE}}(A) \equiv S(\rho_A) = -\operatorname{Tr}_A \rho_A \log \rho_A$$

Remember, $\rho_A \equiv \operatorname{Tr}_B \rho_{AB}$ is the reduced density matrix.

- $S_{\text{EE}}(A)$ measures "how entangled" is A with B.
- If ρ_{AB} is pure, this can be written in terms of the Schmidt coefficients as
$$S_{\text{EE}} = -\sum_{\alpha} \sigma_\alpha^2 \log \sigma_\alpha^2$$
- Remember that ρ_A and ρ_B have the same eigenvalues $\{\sigma_\alpha^2\}$, which implies
$$S_{\text{EE}}(A) = S_{\text{EE}}(B)$$

Entanglement entropy

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Examples:

$$|\psi_1\rangle = 1 \cdot |0\rangle \otimes \left[\frac{1}{\sqrt{2}}\left[|1\rangle + |0\rangle\right]\right]$$

$$\Rightarrow S_{\mathrm{EE}}(A) = -1 \log 1 = 0\,,$$

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$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\,|0\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |0\rangle$$

$$\Rightarrow S_{\mathrm{EE}}(A) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \log 2 \simeq 0.6931\,,$$

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$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\, |0\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}}\, |1\rangle \otimes |0\rangle$$

$$\Rightarrow S_{\rm EE}(A) = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = \log 2 \simeq 0.6931\,,$$

$$|\psi_4\rangle = \sqrt{\frac{1}{6}\left[3+\sqrt{5}\right]}\, |v_1\rangle \otimes |w_1\rangle - \sqrt{\frac{1}{6}\left[3-\sqrt{5}\right]}\, |v_2\rangle \otimes |w_2\rangle$$

$$\Rightarrow S_{\rm EE}(A) = -\frac{1}{6}\left[3+\sqrt{5}\right] \log \frac{1}{6}\left[3+\sqrt{5}\right] - \frac{1}{6}\left[3-\sqrt{5}\right] \log \frac{1}{6}\left[3-\sqrt{5}\right]$$

$$= \log\left[\frac{6}{3+\sqrt{5}}\right] \simeq 0.1362\,.$$

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This is the **first-law of entanglement entropy**. In particular, for a thermal state:

$$\delta \langle H \rangle = T\delta S$$

quantum version of the first-law of thermodynamics!

Additional entanglement measures and inequalities

Relative entropy

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$$\rho = |0\rangle\langle 0|\,, \quad \sigma = \frac{1}{\alpha}\left[|0\rangle\langle 0| + (\alpha-1)|1\rangle\langle 1|\right] \quad \text{where} \quad \alpha \geq 1$$

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$$\Rightarrow \operatorname{Tr}\rho\log\rho = 0\,, \quad -\operatorname{Tr}\rho\log\sigma = -\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\log\begin{bmatrix}\frac{1}{\alpha} & 0\\ 0 & 1-\frac{1}{\alpha}\end{bmatrix} = -\log\frac{1}{\alpha} = \log\alpha$$

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Hence:

$$S_{\text{rel.}}(\rho||\sigma) = \log\alpha \Rightarrow \begin{cases} S_{\text{rel.}}(\rho||\sigma) \to 0 & \text{as} \quad \alpha \to 1 \\ S_{\text{rel.}}(\rho||\sigma) \to \infty & \text{as} \quad \alpha \to \infty \end{cases}$$

Mutual information

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Another important measure is the so-called **mutual information**. This can be defined in terms of the EE or, alternatively, in terms of the relative entropy, as:

$$I(A,B) \equiv S_{\text{EE}}(A) + S_{\text{EE}}(B) - S_{\text{EE}}(AB)\,,$$
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"How much information is shared between A and B"

- If ρ_{AB} pure $\Rightarrow I(A,B) = 2S_{\text{EE}}(A) = 2S_{\text{EE}}(B)$
- If ρ_{AB} mixed, $I(A,B)$ also captures classical correlations

Mutual information

Example:

$$\rho_{AB} = \frac{1}{\alpha}\left[|00\rangle\langle 00| + (\alpha-1)\,|11\rangle\langle 11|\right], \quad \alpha \geq 1$$

$$\Rightarrow \rho_A = \rho_B = \frac{1}{\alpha}\left[|0\rangle\langle 0| + (\alpha-1)\,|1\rangle\langle 1|\right]$$

