Entanglement in QM and QFT

2/5 & 3/5
Entanglement in QFT

Pablo Bueno

It from Qubit
Simons Collaboration on
Quantum Fields, Gravity and Information

Instituto
Balseiro

— **II School of Holography and Entanglement Entropy** —
December, 2020

Outline

Some References

- Interesting reviews on entanglement entropy in QFT can be found in https://arxiv.org/abs/1803.04993 (Witten; more advanced, algebraic-oriented, more about "fundamentals") and https://arxiv.org/pdf/1801.10352.pdf (Nishioka; more basic, with more explicit calculations and methods).
- The algebraic/axiomatic approach to QFT is extensively discussed in R. Haag's, *Local quantum physics: Fields, particles, algebras.* 1992. This is a pretty advanced book, but at least some sections should be reasonably followable by hep-th M.Sc./Ph.D. students.
- The axiomatic formulation of QFT presented here is due to Wightman, and it also appears discussed *e.g.,* in Haag's book.
- The Reeh-Schlieder theorem is an old result in algebraic QFT (1961). It appears nicely discussed in Witten's review and in Haag's book.
- EE in the context of QFT was first considered by Sorkin et al https://journals.aps.org/prd/abstract/10.1103/PhysRevD.34.373, https://arxiv.org/pdf/1402.3589.pdf and Srednicki https://arxiv.org/pdf/hep-th/9303048.pdf. The area-law of EE was also first discussed in these papers.
- An interesting paper on the general structure of EE in QFT is https://arxiv.org/pdf/1202.2070.pdf.
- The use of mutual information as a geometric regulator for EE is discussed *e.g.,* in https://arxiv.org/pdf/1506.06195.pdf.
- The standard review for entanglement entropy for free QFTs is Casini and Huerta's https://arxiv.org/pdf/0905.2562.pdf.
- The RG flow approach to QFTs (Wilson, etc.) is discussed *e.g.,* in Rychkov's lectures https://arxiv.org/pdf/1601.05000.pdf, where CFTs in $d \geq 3$ are also extensively discussed.
- The entropic c-theorem proof appeared in Casini and Huerta's https://arxiv.org/pdf/cond-mat/0610375.pdf. A general account of entropic monotonicity theorems in various dimensions can be found in https://arxiv.org/pdf/1704.01870.pdf.
- The quantum version of the Bekenstein bound was proven by Casini in https://arxiv.org/pdf/0804.2182.pdf. The original Bekenstein paper is https://journals.aps.org/prd/abstract/10.1103/PhysRevD.23.287 —see also https://arxiv.org/pdf/1810.01880.pdf.

Aspects of quantum fields, algebras, etc.

A crash course on quantum field theory

So far we have been dealing with standard quantum mechanics of discrete systems. Now we consider quantum fields...

A crash course on quantum field theory

So far we have been dealing with standard quantum mechanics of discrete systems. Now we consider quantum fields...

- Relativistic theories $\Rightarrow$ fixed background: Minkowski spacetime.

A crash course on quantum field theory

So far we have been dealing with standard quantum mechanics of discrete systems. Now we consider quantum fields...

- Relativistic theories $\Rightarrow$ fixed background: Minkowski spacetime.
- We also have a Hilbert space of states $\mathcal{H}$. Special state $|\Omega\rangle \in \mathcal{H}$ of minimal energy (vacuum state).

A crash course on quantum field theory

Instituto Balseiro

So far we have been dealing with standard quantum mechanics of discrete systems. Now we consider quantum fields...

- Relativistic theories $\Rightarrow$ fixed background: Minkowski spacetime.
- We also have a Hilbert space of states $\mathcal{H}$. Special state $|\Omega\rangle \in \mathcal{H}$ of minimal energy (vacuum state).
- Fundamental objects are "fields", $\Phi(x)$. These are "operator-valued distributions". Relevant (quasi-local) operators obtained by smearing out the fields over regions:

$$\Phi(f) \equiv \int \Phi(x) f(x) \mathrm{d}^4 x$$

for test functions with fast fall-offs (like Gaussians).

A crash course on quantum field theory

So far we have been dealing with standard quantum mechanics of discrete systems. Now we consider quantum fields...

- Relativistic theories $\Rightarrow$ fixed background: Minkowski spacetime.
- We also have a Hilbert space of states $\mathcal{H}$. Special state $|\Omega\rangle \in \mathcal{H}$ of minimal energy (vacuum state).
- Fundamental objects are "fields", $\Phi(x)$. These are "operator-valued distributions". Relevant (quasi-local) operators obtained by smearing out the fields over regions:

$$\Phi(f) \equiv \int \Phi(x) f(x) \mathrm{d}^4 x$$

  for test functions with fast fall-offs (like Gaussians).
- All states in $\mathcal{H}$ can be created by some linear combination of products of $\Phi(f)$ acting on the vacuum: $|\psi\rangle = \Phi(f_1) \cdots \Phi(f_n) |\Omega\rangle$ generate the full $\mathcal{H}$.

A crash course on quantum field theory

Instituto Balseiro

- States and fields "respect" the symmetries of Minkowski spacetime (they transform under certain "representations" of the Poincaré and Lorentz groups). This has to do with the spin of the fields.

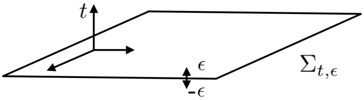
A crash course on quantum field theory

- States and fields "respect" the symmetries of Minkowski spacetime (they transform under certain "representations" of the Poincaré and Lorentz groups). This has to do with the spin of the fields.
- Fields respect causality: if supports of $f(x)$ and $h(x)$ spacelike to each other, then: either $[\Phi(f), \Phi(h)] = 0$ or $\{\Phi(f), \Phi(h)\} = 0$ (fermions).

A crash course on quantum field theory

Instituto Balseiro

- States and fields "respect" the symmetries of Minkowski spacetime (they transform under certain "representations" of the Poincaré and Lorentz groups). This has to do with the spin of the fields.
- Fields respect causality: if supports of $f(x)$ and $h(x)$ spacelike to each other, then: either $[\Phi(f), \Phi(h)] = 0$ or $\{\Phi(f), \Phi(h)\} = 0$ (fermions).
- There is some "dynamical law" which allows to compute fields at any time in terms of fields in a small time slice $\Sigma_{t,\epsilon} = \{x : |x^0 - t| < \epsilon\}$. This means that we can actually obtain any state in $\mathcal{H}$ using test functions restricted to $\Sigma_{t,\epsilon}$. This is similar to the classical statement that we can obtain $x(t)$ for any t if we know $x(0)$ and $\dot{x}(0) = [x(\epsilon) - x(0)]/\epsilon$.



Wightman's reconstruction theorem states that the full information about the QFT (fields and Hilbert space) is contained in the vacuum fluctutations:

$$\{\Phi(x), \mathcal{H}\} \Leftrightarrow \langle\Omega|\Phi(x_1), \ldots, \Phi(x_n)|\Omega\rangle$$

Algebras of operators

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Algebras of operators

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras?

Algebras of operators

Instituto Balseiro

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras? Von Neumann theorem: Let $\mathcal{A}' \equiv \{b : [b, a] = 0, \forall a \in \mathcal{A}\}$ be the "commutant" of $\mathcal{A}$.

- Whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra
- $\mathcal{A}$ is an algebra $\Leftrightarrow \mathcal{A} = \mathcal{A}''$

Algebras of operators

Instituto Balseiro

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras? Von Neumann theorem: Let $\mathcal{A}' \equiv \{b : [b,a] = 0, \forall a \in \mathcal{A}\}$ be the "commutant" of $\mathcal{A}$.

- Whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra
- $\mathcal{A}$ is an algebra $\Leftrightarrow$ $\mathcal{A} = \mathcal{A}''$

It is useful to think of states as operations which take operators and give numbers (expectation values). E.g., $\omega : \mathcal{A} \to \mathbb{C}$.

Algebras of operators

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras? Von Neumann theorem: Let $\mathcal{A}' \equiv \{b : [b, a] = 0, \forall a \in \mathcal{A}\}$ be the "commutant" of $\mathcal{A}$.

- Whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra
- $\mathcal{A}$ is an algebra $\Leftrightarrow \mathcal{A} = \mathcal{A}''$

It is useful to think of states as operations which take operators and give numbers (expectation values). E.g., $\omega : \mathcal{A} \to \mathbb{C}$. They should be normalized and when applied to Hermitian operators with positive spectrum they should produce positive numbers:

$$\omega(\alpha a + \beta b) = \alpha\omega(a) + \beta\omega(b)\,, \quad \omega(aa^\dagger) \geq 0\,, \quad \omega(1) = 1$$

Algebras of operators

Instituto Balseiro

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras? Von Neumann theorem: Let $\mathcal{A}' \equiv \{b : [b, a] = 0, \forall a \in \mathcal{A}\}$ be the "commutant" of $\mathcal{A}$.

- Whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra
- $\mathcal{A}$ is an algebra $\Leftrightarrow \mathcal{A} = \mathcal{A}''$

It is useful to think of states as operations which take operators and give numbers (expectation values). E.g., $\omega : \mathcal{A} \to \mathbb{C}$. They should be normalized and when applied to Hermitian operators with positive spectrum they should produce positive numbers:

$$\omega(\alpha a + \beta b) = \alpha\omega(a) + \beta\omega(b)\,, \quad \omega(aa^\dagger) \geq 0\,, \quad \omega(1) = 1$$

For any state ω acting on $\mathcal{A}$, $\exists!$ density matrix $\rho_\omega \in \mathcal{A}$ such that

$$\omega(a) = \mathrm{Tr}(\rho_\omega a) \quad \forall a \in \mathcal{A}$$

In words, a state in an algebra selects an operator in the algebra itself (the density matrix).

Algebras of operators

Instituto Balseiro

An algebra is a set of operators (matrices) closed under linear combinations, products and taking adjoints. Multiples of the identity are also included:

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

Which sets of operators form algebras? Von Neumann theorem: Let $\mathcal{A}' \equiv \{b : [b, a] = 0, \forall a \in \mathcal{A}\}$ be the "commutant" of $\mathcal{A}$.

- Whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra
- $\mathcal{A}$ is an algebra $\Leftrightarrow \mathcal{A} = \mathcal{A}''$

It is useful to think of states as operations which take operators and give numbers (expectation values). E.g., $\omega : \mathcal{A} \to \mathbb{C}$. They should be normalized and when applied to Hermitian operators with positive spectrum they should produce positive numbers:

$$\omega(\alpha a + \beta b) = \alpha\omega(a) + \beta\omega(b)\,, \quad \omega(aa^\dagger) \geq 0\,, \quad \omega(1) = 1$$

For any state ω acting on $\mathcal{A}$, $\exists!$ density matrix $\rho_\omega \in \mathcal{A}$ such that

$$\omega(a) = \mathrm{Tr}(\rho_\omega a) \quad \forall a \in \mathcal{A}$$

In words, a state in an algebra selects an operator in the algebra itself (the density matrix). Once we have the density matrix representation, we can compute functionals to get numbers out of it (like EE, $S = -\operatorname{Tr}\rho\log\rho$). These functionals will be an intrinsic property of the state and the algebra (and nothing else!).

Algebras of operators

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W.

Algebras of operators

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W. There are a couple of basic properties that these must satisfy

Algebras of operators

Instituto Balseiro

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W. There are a couple of basic properties that these must satisfy

- Isotony: if $V \subseteq W \Rightarrow \mathcal{A}(V) \subseteq \mathcal{A}(W)$

Algebras of operators

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W. There are a couple of basic properties that these must satisfy

- Isotony: if $V \subseteq W \Rightarrow \mathcal{A}(V) \subseteq \mathcal{A}(W)$
- Causality: let $V' \equiv \{x : x \text{ spacelike to } y, \forall y \in V\}$. Then, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Operators in spatially separated regions (anti)commute with each other.

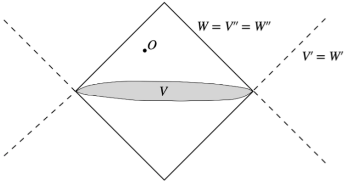
Algebras of operators

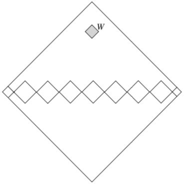
Instituto Balseiro

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W. There are a couple of basic properties that these must satisfy

- Isotony: if $V \subseteq W \Rightarrow \mathcal{A}(V) \subseteq \mathcal{A}(W)$
- Causality: let $V' \equiv \{x : x \text{ spacelike to } y, \forall y \in V\}$. Then, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Operators in spatially separated regions (anti)commute with each other.

Different regions may have the same algebra. The natural fundamental regions are causal diamonds (domain of dependence of pieces of space-like regions). These are defined by the property $W = W''$.





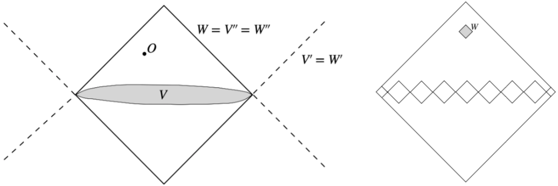
Algebras of operators

Instituto Balseiro

Operators in QFT organize themselves in algebras. In fact, the Haag-Kastler formulation of QFT takes as fundamental objects algebras $\mathcal{A}(W)$ of operators localized at spacetime regions W. There are a couple of basic properties that these must satisfy

- Isotony: if $V \subseteq W \Rightarrow \mathcal{A}(V) \subseteq \mathcal{A}(W)$
- Causality: let $V' \equiv \{x : x \text{ spacelike to } y, \forall y \in V\}$. Then, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Operators in spatially separated regions (anti)commute with each other.

Different regions may have the same algebra. The natural fundamental regions are causal diamonds (domain of dependence of pieces of space-like regions). These are defined by the property $W = W''$.



When the causality condition becomes equality, $\mathcal{A}(V) = (\mathcal{A}(V'))'$, the theory is said to satisfy “Haag duality”. This happens for sufficiently complete theories...

Algebras of operators

The information about the QFT is not in the algebras themselves (they are all isomorphic!). It is encoded in the relations between algebras (the way they intersect and share operators).

Algebras of operators

Instituto Balseiro

The information about the QFT is not in the algebras themselves (they are all isomorphic!). It is encoded in the relations between algebras (the way they intersect and share operators). Mutual information between spatially separated regions does this: it measures correlations between algebras. A natural unsolved question reminiscent to Wightman's reconstruction theorem is:

$$\{\mathcal{A}(W), \mathcal{H}\} \overset{?}{\Leftrightarrow} I(V, W)$$

In words: can we reconstruct the full information of the QFT from the mutual information of subregions?

The Reeh-Schlieder theorem

The Reeh-Schlieder theorem

There exists an even more spectacular manifestation of the highly-entangled nature of QFT states...

The Reeh-Schlieder theorem

There exists an even more spectacular manifestation of the highly-entangled nature of QFT states...

- As we mentioned earlier, any state $|\psi\rangle \in \mathcal{H}$ can be obtained by acting on the vacuum with linear combinations of operators on the neighborhood of a Cauchy slice $\Sigma_{t,\epsilon} = \{x : |x^0 - t| < \epsilon\}$.

$$|\psi\rangle = \text{L.C.}[\Phi(f_1)\cdots\Phi(f_n)\,|\Omega\rangle] \quad \text{with} \quad \Phi(f) = \int_{\Sigma_{t,\epsilon}} \Phi(x) f(x) d^4x$$

[L.C. $\equiv$ “linear combin.”, $f(x)$ is essentially zero outside $\Sigma_{t,\epsilon}$].

The Reeh-Schlieder theorem

Instituto Balseiro

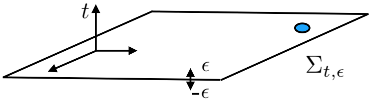
There exists an even more spectacular manifestation of the highly-entangled nature of QFT states...

- As we mentioned earlier, any state $|\psi\rangle \in \mathcal{H}$ can be obtained by acting on the vacuum with linear combinations of operators on the neighborhood of a Cauchy slice $\Sigma_{t,\epsilon} = \{x : |x^0 - t| < \epsilon\}$.

  $$|\psi\rangle = \text{L.C.}[\Phi(f_1)\cdots\Phi(f_n)\,|\Omega\rangle] \quad \text{with} \quad \Phi(f) = \int_{\Sigma_{t,\epsilon}} \Phi(x)f(x)d^4x$$

  [L.C. $\equiv$ "linear combin.", $f(x)$ is essentially zero outside $\Sigma_{t,\epsilon}$].

- **Reeh-Schlieder theorem**: we can actually generate the full Hilbert space $\mathcal{H}$ by restricting the support of the $\Phi(f)$ to an **arbitrarily small** open set of $\Sigma_{t,\epsilon}$!



The Reeh-Schlieder theorem

- "We" can create a peanut in Andromeda by acting with local operators at the university in Lima!

The Reeh-Schlieder theorem

- "We" can create a peanut in Andromeda by acting with local operators at the university in Lima!
- Let $\hat{P}$ be the "peanut operator", an operator supported in Andromeda such that $\langle\psi|\hat{P}|\psi\rangle \approx 0$ for states which do not contain a peanut in Andromeda, and $\langle\psi|\hat{P}|\psi\rangle \approx 1$ for states which do. In the vacuum state $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$ (no peanuts in Andromeda in the vacuum state).

The Reeh-Schlieder theorem

- “We” can create a peanut in Andromeda by acting with local operators at the university in Lima!
- Let $\hat{P}$ be the “peanut operator”, an operator supported in Andromeda such that $\langle\psi|\hat{P}|\psi\rangle \approx 0$ for states which do not contain a peanut in Andromeda, and $\langle\psi|\hat{P}|\psi\rangle \approx 1$ for states which do. In the vacuum state $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$ (no peanuts in Andromeda in the vacuum state).
- According to R.S. theorem, $\exists$ some operator $\hat{a}$ with support in this room such that

$$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle \approx 1$$

namely, such that in that state there is a peanut in Andromeda.

The Reeh-Schlieder theorem

Instituto Balseiro

- "We" can create a peanut in Andromeda by acting with local operators at the university in Lima!
- Let $\hat{P}$ be the "peanut operator", an operator supported in Andromeda such that $\langle\psi|\hat{P}|\psi\rangle \approx 0$ for states which do not contain a peanut in Andromeda, and $\langle\psi|\hat{P}|\psi\rangle \approx 1$ for states which do. In the vacuum state $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$ (no peanuts in Andromeda in the vacuum state).
- According to R.S. theorem, $\exists$ some operator $\hat{a}$ with support in this room such that
$$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle \approx 1$$
namely, such that in that state there is a peanut in Andromeda.
- Since $\hat{a}$ and $\hat{P}$ have support in space-like separated regions, they commute, so
$$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle = \langle\Omega|\hat{a}^\dagger\hat{P}\hat{a}|\Omega\rangle = \langle\Omega|\hat{P}\hat{a}^\dagger\hat{a}|\Omega\rangle \approx 1$$
and at the same time we know that $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$. Weird, isn't it?

The Reeh-Schlieder theorem

Instituto Balseiro

- "We" can create a peanut in Andromeda by acting with local operators at the university in Lima!
- Let $\hat{P}$ be the "peanut operator", an operator supported in Andromeda such that $\langle\psi|\hat{P}|\psi\rangle \approx 0$ for states which do not contain a peanut in Andromeda, and $\langle\psi|\hat{P}|\psi\rangle \approx 1$ for states which do. In the vacuum state $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$ (no peanuts in Andromeda in the vacuum state).
- According to R.S. theorem, $\exists$ some operator $\hat{a}$ with support in this room such that
$$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle \approx 1$$
namely, such that in that state there is a peanut in Andromeda.
- Since $\hat{a}$ and $\hat{P}$ have support in space-like separated regions, they commute, so
$$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle = \langle\Omega|\hat{a}^\dagger\hat{P}\hat{a}|\Omega\rangle = \langle\Omega|\hat{P}\hat{a}^\dagger\hat{a}|\Omega\rangle \approx 1$$
and at the same time we know that $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$. Weird, isn't it?
- It is Ok, but $\hat{a}$ simply cannot be a unitary operator (such that $\hat{a}^\dagger\hat{a} = 1$). Physically, we can only act with unitary operators, so this is not possible in practice...

The Reeh-Schlieder theorem

Instituto Balseiro

- "We" can create a peanut in Andromeda by acting with local operators at the university in Lima!
- Let $\hat{P}$ be the "peanut operator", an operator supported in Andromeda such that $\langle\psi|\hat{P}|\psi\rangle \approx 0$ for states which do not contain a peanut in Andromeda, and $\langle\psi|\hat{P}|\psi\rangle \approx 1$ for states which do. In the vacuum state $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$ (no peanuts in Andromeda in the vacuum state).
- According to R.S. theorem, $\exists$ some operator $\hat{a}$ with support in this room such that

  $$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle \approx 1$$

  namely, such that in that state there is a peanut in Andromeda.
- Since $\hat{a}$ and $\hat{P}$ have support in space-like separated regions, they commute, so

  $$\langle\hat{a}\Omega|\hat{P}|\hat{a}\Omega\rangle = \langle\Omega|\hat{a}^\dagger\hat{P}\hat{a}|\Omega\rangle = \langle\Omega|\hat{P}\hat{a}^\dagger\hat{a}|\Omega\rangle \approx 1$$

  and at the same time we know that $\langle\Omega|\hat{P}|\Omega\rangle \approx 0$. Weird, isn't it?
- It is Ok, but $\hat{a}$ simply cannot be a unitary operator (such that $\hat{a}^\dagger\hat{a} = 1$). Physically, we can only act with unitary operators, so this is not possible in practice...
- Still, manifests strong non-local quantum correlations. Non-separability à la QFT: $\langle\Omega|\hat{P}\hat{a}^\dagger\hat{a}|\Omega\rangle \neq \langle\Omega|\hat{P}|\Omega\rangle\,\langle\Omega|\hat{a}^\dagger\hat{a}|\Omega\rangle$.

The Reeh-Schlieder theorem

The Reeh-Schlieder theorem is a consequence of the analyticity of correlation functions, which in turn follows from the positivity of energy and Lorentz invariance.

The Reeh-Schlieder theorem

Instituto Balseiro

The Reeh-Schlieder theorem is a consequence of the analyticity of correlation functions, which in turn follows from the positivity of energy and Lorentz invariance.

Suppose that acting with operators in the arbitrarily small open set W on the vacuum we did not obtain a dense set of vectors. Then, there would be some vector $|\psi\rangle$ which is orthogonal to the generated set, $\langle\psi|\Phi(x_1)\dots\Phi(x_n)|\Omega\rangle = 0$, $x_1, \dots, x_n \in W$.

The Reeh-Schlieder theorem

Instituto Balseiro

The Reeh-Schlieder theorem is a consequence of the analyticity of correlation functions, which in turn follows from the positivity of energy and Lorentz invariance.

Suppose that acting with operators in the arbitrarily small open set W on the vacuum we did not obtain a dense set of vectors. Then, there would be some vector $|\psi\rangle$ which is orthogonal to the generated set, $\langle\psi|\Phi(x_1)\ldots\Phi(x_n)|\Omega\rangle = 0$, $x_1, \ldots, x_n \in W$. But, since the correlators are analytic and vanish on W, they would have to vanish for every $x_1, \ldots, x_n$ not restricted to W.