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Then:

$$I(A,B) = \log\alpha - \left(1-\frac{1}{\alpha}\right)\log(\alpha-1) \Rightarrow \begin{cases} I(A,B)\to 0 \quad \text{as} \quad \alpha\to 1 \\ I_{\max}(A,B) = \log 2 \quad \text{for} \quad \alpha = 2 \\ I(A,B)\to 0 \quad \text{as} \quad \alpha\to\infty \end{cases}$$

Inequalities

- **Strong subadditivity** (SSA) property:

$$I(A,BC) \geq I(A,B)$$
$$\Leftrightarrow$$
$$S_{\mathrm{EE}}(AB) + S_{\mathrm{EE}}(BC) \geq S_{\mathrm{EE}}(ABC) + S_{\mathrm{EE}}(B)$$

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- And also the Araki-Lieb inequality:

$$S_{\text{EE}}(AB) \geq |S_{\text{EE}}(A) - S_{\text{EE}}(B)|$$

Inequalities

Example:

$$\rho_{ABC} = \frac{1}{4}\left[|000\rangle\langle 000| + |010\rangle\langle 010| + |011\rangle\langle 011| + |111\rangle\langle 111|\right]$$

$$\Rightarrow S_{\mathrm{EE}}(ABC) = \log 4\,,$$

$$\rho_{AB} = \frac{1}{4}\left[|00\rangle\langle 00| + 2\,|01\rangle\langle 01| + |11\rangle\langle 11|\right]\,,\ \rho_{BC} = \frac{1}{4}\left[|00\rangle\langle 00| + |10\rangle\langle 10| + 2\,|11\rangle\langle 11|\right]\,,$$

$$\Rightarrow S_{\mathrm{EE}}(AB) = S_{\mathrm{EE}}(BC) = -\frac{1}{4}\log\frac{1}{4} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} = \frac{3}{2}\log 2\,,$$

$$\rho_{B} = \frac{1}{4}\left[|0\rangle\langle 0| + 3\,|1\rangle\langle 1|\right]\,,\ \rho_{A} = \frac{1}{4}\left[3\,|0\rangle\langle 0| + |1\rangle\langle 1|\right]$$

$$\Rightarrow S_{\mathrm{EE}}(B) = S_{\mathrm{EE}}(A) = -\frac{1}{4}\log\frac{1}{4} - \frac{3}{4}\log\frac{3}{4} = \log 4 - \frac{3}{4}\log 3\,,$$

- SSA?

$$S_{\mathrm{EE}}(BC) + S_{\mathrm{EE}}(AB) = 3\log 2 \simeq 2.0794$$

$$S_{\mathrm{EE}}(ABC) + S_{\mathrm{EE}}(B) = 2\log 4 - \frac{3}{4}\log 3 \simeq 1.9486$$

- Subadditivity?

$$S_{\mathrm{EE}}(A) + S_{\mathrm{EE}}(B) = 2\log 4 - \frac{3}{2}\log 3 \simeq 1.1247\,,\quad S_{\mathrm{EE}}(AB) = \frac{3}{2}\log 2 \simeq 1.0397$$

Rényi entropies

Another interesting family of entanglement measures are the so-called **Rényi entropies**. This are defined by

$$S_n(A) \equiv \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n$$

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Example:

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$$\Rightarrow \operatorname{Tr} \rho_A^n = \left(\frac{1}{\alpha}\right)^n + \left(1-\frac{1}{\alpha}\right)^n \Rightarrow S_n = \frac{1}{1-n} \log\left[\left(\frac{1}{\alpha}\right)^n + \left(1-\frac{1}{\alpha}\right)^n\right]$$

What about the EE?

$$\begin{aligned} S_{n\to 1} &= \frac{1}{1-n}\left(-\left[\frac{\log(\alpha-1)}{\alpha} - \log\left(1-\frac{1}{\alpha}\right)\right](n-1) + \mathcal{O}(n-1)^2\right) \\ &= \frac{\log(\alpha-1)}{\alpha} - \log\left(1-\frac{1}{\alpha}\right) \end{aligned}$$

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- Entanglement = non-separability of quantum states. The individual description of a subsystem corresponding to an entangled system is necessarily incomplete.

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  - Rényi entropies: uniparametric family of generalizations of entanglement entropy.