The Reeh-Schlieder theorem

The Reeh-Schlieder theorem is a consequence of the analyticity of correlation functions, which in turn follows from the positivity of energy and Lorentz invariance.

Suppose that acting with operators in the arbitrarily small open set W on the vacuum we did not obtain a dense set of vectors. Then, there would be some vector $|\psi\rangle$ which is orthogonal to the generated set, $\langle\psi|\Phi(x_1)\ldots\Phi(x_n)|\Omega\rangle = 0$, $x_1,\ldots,x_n \in W$. But, since the correlators are analytic and vanish on W, they would have to vanish for every $x_1,\ldots,x_n$ not restricted to W. But we know that operators defined on the full space acting on the vacuum do generate the whole Hilbert space.

The Reeh-Schlieder theorem

The Reeh-Schlieder theorem is a consequence of the analyticity of correlation functions, which in turn follows from the positivity of energy and Lorentz invariance.

Suppose that acting with operators in the arbitrarily small open set W on the vacuum we did not obtain a dense set of vectors. Then, there would be some vector $|\psi\rangle$ which is orthogonal to the generated set, $\langle\psi|\Phi(x_1)\ldots\Phi(x_n)|\Omega\rangle = 0$, $x_1, \ldots, x_n \in W$. But, since the correlators are analytic and vanish on W, they would have to vanish for every $x_1, \ldots, x_n$ not restricted to W. But we know that operators defined on the full space acting on the vacuum do generate the whole Hilbert space. Then, the only possibility is $|\psi\rangle = 0$, *i.e.*, we do obtain a dense set by acting on any W.

Entanglement entropy in QFT

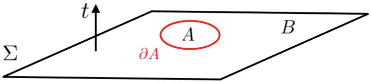
(No) entanglement entropy in QFT

Before, we were considering discrete subsystems, like two electrons.

(No) entanglement entropy in QFT

Before, we were considering discrete subsystems, like two electrons.

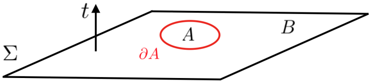
- The natural subsystems in QFT are spatial subregions (or their associated causal diamonds): fix some time-slice Σ, divide it in two regions A and B:



(No) entanglement entropy in QFT

Before, we were considering discrete subsystems, like two electrons.

- The natural subsystems in QFT are spatial subregions (or their associated causal diamonds): fix some time-slice Σ, divide it in two regions A and B:



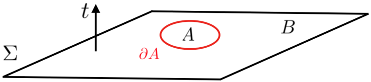
- First surprise: Hilbert space does not factorize! $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$. If it did, there would exist some state $|\psi\rangle$ such that $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B$, which would imply

$$S_{\mathrm{EE}}(A) = 0\,.$$

(No) entanglement entropy in QFT

Before, we were considering discrete subsystems, like two electrons.

- The natural subsystems in QFT are spatial subregions (or their associated causal diamonds): fix some time-slice Σ, divide it in two regions A and B:



- First surprise: Hilbert space does not factorize! $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$. If it did, there would exist some state $|\psi\rangle$ such that $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B$, which would imply

$$S_{\mathrm{EE}}(A) = 0\,.$$

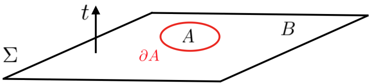
However, in QFT, the entanglement entropy of subregions is divergent in any state, $S_{\mathrm{EE}}(A) = +\infty$. There is infinite entanglement between any pair of adjacent regions.

(No) entanglement entropy in QFT

Instituto Balseiro

Before, we were considering discrete subsystems, like two electrons.

- The natural subsystems in QFT are spatial subregions (or their associated causal diamonds): fix some time-slice Σ, divide it in two regions A and B:



- First surprise: Hilbert space does not factorize! $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$. If it did, there would exist some state $|\psi\rangle$ such that $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B$, which would imply

$$S_{\mathrm{EE}}(A) = 0\,.$$

  However, in QFT, the entanglement entropy of subregions is divergent in any state, $S_{\mathrm{EE}}(A) = +\infty$. There is infinite entanglement between any pair of adjacent regions. This is actually related to the smoothness of spacetime. Something with $S_{\mathrm{EE}}(A) = 0$ would be like a firewall at ∂A...

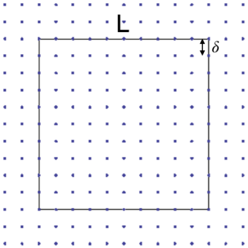
QFTs from the lattice

Often it is useful to think of a QFT as a discrete model, such as a lattice, and then take the continuum limit, putting more and more points in the lattice finally reproducing the results one would obtain doing calculations directly in the continuum.

QFTs from the lattice

Instituto Balseiro

Often it is useful to think of a QFT as a discrete model, such as a lattice, and then take the continuum limit, putting more and more points in the lattice finally reproducing the results one would obtain doing calculations directly in the continuum. There may be many ways to cutoff a theory, but all of them should arrive to the same QFT. Only quantities that are well defined in the limit belong to the continuum theory (are "universal").



The "area-law" of entanglement entropy

- The EE divergence comes from infinite correlations existing between degrees of freedom living in both sides of the interface ∂A separating A and B.

The "area-law" of entanglement entropy

Instituto Balseiro

- The EE divergence comes from infinite correlations existing between degrees of freedom living in both sides of the interface ∂A separating A and B.
- We can analyze this divergence by considering a discretized version of the QFT (imagine a lattice of coupled harmonic oscillators). If we use δ as the lattice spacing, (this is called a "UV cutoff"), one finds for any QFT:

$$S_{\mathrm{EE}}(A) = c_0 \frac{\mathrm{Area}(\partial A)}{\delta^{d-2}} + \dots$$

The "area-law" of entanglement entropy

Instituto Balseiro

- The EE divergence comes from infinite correlations existing between degrees of freedom living in both sides of the interface ∂A separating A and B.
- We can analyze this divergence by considering a discretized version of the QFT (imagine a lattice of coupled harmonic oscillators). If we use δ as the lattice spacing, (this is called a "UV cutoff"), one finds for any QFT:

$$S_{\text{EE}}(A) = c_0 \frac{\text{Area}(\partial A)}{\delta^{d-2}} + \dots$$

This holds in any state (any state looks like the vacuum at sufficiently short distances). It is called the "area law" of entanglement entropy.

The "area-law" of entanglement entropy

- The EE divergence comes from infinite correlations existing between degrees of freedom living in both sides of the interface ∂A separating A and B.

- We can analyze this divergence by considering a discretized version of the QFT (imagine a lattice of coupled harmonic oscillators). If we use δ as the lattice spacing, (this is called a "UV cutoff"), one finds for any QFT:

$$S_{\text{EE}}(A) = c_0 \frac{\text{Area}(\partial A)}{\delta^{d-2}} + \dots$$

  This holds in any state (any state looks like the vacuum at sufficiently short distances). It is called the "area law" of entanglement entropy.

- One might have guessed that $S_{\text{EE}}(A)$ should scale with the volume of A, instead of with the area of ∂A.

The "area-law" of entanglement entropy

Heuristically, one can try to understand the origin of this area law as follows.

The "area-law" of entanglement entropy

Heuristically, one can try to understand the origin of this area law as follows.

- As we argued before, $S_{\rm EE}(A) = S_{\rm EE}(B)$ for global pure states like $|\Omega\rangle$.

The "area-law" of entanglement entropy

Heuristically, one can try to understand the origin of this area law as follows.

- As we argued before, $S_{\mathrm{EE}}(A) = S_{\mathrm{EE}}(B)$ for global pure states like $|\Omega\rangle$.
- In the QFT context, this means that the entanglement entropy is not extensive at all. Regardless of how small we make A, $S_{\mathrm{EE}}(A) = S_{\mathrm{EE}}(B)$ holds.

THE "AREA-LAW" OF ENTANGLEMENT ENTROPY

Instituto Balseiro

Heuristically, one can try to understand the origin of this area law as follows.

- As we argued before, $S_{\text{EE}}(A) = S_{\text{EE}}(B)$ for global pure states like $|\Omega\rangle$.
- In the QFT context, this means that the entanglement entropy is not extensive at all. Regardless of how small we make A, $S_{\text{EE}}(A) = S_{\text{EE}}(B)$ holds.
- One could think that S_{EE} should depend on something which is common to A and B, and the only thing available is precisely the interface between both regions $\partial A = \partial B$.

EE general structure

Given some region A and a regulator δ, the entanglement entropy has the general structure

$$S_{\text{EE}}(A) = \sum_i C_i(\partial A) \cdot \delta^{-\lambda_i} + S_0(A)$$

where λ_i are dimensions, *e.g.*, for smooth ∂A, $\lambda_i = (d-2), (d-4), \ldots$

EE general structure

Given some region A and a regulator δ, the entanglement entropy has the general structure

$$S_{\mathrm{EE}}(A) = \sum_i {\color{red} C_i(\partial A)} \cdot \delta^{-\lambda_i} + {\color{blue} S_0(A)}$$

where λ_i are dimensions, *e.g.*, for smooth ∂A, $\lambda_i = (d-2), (d-4), \ldots$

${\color{red} C_i(\partial A)}$:

- Are terms additive on the entangling surface ∂A, given by integrals over ∂A
- They depend only on UV physics and the geometry of ∂A
- They are state-independent

EE general structure

Given some region A and a regulator δ, the entanglement entropy has the general structure

$$S_{\mathrm{EE}}(A) = \sum_i C_i(\partial A) \cdot \delta^{-\lambda_i} + S_0(A)$$

where λ_i are dimensions, *e.g.*, for smooth ∂A, $\lambda_i = (d-2), (d-4), \ldots$

$C_i(\partial A)$:

- Are terms additive on the entangling surface ∂A, given by integrals over ∂A
- They depend only on UV physics and the geometry of ∂A
- They are state-independent

$S_0(A)$:

- Are non-local (not given by integrals over ∂A, but rather depending on the whole A)
- They depend on the state (*e.g.*, if the state is thermal, T would appear here

Quantum information measures in QFT

If EE is not well defined in the continuum, what can we do?? Other measures are... Mutual information between two regions

Quantum information measures in QFT

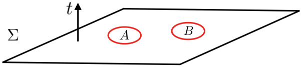
If EE is not well defined in the continuum, what can we do?? Other measures are... Mutual information between two regions (can be defined without reference to EE in terms of relative entropy)

$$I(A,B) \equiv S_{\text{EE}}(A) + S_{\text{EE}}(B) - S_{\text{EE}}(A \cup B)$$

Quantum information measures in QFT

If EE is not well defined in the continuum, what can we do?? Other measures are... Mutual information between two regions (can be defined without reference to EE in terms of relative entropy)

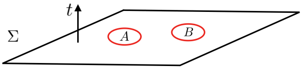
$$I(A,B) \equiv S_{\text{EE}}(A) + S_{\text{EE}}(B) - S_{\text{EE}}(A \cup B)$$



Quantum information measures in QFT

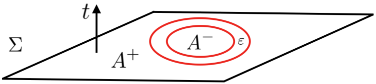
If EE is not well defined in the continuum, what can we do?? Other measures are... Mutual information between two regions (can be defined without reference to EE in terms of relative entropy)

$$I(A,B) \equiv S_{\rm EE}(A) + S_{\rm EE}(B) - S_{\rm EE}(A\cup B)$$



May be used as a regulator for entanglement entropy

$$I_\varepsilon(A^+,A^-) \xrightarrow{\varepsilon\to 0} 2S_{\rm EE}^{(\varepsilon)}(A)$$



Certain pieces in EE survive the continuum limit and have well-defined information. More about those universal terms in Lecture 4.

How do we actually compute EE?

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

How do we actually compute EE?

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

- Replica trick: the n-th Rényi entropy is obtained from a certain partition function on a complicated manifold obtained from considering n copies of the original Euclidean manifold and sewing them successively through the entangling surface. Analytically continuing the result and taking $n \to 1$ one obtains the EE. This approach has many sub-approaches, ramifications and applications...

How do we actually compute EE?

Instituto Balseiro

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

- Replica trick: the n-th Rényi entropy is obtained from a certain partition function on a complicated manifold obtained from considering n copies of the original Euclidean manifold and sewing them successively through the entangling surface. Analytically continuing the result and taking $n \to 1$ one obtains the EE. This approach has many sub-approaches, ramifications and applications...
- Real-time methods for Gaussian systems: the idea is to relate the density matrix to the expectation value of the fundamental fields (scalars or fermions), something that can be done for Gaussian systems. Those expectation values can be then obtained from different methods and the EE can be calculated.

How do we actually compute EE?

Instituto Balseiro

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

- Replica trick: the n-th Rényi entropy is obtained from a certain partition function on a complicated manifold obtained from considering n copies of the original Euclidean manifold and sewing them successively through the entangling surface. Analytically continuing the result and taking $n \to 1$ one obtains the EE. This approach has many sub-approaches, ramifications and applications...
- Real-time methods for Gaussian systems: the idea is to relate the density matrix to the expectation value of the fundamental fields (scalars or fermions), something that can be done for Gaussian systems. Those expectation values can be then obtained from different methods and the EE can be calculated.
- Hard-core numerical methods.

How do we actually compute EE?

Instituto Balseiro

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

- Replica trick: the n-th Rényi entropy is obtained from a certain partition function on a complicated manifold obtained from considering n copies of the original Euclidean manifold and sewing them successively through the entangling surface. Analytically continuing the result and taking $n \to 1$ one obtains the EE. This approach has many sub-approaches, ramifications and applications...
- Real-time methods for Gaussian systems: the idea is to relate the density matrix to the expectation value of the fundamental fields (scalars or fermions), something that can be done for Gaussian systems. Those expectation values can be then obtained from different methods and the EE can be calculated.
- Hard-core numerical methods.
- Holographic theories: for CFTs which admit a (semi)classical gravity dual in some regime, there is an additional geometric route for obtaining EE (the Ryu-Takayanagi prescription and various generalizations).

How do we actually compute EE?

Instituto Balseiro

Manipulating density matrices, algebras, taking traces, etc. is not an easy task in QFT. So, how do we actually compute EE in (regulated) QFTs?

- Replica trick: the n-th Rényi entropy is obtained from a certain partition function on a complicated manifold obtained from considering n copies of the original Euclidean manifold and sewing them successively through the entangling surface. Analytically continuing the result and taking $n \to 1$ one obtains the EE. This approach has many sub-approaches, ramifications and applications...
- Real-time methods for Gaussian systems: the idea is to relate the density matrix to the expectation value of the fundamental fields (scalars or fermions), something that can be done for Gaussian systems. Those expectation values can be then obtained from different methods and the EE can be calculated.
- Hard-core numerical methods.
- Holographic theories: for CFTs which admit a (semi)classical gravity dual in some regime, there is an additional geometric route for obtaining EE (the Ryu-Takayanagi prescription and various generalizations).
- Other special theories.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.To any state acting on an algebra corresponds a unique density matrix. Functionals like EE are intrinsic properties of states and algebras.

Summary of the first part

Instituto Balseiro

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.To any state acting on an algebra corresponds a unique density matrix. Functionals like EE are intrinsic properties of states and algebras. Algebras of operators defined on spacetime regions respect causality: operators in a given region commute with those defined in its causal complement, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Natural fundamental regions are causal diamonds, for which $W = W''$.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.To any state acting on an algebra corresponds a unique density matrix. Functionals like EE are intrinsic properties of states and algebras. Algebras of operators defined on spacetime regions respect causality: operators in a given region commute with those defined in its causal complement, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Natural fundamental regions are causal diamonds, for which $W = W''$.
- The vacuum is full of entanglement. We can generate the full Hilbert space acting on the vacuum with local operators with support in an arbitrarily small spacetime region.

Summary of the first part

Instituto Balseiro

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.To any state acting on an algebra corresponds a unique density matrix. Functionals like EE are intrinsic properties of states and algebras. Algebras of operators defined on spacetime regions respect causality: operators in a given region commute with those defined in its causal complement, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Natural fundamental regions are causal diamonds, for which $W = W''$.
- The vacuum is full of entanglement. We can generate the full Hilbert space acting on the vacuum with local operators with support in an arbitrarily small spacetime region.
- The EE of subregions is divergent in QFT. Entanglement between degrees of freedom at both sides of the interface dominates it, giving rise to an "area-law" for any state.

Summary of the first part

- QFTs are relativistic quantum theories (in flat space). Relevant operators obtained from smearings of fundamental local quantum fields.States live in some Hilbert space spanned by the action of the fields on the vacuum state. Causality, Lorentz invariance and unitary evolution implemented.
- Operators in QFT organize themselves in algebras. From this perspective, states can be thought as objects which act on algebras of operators to produce numbers.To any state acting on an algebra corresponds a unique density matrix. Functionals like EE are intrinsic properties of states and algebras. Algebras of operators defined on spacetime regions respect causality: operators in a given region commute with those defined in its causal complement, $\mathcal{A}(V) \subseteq (\mathcal{A}(V'))'$. Natural fundamental regions are causal diamonds, for which $W = W''$.
- The vacuum is full of entanglement. We can generate the full Hilbert space acting on the vacuum with local operators with support in an arbitrarily small spacetime region.
- The EE of subregions is divergent in QFT. Entanglement between degrees of freedom at both sides of the interface dominates it, giving rise to an "area-law" for any state.
- Regulating our QFT by putting it in a lattice, one can see that besides the area-law there are extra local and non-local pieces, some of which contain meaningful information about the continuum theory.

Outline

Instituto Balseiro

Some References

Instituto Balseiro

- Interesting reviews on entanglement entropy in QFT can be found in https://arxiv.org/abs/1803.04993 (Witten; more advanced, algebraic-oriented, more about "fundamentals") and https://arxiv.org/pdf/1801.10352.pdf (Nishioka; more basic, with more explicit calculations and methods).
- The algebraic/axiomatic approach to QFT is extensively discussed in R. Haag's, *Local quantum physics: Fields, particles, algebras.* 1992. This is a pretty advanced book, but at least some sections should be reasonably followable by hep-th M.Sc./Ph.D. students.
- The axiomatic formulation of QFT presented here is due to Wightman, and it also appears discussed *e.g.,* in Haag's book.
- The Reeh-Schlieder theorem is an old result in algebraic QFT (1961). It appears nicely discussed in Witten's review and in Haag's book.
- EE in the context of QFT was first considered by Sorkin et al https://journals.aps.org/prd/abstract/10.1103/PhysRevD.34.373, https://arxiv.org/pdf/1402.3589.pdf and Srednicki https://arxiv.org/pdf/hep-th/9303048.pdf. The area-law of EE was also first discussed in these papers.
- An interesting paper on the general structure of EE in QFT is https://arxiv.org/pdf/1202.2070.pdf.
- The use of mutual information as a geometric regulator for EE is discussed *e.g.,* in https://arxiv.org/pdf/1506.06195.pdf.
- The standard review for entanglement entropy for free QFTs is Casini and Huerta's https://arxiv.org/pdf/0905.2562.pdf.
- The RG flow approach to QFTs (Wilson, etc.) is discussed *e.g.,* in Rychkov's lectures https://arxiv.org/pdf/1601.05000.pdf, where CFTs in $d \geq 3$ are also extensively discussed.
- The entropic c-theorem proof appeared in Casini and Huerta's https://arxiv.org/pdf/cond-mat/0610375.pdf. A general account of entropic monotonicity theorems in various dimensions can be found in https://arxiv.org/pdf/1704.01870.pdf.
- The quantum version of the Bekenstein bound was proven by Casini in https://arxiv.org/pdf/0804.2182.pdf. The original Bekenstein paper is https://journals.aps.org/prd/abstract/10.1103/PhysRevD.23.287 —see also https://arxiv.org/pdf/1810.01880.pdf.

Free field calculations

Free fields

The simplifications provided by assuming that the fields do not interact allow for the application of various methods suitable for the evaluation of EE and related quantities.

Free fields

The simplifications provided by assuming that the fields do not interact allow for the application of various methods suitable for the evaluation of EE and related quantities.

Sometimes the approaches are divided into "Euclidean" and "Real time" due to the nature of the tools involved.

Free fields

The simplifications provided by assuming that the fields do not interact allow for the application of various methods suitable for the evaluation of EE and related quantities.

Sometimes the approaches are divided into “Euclidean” and “Real time” due to the nature of the tools involved.

In the Euclidean approach, one uses a representation of the vacuum state in terms of an Euclidean path integral and constructs the reduced density matrix in terms of similar objects.

Free fields

The simplifications provided by assuming that the fields do not interact allow for the application of various methods suitable for the evaluation of EE and related quantities.

Sometimes the approaches are divided into "Euclidean" and "Real time" due to the nature of the tools involved.

In the Euclidean approach, one uses a representation of the vacuum state in terms of an Euclidean path integral and constructs the reduced density matrix in terms of similar objects.

In the real time approach one aims at computing directly the reduced density matrix corresponding to the global vacuum state in terms of correlators of the fields.

Here I will give you a flavor of the second type of methods.

Free bosons

Consider a system of N scalar fields and momenta in a lattice.

Free bosons

Consider a system of N scalar fields and momenta in a lattice. By definition, they satisfying canonical commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij} \,, \quad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0 \,.$$

Free bosons

Consider a system of N scalar fields and momenta in a lattice. By definition, they satisfying canonical commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij} \,, \quad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0 \,.$$

Define the correlation functions associated to ρ_A as

$$\langle \phi_i \phi_j \rangle \equiv X_{ij} \,, \quad \langle \pi_i \pi_j \rangle \equiv P_{ij} \,, \quad \langle \phi_i \pi_j \rangle = \langle \pi_j \phi_i \rangle^* = \frac{i}{2}\delta_{ij} \,.$$

Free bosons

Consider a system of N scalar fields and momenta in a lattice. By definition, they satisfying canonical commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij} \, , \quad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0 \, .$$

Define the correlation functions associated to ρ_A as

$$\langle \phi_i \phi_j \rangle \equiv X_{ij} \, , \quad \langle \pi_i \pi_j \rangle \equiv P_{ij} \, , \quad \langle \phi_i \pi_j \rangle = \langle \pi_j \phi_i \rangle^* = \frac{i}{2} \delta_{ij} \, .$$

We are interested in Gaussian states (*i.e.*, those for which all other non-zero correlators follow from the two-point functions of the fields).

Free bosons

Consider a system of N scalar fields and momenta in a lattice. By definition, they satisfying canonical commutation relations

$$[\phi_i, \pi_j] = i\delta_{ij} \,, \quad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0 \,.$$

Define the correlation functions associated to ρ_A as

$$\langle \phi_i \phi_j \rangle \equiv X_{ij} \,, \quad \langle \pi_i \pi_j \rangle \equiv P_{ij} \,, \quad \langle \phi_i \pi_j \rangle = \langle \pi_j \phi_i \rangle^* = \frac{i}{2}\delta_{ij} \,.$$

We are interested in Gaussian states (*i.e.*, those for which all other non-zero correlators follow from the two-point functions of the fields). We can write in general

$$\rho_A = K e^{-\sum_l \varepsilon_l a_l^\dagger a_l} \,, \quad K \equiv \prod_l (1 - e^{-\varepsilon_l})$$

where we already diagonalized the modular Hamiltonian introducing creation and annihilation operators (just like for the usual harmonic oscillator), $[a_i, a_j^\dagger] = \delta_{ij}$.

Free bosons

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$.

Free bosons

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$. As a consequence, we can write the EE in terms of C.

Free bosons

Instituto Balseiro

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$. As a consequence, we can write the EE in terms of C. The result reads

$$S_{\text{EE}}(A) = \text{Tr}[(C + 1/2)\log(C + 1/2) - (C - 1/2)\log(C - 1/2)] \,.$$

Free bosons

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$. As a consequence, we can write the EE in terms of C. The result reads

$$S_{\text{EE}}(A) = \text{Tr}[(C + 1/2)\log(C + 1/2) - (C - 1/2)\log(C - 1/2)] \, .$$

In terms of the eigenvalues of C, $\{\lambda_k\}$, this can be alternatively written as

$$S_{\text{EE}}(A) = \sum_k [(\lambda_k + 1/2)\log(\lambda_k + 1/2) - (\lambda_k - 1/2)\log(\lambda_k - 1/2)]$$

Free bosons

Instituto Balseiro

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$. As a consequence, we can write the EE in terms of C. The result reads

$$S_{\rm EE}(A) = \mathrm{Tr}[(C+1/2)\log(C+1/2) - (C-1/2)\log(C-1/2)]\,.$$

In terms of the eigenvalues of C, $\{\lambda_k\}$, this can be alternatively written as

$$S_{\rm EE}(A) = \sum_k [(\lambda_k + 1/2)\log(\lambda_k + 1/2) - (\lambda_k - 1/2)\log(\lambda_k - 1/2)]$$

The thing is that computing correlators like X_{ij} and P_{ij} (and consequently, C_{ij}) is usually something rather doable, so we can evaluate $S_{\rm EE}(A)$ using the above formula. We will see examples later...

Free bosons

Instituto Balseiro

Now, a key point is that one can show that the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of $C \equiv \sqrt{XP}$. As a consequence, we can write the EE in terms of C. The result reads

$$S_{\text{EE}}(A) = \text{Tr}[(C+1/2)\log(C+1/2) - (C-1/2)\log(C-1/2)] \,.$$

In terms of the eigenvalues of C, $\{\lambda_k\}$, this can be alternatively written as

$$S_{\text{EE}}(A) = \sum_k [(\lambda_k + 1/2)\log(\lambda_k + 1/2) - (\lambda_k - 1/2)\log(\lambda_k - 1/2)]$$

The thing is that computing correlators like X_{ij} and P_{ij} (and consequently, C_{ij}) is usually something rather doable, so we can evaluate $S_{\text{EE}}(A)$ using the above formula. We will see examples later...

A prototypical case is when we consider the vacuum state, and a global Hamiltonian of the form $H = \frac{1}{2}\sum_i \pi_i^2 + \frac{1}{2}\phi_i K_{ij}\phi_j$. Then, the correlators read $X_{ij} = \frac{1}{2}(K^{-1/2})_{ij}$, $P_{ij} = \frac{1}{2}(K^{1/2})_{ij}$.

Free fermions

The story is very similar for free fermions. In that case we start with N fields satisfying anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \ ,$$

Free fermions

The story is very similar for free fermions. In that case we start with N fields satisfying anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \,,$$

We define the correlators in the state ρ_A by

$$\langle \psi_i \psi_j^\dagger \rangle \equiv C_{ij} \,, \quad \langle \psi_i \psi_j \rangle = \langle \psi_i^\dagger \psi_j^\dagger \rangle = 0 \,.$$

Free fermions

The story is very similar for free fermions. In that case we start with N fields satisfying anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \, ,$$

We define the correlators in the state ρ_A by

$$\langle \psi_i \psi_j^\dagger \rangle \equiv C_{ij} \, , \quad \langle \psi_i \psi_j \rangle = \langle \psi_i^\dagger \psi_j^\dagger \rangle = 0 \, .$$

Again we are interested in Gaussian states of the form

$$\rho_A = K e^{-\varepsilon_l d_l^\dagger d_l} \, , \quad K \equiv (1 + e^{-\varepsilon_l})$$

where again we already diagonalized the modular Hamiltonian.

Free fermions

The story is very similar for free fermions. In that case we start with N fields satisfying anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \, ,$$

We define the correlators in the state ρ_A by

$$\langle \psi_i \psi_j^\dagger \rangle \equiv C_{ij} \, , \quad \langle \psi_i \psi_j \rangle = \langle \psi_i^\dagger \psi_j^\dagger \rangle = 0 \, .$$

Again we are interested in Gaussian states of the form

$$\rho_A = K e^{-\varepsilon_l d_l^\dagger d_l} \, , \quad K \equiv (1 + e^{-\varepsilon_l})$$

where again we already diagonalized the modular Hamiltonian.

Similarly to the scalars case, the eigenvalues of ρ_A can be obtained in terms of the eigenvalues of C, so we can write the EE in terms of that correlators matrix. The result is:

$$S_{\text{EE}}(A) = -\operatorname{Tr}[(1 - C)\log(1 - C) + C \log C]$$

Ex 1: free fermion in $d = 2$

Let us consider first the case of a free fermion in $d = 2$, with A being a single interval of length L_A.

Ex 1: free fermion in $d = 2$

Let us consider first the case of a free fermion in $d = 2$, with A being a single interval of length L_A. The lattice Hamiltonian reads

$$H^{\text{ferm.}}_{\text{latt.}} = -\frac{i}{2} \sum_{j} \left[\psi_j^{\dagger} \psi_{j+1} - \psi_{j+1}^{\dagger} \psi_j \right] ,$$

Ex 1: free fermion in $d = 2$

Let us consider first the case of a free fermion in $d = 2$, with A being a single interval of length L_A. The lattice Hamiltonian reads

$$H^{\text{ferm.}}_{\text{latt.}} = -\frac{i}{2}\sum_j \left[\psi_j^\dagger \psi_{j+1} - \psi_{j+1}^\dagger \psi_j\right] ,$$

The ground state correlators, $C_{jl} \equiv \langle \psi_j \psi_l^\dagger \rangle$, read

Ex 1: free fermion in $d = 2$

Let us consider first the case of a free fermion in $d = 2$, with A being a single interval of length L_A. The lattice Hamiltonian reads

$$H_{\text{latt.}}^{\text{ferm.}} = -\frac{i}{2}\sum_{j}\left[\psi_j^\dagger \psi_{j+1} - \psi_{j+1}^\dagger \psi_j\right] ,$$

The ground state correlators, $C_{jl} \equiv \langle \psi_j \psi_l^\dagger \rangle$, read

$$C_{jl} = \begin{cases} \frac{(-1)^{(j-l)}-1}{2\pi i (j-l)} & j \neq l\,, \\ \frac{1}{2} & j = l\,, \end{cases} .$$

Ex 1: free fermion in $d=2$

Let us consider first the case of a free fermion in $d=2$, with A being a single interval of length L_A. The lattice Hamiltonian reads

$$H_{\text{latt.}}^{\text{ferm.}} = -\frac{i}{2}\sum_j \left[\psi_j^\dagger \psi_{j+1} - \psi_{j+1}^\dagger \psi_j\right] ,$$

The ground state correlators, $C_{jl} \equiv \langle \psi_j \psi_l^\dagger \rangle$, read

$$C_{jl} = \begin{cases} \frac{(-1)^{(j-l)}-1}{2\pi i (j-l)} & j \neq l \, , \\ \frac{1}{2} & j = l \, , \end{cases} .$$

For a general CFT_2, the result for the EE of an interval reads

$$S_{\text{EE}} = \frac{c}{3}\log(L_A/\delta) + \mathcal{O}(\delta^0)$$

where c is the “Virasoro central charge” of the theory. In the case of the free fermion, $c = 1/2$...

Ex 1: free fermion in $d = 2$

For technical reasons, when performing lattice calculations for fermions there is an extra factor 2 which needs to be removed.

Ex 1: free fermion in $d = 2$

Instituto Balseiro

For technical reasons, when performing lattice calculations for fermions there is an extra factor 2 which needs to be removed. A small program in Mathematica yields perfect agreement:

```
In[258]:= c[x_] := If[x == 0, 1/2, N[((-1)^(x) - 1)/(2 Pi I x)]]

In[259]:= entro[reg_] :=
  Module[{corr, v},
   corr = Table[c[reg[[i]] - reg[[j]]], {i, 1, Length[reg]}, {j, 1, Length[reg]}];
   v = Re[Eigenvalues[corr]];
   Re[-v.Log[v + 10^(-11)] - (1 - v).Log[1 - v - 10^(-11)]]]

In[270]:= entropia = Table[entro[Table[j, {j, 1, i * 10}]], {i, 1, 25}]

Out[270]= {1.49342, 1.7246, 1.85978, 1.95568, 2.03007, 2.09084, 2.14223, 2.18674,
  2.226, 2.26112, 2.29289, 2.3219, 2.34858, 2.37328, 2.39628, 2.41779, 2.438,
  2.45705, 2.47507, 2.49217, 2.50844, 2.52394, 2.53876, 2.55295, 2.56655}

In[271]:= Fit[entropia, {Log[x], 1}, x]

Out[271]= 1.49351 + 0.333363 Log[x]
```

Ex 2: free scalar in $d = 3$

Consider now a free scalar in $d = 3$

$$H^{\text{scal.}}_{\text{latt.}} = +\frac{1}{2}\sum_{n,m}\left[\pi^2_{n,m} + (\phi_{n+1,m} - \phi_{n,m})^2 + (\phi_{n,m+1} - \phi_{n,m})^2\right] ,$$

Ex 2: free scalar in $d = 3$

Instituto Balseiro

Consider now a free scalar in $d = 3$

$$H^{\text{scal.}}_{\text{latt.}} = +\frac{1}{2}\sum_{n,m}\left[\pi^2_{n,m} + (\phi_{n+1,m} - \phi_{n,m})^2 + (\phi_{n,m+1} - \phi_{n,m})^2\right] ,$$

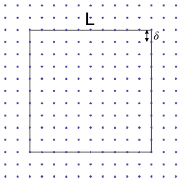
Ground state correlators

$$\langle \phi_{0,0}\phi_{i,j}\rangle = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} \mathrm{d}x \int_{-\pi}^{\pi} \mathrm{d}y \frac{\cos(ix)\cos(jy)}{\sqrt{2(1-\cos(x)) + 2(1-\cos(y))}} ,$$

$$\langle \pi_{0,0}\pi_{i,j}\rangle = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} \mathrm{d}x \int_{-\pi}^{\pi} \mathrm{d}y \cos(ix)\cos(jy)\sqrt{2(1-\cos x) + 2(1-\cos y)} .$$

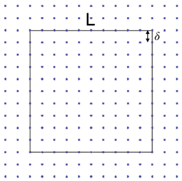
Ex 2: free scalar in $d = 3$

Take first A to be a square region



Ex 2: free scalar in $d = 3$

Take first A to be a square region



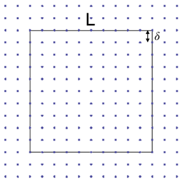
With a slightly more complicated Mathematica program, one finds

$$S_{\text{EE}} \simeq 0.077 \times \frac{4L}{\delta} - 0.0116 \times \mathbf{4} \log(L/\delta) + \mathcal{O}(\delta^0)$$

We get an "area-law" piece plus a logarithmic correction, both of them divergent in the continuum limit.

Ex 2: free scalar in $d = 3$

Take first A to be a square region



With a slightly more complicated Mathematica program, one finds

$$S_{\text{EE}} \simeq 0.077 \times \frac{4L}{\delta} - 0.0116 \times \mathbf{4} \log(L/\delta) + \mathcal{O}(\delta^0)$$

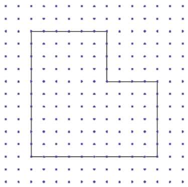
We get an "area-law" piece plus a logarithmic correction, both of them divergent in the continuum limit. The logarithmic piece is related to the presence of corners in A.

Ex 2: free scalar in $d = 3$

What happens if we consider now a region with the same “area” (length$= 4L$) but more corners?

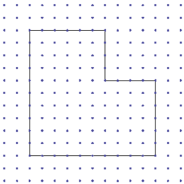
Ex 2: free scalar in $d = 3$

What happens if we consider now a region with the same "area" (length$= 4L$) but more corners?



Ex 2: free scalar in $d = 3$

What happens if we consider now a region with the same "area" (length$= 4L$) but more corners?



Now the result reads

$$S_{\text{EE}} \simeq 0.077 \times \frac{4L}{\delta} - 0.0116 \times \mathbf{6} \log(L/\delta) + \mathcal{O}(\delta^0)$$

The coefficient of the "area-law" does not change, but now we get a different coefficient for the log term, proportional to the number of corners.

Ex 2: free scalar in $d=3$

This behavior is in fact general:

$$S_{\text{EE}} = c_1 \frac{L}{\delta} + \sum_{\text{corner}_j} a_j(\theta) \log(L/\delta) + \mathcal{O}(\delta^0)$$

where $a(\theta)$ is universal, *i.e.*, well-defined in the continuum theory. More in Lecture 4.

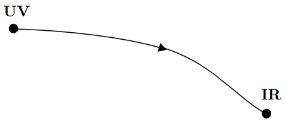
Monotonicity theorems

QFT RELOADED

Before I presented an axiomatic formulation of QFTs.

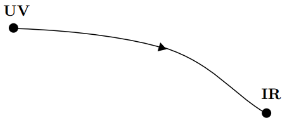
QFT reloaded

Before I presented an axiomatic formulation of QFTs. A somewhat complementary approach establishes that QFT is the study of Renormalization Group (RG) flows, *i.e.*, how the theory evolves from the Ultraviolet (UV) to the Infrared (IR) regimes...



QFT RELOADED

Before I presented an axiomatic formulation of QFTs. A somewhat complementary approach establishes that QFT is the study of Renormalization Group (RG) flows, *i.e.*, how the theory evolves from the Ultraviolet (UV) to the Infrared (IR) regimes...



What kinds of theories there exist at low energies?

- Theories with a mass gap (*e.g.*, QCD)
- Theories with massless particles (*e.g.*, QED)
- Scale invariant theories with continuous spectrum: CFTs $\Leftrightarrow$ fixed points of the RG flow (can be stable or unstable)

Monotonicity of RG flows

RG flow $\Leftrightarrow$ coarse graining of microscopic degrees of freedom heavier than the relevant energy scale. As we move to lower energies and "integrate out" higher-energy degrees of freedom, we loose information about the theory. This results in a "trajectory of theories" in the space of coupling constants $\{g_i\}$.

Monotonicity of RG flows

RG flow $\Leftrightarrow$ coarse graining of microscopic degrees of freedom heavier than the relevant energy scale. As we move to lower energies and “integrate out” higher-energy degrees of freedom, we loose information about the theory. This results in a “trajectory of theories” in the space of coupling constants $\{g_i\}$.

Intuitively, physical degrees of freedom should decrease monotonically under RG flows, since massive degrees of freedom are integrated out once the energy scale of the flow becomes below the scale set by the masses.

Monotonicity of RG flows

RG flow $\Leftrightarrow$ coarse graining of microscopic degrees of freedom heavier than the relevant energy scale. As we move to lower energies and "integrate out" higher-energy degrees of freedom, we loose information about the theory. This results in a "trajectory of theories" in the space of coupling constants $\{g_i\}$.

Intuitively, physical degrees of freedom should decrease monotonically under RG flows, since massive degrees of freedom are integrated out once the energy scale of the flow becomes below the scale set by the masses.

Finding functions $c(\lambda)$ which quantify the effective number of degrees of freedom (*i.e.*, such that $c(\lambda)$ decreases monotonically along the RG flow) is an important problem in QFT. When they exist, they are customarily called "c-functions". In particular, they must satisfy $c_{\mathrm{UV}} > c_{\mathrm{IR}}$ for the fixed-point theories.

Monotonicity theorems

Throughout the years, c-functions have been found and monotonicity theorems have been proven in various dimensions:

Monotonicity theorems

Instituto Balseiro

Throughout the years, c-functions have been found and monotonicity theorems have been proven in various dimensions:

- $d = 2$ theories, called the "c-theorem". At the fixed points the c-function coincides with the Virasoro central charges of the fixed point CFTs.

Monotonicity theorems

Throughout the years, c-functions have been found and monotonicity theorems have been proven in various dimensions:

- $d = 2$ theories, called the "c-theorem". At the fixed points the c-function coincides with the Virasoro central charges of the fixed point CFTs.
- $d = 3$ theories, called the "F-theorem". At the fixed points the c-function coincides with the free energy of the CFTs on $\mathbb{S}^3$ which is the same as the EE universal term across a $\mathbb{S}^1$.

Monotonicity theorems

Instituto Balseiro

Throughout the years, c-functions have been found and monotonicity theorems have been proven in various dimensions:

- $d = 2$ theories, called the "c-theorem". At the fixed points the c-function coincides with the Virasoro central charges of the fixed point CFTs.
- $d = 3$ theories, called the "F-theorem". At the fixed points the c-function coincides with the free energy of the CFTs on $\mathbb{S}^3$ which is the same as the EE universal term across a $\mathbb{S}^1$.
- $d = 4$ theories, called the "a-theorem". At the fixed points the c-function coincides with the trace-anomaly coefficient a.

Monotonicity theorems

Throughout the years, c-functions have been found and monotonicity theorems have been proven in various dimensions:

- $d = 2$ theories, called the "c-theorem". At the fixed points the c-function coincides with the Virasoro central charges of the fixed point CFTs.
- $d = 3$ theories, called the "F-theorem". At the fixed points the c-function coincides with the free energy of the CFTs on $\mathbb{S}^3$ which is the same as the EE universal term across a $\mathbb{S}^1$.
- $d = 4$ theories, called the "a-theorem". At the fixed points the c-function coincides with the trace-anomaly coefficient a.

In all cases there exist versions of the theorems which make crucial use of EE, but there are also alternative versions which do not. In $d = 3$ the only available proof uses EE.

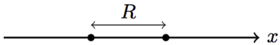
Entropic c-theorem in $d = 2$

As an illustration, I will explicitly proof a c-theorem using EE methods for $d = 2$ QFTs.

Entropic c-theorem in $d = 2$

As an illustration, I will explicitly proof a c-theorem using EE methods for $d = 2$ QFTs.

- We start with an interval in a time slice of the theory. When we make the interval very small ($R \ll$), we are probing UV physics, and when we make it very large, we are probing IR physics ($R \gg$).

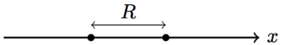


ENTROPIC C-THEOREM IN $d = 2$

Instituto Balseiro

As an illustration, I will explicitly proof a c-theorem using EE methods for $d = 2$ QFTs.

- We start with an interval in a time slice of the theory. When we make the interval very small ($R \ll$), we are probing UV physics, and when we make it very large, we are probing IR physics ($R \gg$).

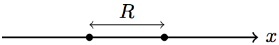


- We need to find a function $c_{\text{EE}}(R)$ such that:
  1) it coincides with the central charge c of the CFT at the fixed points
  $$c_{\text{EE}}(R)|_{\text{CFT}} = c$$

Entropic c-theorem in $d = 2$

As an illustration, I will explicitly proof a c-theorem using EE methods for $d = 2$ QFTs.

- We start with an interval in a time slice of the theory. When we make the interval very small ($R \ll$), we are probing UV physics, and when we make it very large, we are probing IR physics ($R \gg$).



- We need to find a function $c_{\mathrm{EE}}(R)$ such that:
  1) it coincides with the central charge c of the CFT at the fixed points
  $$c_{\mathrm{EE}}(R)|_{\mathrm{CFT}} = c$$
  2) it is monotonically decreasing under any RG flow,
  $$c'_{\mathrm{EE}}(R) \leq 0\,.$$

ENTROPIC C-THEOREM IN $d = 2$

The proposal for EE-based c-function reads

$$c_{\mathrm{EE}}(R) \equiv 3RS'_{\mathrm{EE}}(R)$$

where $S_{\mathrm{EE}}(R)$ is the EE of an interval of length R.

Entropic c-theorem in $d = 2$

The proposal for EE-based c-function reads

$$c_{\mathrm{EE}}(R) \equiv 3RS'_{\mathrm{EE}}(R)$$

where $S_{\mathrm{EE}}(R)$ is the EE of an interval of length R. This is well defined for any QFT, and note also that by definition it is always finite (free of divergences).

Entropic c-theorem in $d = 2$

The proposal for EE-based c-function reads

$$c_{\mathrm{EE}}(R) \equiv 3RS'_{\mathrm{EE}}(R)$$

where $S_{\mathrm{EE}}(R)$ is the EE of an interval of length R. This is well defined for any QFT, and note also that by definition it is always finite (free of divergences). Observe also that it satisfies requirement 1), since for a CFT, the EE of an interval reads

$$S_{\mathrm{EE}}|_{\mathrm{CFT}} = \frac{c}{3}\log(R/\delta) \quad \Rightarrow \quad c_{\mathrm{EE}}(R)|_{\mathrm{CFT}} = c\,.$$

Entropic c-theorem in $d = 2$

The proposal for EE-based c-function reads

$$c_{\text{EE}}(R) \equiv 3RS'_{\text{EE}}(R)$$

where $S_{\text{EE}}(R)$ is the EE of an interval of length R. This is well defined for any QFT, and note also that by definition it is always finite (free of divergences). Observe also that it satisfies requirement 1), since for a CFT, the EE of an interval reads

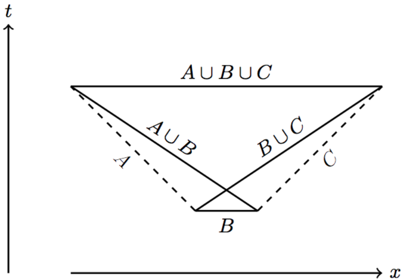
$$S_{\text{EE}}|_{\text{CFT}} = \frac{c}{3}\log(R/\delta) \quad \Rightarrow \quad c_{\text{EE}}(R)|_{\text{CFT}} = c\,.$$

Now, the hard part is to prove requirement 2). For that, we will use the strong subadditivity (SSA) property of EE,

$$S_{\text{EE}}(A\cup B\cup C) + S_{\text{EE}}(B) \leq S_{\text{EE}}(A\cup B) + S_{\text{EE}}(B\cup C)\,.$$

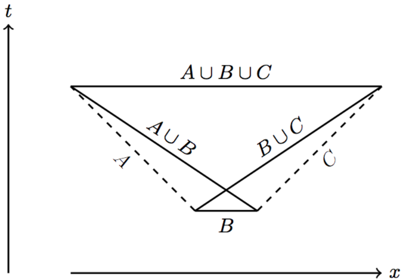
Entropic c-theorem in $d = 2$

Consider two intervals A, C on the light rays $t = \pm x$ and an interval B of width r on a time slice $t = 0$: $A = \{t = -x, -R/2 \leq x \leq -r/2\}$, $B = \{t = 0, -r/2 \leq x \leq r/2\}$, $C = \{t = x, r/2 \leq x \leq R/2\}$.



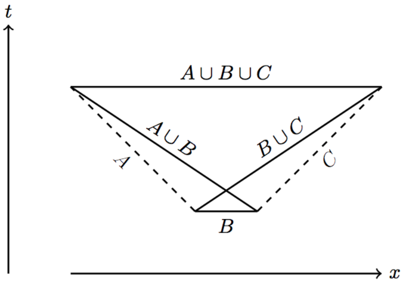
Entropic c-theorem in $d = 2$

Consider two intervals A, C on the light rays $t = \pm x$ and an interval B of width r on a time slice $t = 0$: $A = \{t = -x, -R/2 \leq x \leq -r/2\}$, $B = \{t = 0, -r/2 \leq x \leq r/2\}$, $C = \{t = x, r/2 \leq x \leq R/2\}$.

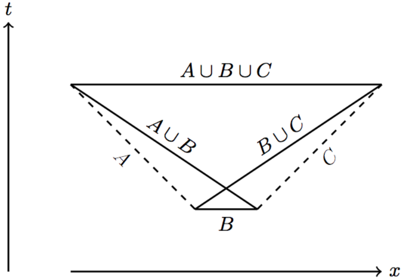


EE is invariant under unitary time evolution, so we can boost our intervals and use, instead of $A \cup B$, the straight interval which appears in the figure, and the same for $B \cup C$.

Entropic c-theorem in $d = 2$



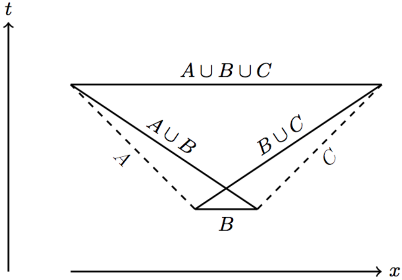
ENTROPIC C-THEOREM IN $d = 2$



The starting and ending points of the boosted $A \cup B$ interval are $(t = 0, x = r/2)$ and $(t = (R-r)/2, x = -R/2)$ respectively. Then, the invariant length between the two points reads $\Delta s = \sqrt{-\Delta t^2 + \Delta x^2} = \sqrt{rR}$. The length of the boosted $B \cup C$ is the also $\sqrt{rR}$.

Entropic c-theorem in $d = 2$

Instituto Balseiro



The starting and ending points of the boosted $A \cup B$ interval are $(t = 0, x = r/2)$ and $(t = (R-r)/2, x = -R/2)$ respectively. Then, the invariant length between the two points reads $\Delta s = \sqrt{-\Delta t^2 + \Delta x^2} = \sqrt{rR}$. The length of the boosted $B \cup C$ is the also $\sqrt{rR}$. For $A \cup B \cup C$ we can use the boosted straight interval shown in the figure, and for that one, the length is R.

Entropic c-theorem in $d = 2$

Now, let us use the SSA inequality:

$$S_{\text{EE}}(A \cup B \cup C) + S_{\text{EE}}(B) \leq S_{\text{EE}}(A \cup B) + S_{\text{EE}}(B \cup C)$$
$$\Rightarrow S_{\text{EE}}(R) + S_{\text{EE}}(r) \leq 2S_{\text{EE}}(\sqrt{rR})$$

Entropic c-theorem in $d = 2$

Now, let us use the SSA inequality:

$$
\begin{aligned}
&S_{\mathrm{EE}}(A \cup B \cup C) + S_{\mathrm{EE}}(B) \leq S_{\mathrm{EE}}(A \cup B) + S_{\mathrm{EE}}(B \cup C) \\
&\Rightarrow S_{\mathrm{EE}}(R) + S_{\mathrm{EE}}(r) \leq 2 S_{\mathrm{EE}}(\sqrt{rR})
\end{aligned}
$$

The last step is now to consider R to be slightly larger than r, namely, $R = r + \epsilon$ with $\epsilon \ll r$.

Entropic c-theorem in $d = 2$

Now, let us use the SSA inequality:

$$S_{\text{EE}}(A \cup B \cup C) + S_{\text{EE}}(B) \leq S_{\text{EE}}(A \cup B) + S_{\text{EE}}(B \cup C)$$
$$\Rightarrow S_{\text{EE}}(R) + S_{\text{EE}}(r) \leq 2S_{\text{EE}}(\sqrt{rR})$$

The last step is now to consider R to be slightly larger than r, namely, $R = r + \epsilon$ with $\epsilon \ll r$. Then, we can expand

$$S_{\text{EE}}(R) + S_{\text{EE}}(r) = 2S_{\text{EE}}(r) + \epsilon S'_{\text{EE}}(r) + \frac{1}{2}\epsilon^2 S''_{\text{EE}}(r) + \dots$$
$$2S_{\text{EE}}(\sqrt{rR}) = 2S_{\text{EE}}(r) + \epsilon S'_{\text{EE}}(r) + \frac{1}{4}\epsilon^2 \left[-\frac{S'_{\text{EE}}(r)}{r} + S''_{\text{EE}}(r)\right] + \dots$$

Entropic c-theorem in $d = 2$

Now, let us use the SSA inequality:

$$S_{\text{EE}}(A \cup B \cup C) + S_{\text{EE}}(B) \leq S_{\text{EE}}(A \cup B) + S_{\text{EE}}(B \cup C)$$
$$\Rightarrow S_{\text{EE}}(R) + S_{\text{EE}}(r) \leq 2S_{\text{EE}}(\sqrt{rR})$$

The last step is now to consider R to be slightly larger than r, namely, $R = r + \epsilon$ with $\epsilon \ll r$. Then, we can expand

$$S_{\text{EE}}(R) + S_{\text{EE}}(r) = 2S_{\text{EE}}(r) + \epsilon S'_{\text{EE}}(r) + \frac{1}{2}\epsilon^2 S''_{\text{EE}}(r) + \ldots$$
$$2S_{\text{EE}}(\sqrt{rR}) = 2S_{\text{EE}}(r) + \epsilon S'_{\text{EE}}(r) + \frac{1}{4}\epsilon^2 \left[-\frac{S'_{\text{EE}}(r)}{r} + S''_{\text{EE}}(r)\right] + \ldots$$

Finally, the SSA inequality implies:

$$S_{\text{EE}}(R) + S_{\text{EE}}(r) \leq 2S_{\text{EE}}(\sqrt{rR}) \Rightarrow \frac{\epsilon^2}{4r}\left[S'_{\text{EE}}(r) + rS''_{\text{EE}}(r)\right] \leq 0$$

Entropic c-theorem in $d = 2$

Instituto Balseiro

Now, remember that what we wanted to prove for requirement 2) was that $c'_{\mathrm{EE}}(R) \leq 0$.

Entropic c-theorem in $d = 2$

Now, remember that what we wanted to prove for requirement 2) was that $c'_{\mathrm{EE}}(R) \leq 0$. Using its definition, we have

$$c'_{\mathrm{EE}}(r) = 3\left[S'_{\mathrm{EE}}(r) + rS''_{\mathrm{EE}}(r)\right] \overset{?}{\leq} 0$$

Entropic c-theorem in $d = 2$

Now, remember that what we wanted to prove for requirement 2) was that $c'_{\mathrm{EE}}(R) \leq 0$. Using its definition, we have

$$c'_{\mathrm{EE}}(r) = 3\left[S'_{\mathrm{EE}}(r) + rS''_{\mathrm{EE}}(r)\right] \overset{?}{\leq} 0$$

But this is exactly what we have just proven using SSA!!

Entropic c-theorem in $d = 2$

Now, remember that what we wanted to prove for requirement 2) was that $c'_{\mathrm{EE}}(R) \leq 0$. Using its definition, we have

$$c'_{\mathrm{EE}}(r) = 3\left[S'_{\mathrm{EE}}(r) + rS''_{\mathrm{EE}}(r)\right] \overset{?}{\leq} 0$$

But this is exactly what we have just proven using SSA!!

Hence, $c_{\mathrm{EE}}(R) = 3RS'_{\mathrm{EE}}(R)$ is a c-function for general QFTs in $d = 2$, and we have just proven the monotonicity of RG flows in that number of dimensions. The proof relies exclusively on SSA of EE and Lorentz invariance.

Entropic c-theorem in $d = 2$

Now, remember that what we wanted to prove for requirement 2) was that $c'_{\rm EE}(R) \leq 0$. Using its definition, we have

$$c'_{\rm EE}(r) = 3\left[S'_{\rm EE}(r) + rS''_{\rm EE}(r)\right] \overset{?}{\leq} 0$$

But this is exactly what we have just proven using SSA!!

Hence, $c_{\rm EE}(R) = 3RS'_{\rm EE}(R)$ is a c-function for general QFTs in $d = 2$, and we have just proven the monotonicity of RG flows in that number of dimensions. The proof relies exclusively on SSA of EE and Lorentz invariance.

Currently there exists a unified framework for EE-based monotonicity theorems in $d = 2, 3, 4$.

Quantum Bekenstein bound

The Bekenstein bound

Instituto Balseiro

The Bekenstein bound is a surprising result which states that the entropy of an object with total energy E and with characteristic size R (*e.g.*, the size of the smallest sphere circumscribing it) cannot exceed $2\pi ER$,

$$\boxed{S(\text{object}) \leq 2\pi ER}$$

The Bekenstein bound

Instituto Balseiro

The Bekenstein bound is a surprising result which states that the entropy of an object with total energy E and with characteristic size R (*e.g.*, the size of the smallest sphere circumscribing it) cannot exceed $2\pi ER$,

$$\boxed{S(\text{object}) \leq 2\pi ER}$$

The argument uses a so-called "Geroch process".

The Bekenstein bound

The Bekenstein bound is a surprising result which states that the entropy of an object with total energy E and with characteristic size R (*e.g.*, the size of the smallest sphere circumscribing it) cannot exceed $2\pi ER$,

$$\boxed{S(\text{object}) \leq 2\pi ER}$$

The argument uses a so-called “Geroch process”. The idea is as follows: when a stationary black hole absorbs an object of energy E and radius R, its mass changes as

$$\delta M = E \times \frac{R}{4GM}$$

(the second term is a red-shift factor), where G is the Newton constant.

The Bekenstein bound

Instituto Balseiro

The Bekenstein bound is a surprising result which states that the entropy of an object with total energy E and with characteristic size R (*e.g.*, the size of the smallest sphere circumscribing it) cannot exceed $2\pi ER$,

$$\boxed{S(\text{object}) \leq 2\pi ER}$$

The argument uses a so-called "Geroch process". The idea is as follows: when a stationary black hole absorbs an object of energy E and radius R, its mass changes as

$$\delta M = E \times \frac{R}{4GM}$$

(the second term is a red-shift factor), where G is the Newton constant. As a consequence, its area changes as $\delta A = 8\pi ERG$.ă

The Bekenstein bound

Instituto Balseiro

The Bekenstein bound is a surprising result which states that the entropy of an object with total energy E and with characteristic size R (*e.g.*, the size of the smallest sphere circumscribing it) cannot exceed $2\pi ER$,

$$\boxed{S(\text{object}) \leq 2\pi ER}$$

The argument uses a so-called "Geroch process". The idea is as follows: when a stationary black hole absorbs an object of energy E and radius R, its mass changes as

$$\delta M = E \times \frac{R}{4GM}$$

(the second term is a red-shift factor), where G is the Newton constant. As a consequence, its area changes as $\delta A = 8\pi ERG$.ăThen, the entropy of the black hole increases by $\delta S_{\mathrm{BH}} = \delta A/(4G) = 2\pi ER$.

The Bekenstein bound

Now, from the second-law of thermodynamics the total entropy of the combined black hole + object should increase, so

$$S_{\text{final}} - S_{\text{initial}} \geq 0$$

The Bekenstein bound

Now, from the second-law of thermodynamics the total entropy of the combined black hole + object should increase, so

$$S_{\text{final}} - S_{\text{initial}} \geq 0$$

where $S_{\text{final}} = S_{\text{BH}} + \delta S_{\text{BH}}$ and $S_{\text{initial}} = S(\text{object}) + S_{\text{BH}}$.

The Bekenstein bound

Now, from the second-law of thermodynamics the total entropy of the combined black hole + object should increase, so

$$S_{\text{final}} - S_{\text{initial}} \geq 0$$

where $S_{\text{final}} = S_{\text{BH}} + \delta S_{\text{BH}}$ and $S_{\text{initial}} = S(\text{object}) + S_{\text{BH}}$. Therefore, $S(\text{object}) \leq 2\pi ER$ follows.

The Bekenstein bound

Now, from the second-law of thermodynamics the total entropy of the combined black hole + object should increase, so

$$S_{\text{final}} - S_{\text{initial}} \geq 0$$

where $S_{\text{final}} = S_{\text{BH}} + \delta S_{\text{BH}}$ and $S_{\text{initial}} = S(\text{object}) + S_{\text{BH}}$. Therefore, $S(\text{object}) \leq 2\pi E R$ follows.

It is interesting that while the derivation uses a gravitational process, Newton's constant G disappears from the final expression... This suggests a broader/more fundamental origin for the bound.

Quantum Bekenstein bound

There are various issues with Bekenstein's bound when considered within the quantum framework:

Quantum Bekenstein bound

There are various issues with Bekenstein's bound when considered within the quantum framework:

- The entropy, if localized, should include vacuum entanglement, which can be divergent

Quantum Bekenstein bound

Instituto Balseiro

There are various issues with Bekenstein's bound when considered within the quantum framework:

- The entropy, if localized, should include vacuum entanglement, which can be divergent
- The energy, if not global, could be negative (while the entropy is always positive)

Quantum Bekenstein bound

There are various issues with Bekenstein's bound when considered within the quantum framework:

- The entropy, if localized, should include vacuum entanglement, which can be divergent
- The energy, if not global, could be negative (while the entropy is always positive)
- The size of the object is ill defined

Quantum Bekenstein bound

There are various issues with Bekenstein's bound when considered within the quantum framework:

- The entropy, if localized, should include vacuum entanglement, which can be divergent
- The energy, if not global, could be negative (while the entropy is always positive)
- The size of the object is ill defined

These problems were historically preventing a better interpretation of the bound.

Quantum Bekenstein bound

A quantum version of the Bekenstein bound can be proven by using the positivity of relative entropy.

Quantum Bekenstein bound

A quantum version of the Bekenstein bound can be proven by using the positivity of relative entropy. Recall that for any two states ρ and ρ_0, we had

$$S_{\text{rel}}(\rho||\rho_0) = \operatorname{Tr}\rho(\log\rho - \log\rho_0) \geq 0$$

Quantum Bekenstein bound

A quantum version of the Bekenstein bound can be proven by using the positivity of relative entropy. Recall that for any two states ρ and ρ_0, we had

$$S_{\rm rel}(\rho||\rho_0) = \operatorname{Tr} \rho(\log \rho - \log \rho_0) \geq 0$$

We can rewrite this formula as an all-order version of the first-law of EE (it becomes equality when $\rho = \rho_0 + \delta\rho$)

$$S_{\rm rel}(\rho||\rho_0) = \operatorname{Tr}(\rho \log \rho - \rho_0 \log \rho_0 + \rho_0 \log \rho_0 - \rho \log \rho_0) = \Delta \langle H \rangle - \Delta S_{\rm EE} \geq 0$$

where we used the definition of the modular Hamiltonian for the original state $H = -\log \rho_0$.

Quantum Bekenstein bound

Instituto Balseiro

A quantum version of the Bekenstein bound can be proven by using the positivity of relative entropy. Recall that for any two states ρ and ρ_0, we had

$$S_{\mathrm{rel}}(\rho||\rho_0) = \operatorname{Tr}\rho(\log\rho - \log\rho_0) \geq 0$$

We can rewrite this formula as an all-order version of the first-law of EE (it becomes equality when $\rho = \rho_0 + \delta\rho$)

$$S_{\mathrm{rel}}(\rho||\rho_0) = \operatorname{Tr}(\rho\log\rho - \rho_0\log\rho_0 + \rho_0\log\rho_0 - \rho\log\rho_0) = \Delta\langle H\rangle - \Delta S_{\mathrm{EE}} \geq 0$$

where we used the definition of the modular Hamiltonian for the original state $H = -\log\rho_0$. So, for any pair of finite-energy states we have

$$\boxed{\Delta S_{\mathrm{EE}} \leq \Delta\langle H\rangle}$$

Quantum Bekenstein bound

Instituto Balseiro

A quantum version of the Bekenstein bound can be proven by using the positivity of relative entropy. Recall that for any two states ρ and ρ_0, we had

$$S_{\text{rel}}(\rho||\rho_0) = \text{Tr}\,\rho(\log\rho - \log\rho_0) \geq 0$$

We can rewrite this formula as an all-order version of the first-law of EE (it becomes equality when $\rho = \rho_0 + \delta\rho$)

$$S_{\text{rel}}(\rho||\rho_0) = \text{Tr}(\rho\log\rho - \rho_0\log\rho_0 + \rho_0\log\rho_0 - \rho\log\rho_0) = \Delta\langle H\rangle - \Delta S_{\text{EE}} \geq 0$$

where we used the definition of the modular Hamiltonian for the original state $H = -\log\rho_0$. So, for any pair of finite-energy states we have

$$\boxed{\Delta S_{\text{EE}} \leq \Delta\langle H\rangle}$$

This can be thought of as a mega-generalized quantum version of the Bekenstein bound.

Quantum Bekenstein bound

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\mathrm{EE}} \leq \Delta \langle H \rangle$.

Quantum Bekenstein bound

Let us compare the two versions: $S \leq 2\pi E R$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.

Quantum Bekenstein bound

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.
- The natural reference state ρ_0 is the reduction of the vacuum state to the corresponding region.

Quantum Bekenstein bound

Instituto Balseiro

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.
- The natural reference state ρ_0 is the reduction of the vacuum state to the corresponding region.
- The piece $2\pi ER$ is interpreted in terms of the modular Hamiltonian of the reduced density matrix ρ_0.

Quantum Bekenstein bound

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.
- The natural reference state ρ_0 is the reduction of the vacuum state to the corresponding region.
- The piece $2\pi ER$ is interpreted in terms of the modular Hamiltonian of the reduced density matrix ρ_0. In general, modular Hamiltonians are very complicated objects.

Quantum Bekenstein bound

Instituto Balseiro

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.
- The natural reference state ρ_0 is the reduction of the vacuum state to the corresponding region.
- The piece $2\pi ER$ is interpreted in terms of the modular Hamiltonian of the reduced density matrix ρ_0. In general, modular Hamiltonians are very complicated objects. However, for the vacuum state, if we take for instance the entangling region to be half space ($x_1 > 0$) in Minkowski space (this is called the "Rindler wedge"), it takes the form

$$H = 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 T_{00}(x)$$

for any QFT in d dimensions, where $T_{00}(x)$ is the energy density operator.

Quantum Bekenstein bound

Instituto Balseiro

Let us compare the two versions: $S \leq 2\pi ER$ vs $\Delta S_{\text{EE}} \leq \Delta \langle H \rangle$.

- Observe first that the quantum version requires a substraction of entropies, which removes the UV divergences present in the EE.
- The natural reference state ρ_0 is the reduction of the vacuum state to the corresponding region.
- The piece $2\pi ER$ is interpreted in terms of the modular Hamiltonian of the reduced density matrix ρ_0. In general, modular Hamiltonians are very complicated objects. However, for the vacuum state, if we take for instance the entangling region to be half space ($x_1 > 0$) in Minkowski space (this is called the "Rindler wedge"), it takes the form

$$H = 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 T_{00}(x)$$

for any QFT in d dimensions, where $T_{00}(x)$ is the energy density operator. (It is a remarkable fact that by reducing the vacuum state to half space, we can learn about the energy density operator)

Quantum Bekenstein bound

- Now, using the fact that $\langle\Omega|T_{00}|\Omega\rangle = 0$ (the expectation value of the energy density in the vacuum is zero), we find

Quantum Bekenstein bound

Instituto Balseiro

- Now, using the fact that $\langle \Omega | T_{00} | \Omega \rangle = 0$ (the expectation value of the energy density in the vacuum is zero), we find

$$\langle H \rangle = 2\pi \int_{x_1 > 0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x) \rangle$$

where $\langle T_{00}(x) \rangle \equiv \mathrm{Tr}(\rho T_{00}(x))$ is computed with respect to the state ρ.

Quantum Bekenstein bound

- Now, using the fact that $\langle\Omega|T_{00}|\Omega\rangle = 0$ (the expectation value of the energy density in the vacuum is zero), we find

$$\langle H\rangle = 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x)\rangle$$

where $\langle T_{00}(x)\rangle \equiv \mathrm{Tr}(\rho T_{00}(x))$ is computed with respect to the state ρ. Our quantum version of the Bekenstein bound becomes

$$\Delta S_{\text{EE}} \leq 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x)\rangle$$

This looks pretty similar to $S \leq 2\pi ER$.

Quantum Bekenstein bound

- Now, using the fact that $\langle \Omega | T_{00} | \Omega \rangle = 0$ (the expectation value of the energy density in the vacuum is zero), we find

$$\langle H \rangle = 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x) \rangle$$

where $\langle T_{00}(x) \rangle \equiv \mathrm{Tr}(\rho T_{00}(x))$ is computed with respect to the state ρ. Our quantum version of the Bekenstein bound becomes

$$\Delta S_{\mathrm{EE}} \leq 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x) \rangle$$

This looks pretty similar to $S \leq 2\pi E R$.

- If the system is sharply localized at a distance R from $x_1 = 0$, the RHS can be actually approximated by $2\pi E R$, where $E \approx \int_{x_1>0} \mathrm{d}^{d-1}x \langle T_{00}(x) \rangle$.

Quantum Bekenstein bound

- Now, using the fact that $\langle\Omega|T_{00}|\Omega\rangle = 0$ (the expectation value of the energy density in the vacuum is zero), we find

$$\langle H\rangle = 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \,\langle T_{00}(x)\rangle$$

where $\langle T_{00}(x)\rangle \equiv \mathrm{Tr}(\rho T_{00}(x))$ is computed with respect to the state ρ. Our quantum version of the Bekenstein bound becomes

$$\Delta S_{\text{EE}} \leq 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \,\langle T_{00}(x)\rangle$$

This looks pretty similar to $S \leq 2\pi ER$.

- If the system is sharply localized at a distance R from $x_1 = 0$, the RHS can be actually approximated by $2\pi ER$, where $E \approx \int_{x_1>0} \mathrm{d}^{d-1}x \,\langle T_{00}(x)\rangle$.
- Other Bekenstein-type bounds involving energy and entropy can be obtained whenever H is given in terms of the stress tensor, like in the case of spheres.

Quantum Bekenstein bound

The interpretation of the original bound was that entropy cannot increase too much for a fixed energy. It would be a kind of universal bound on the number of degrees of freedom.

Quantum Bekenstein bound

The interpretation of the original bound was that entropy cannot increase too much for a fixed energy. It would be a kind of universal bound on the number of degrees of freedom.

The interpretation of the quantum bound is very different: there is no bound on degrees of freedom,

Quantum Bekenstein bound

The interpretation of the original bound was that entropy cannot increase too much for a fixed energy. It would be a kind of universal bound on the number of degrees of freedom.

The interpretation of the quantum bound is very different: there is no bound on degrees of freedom, it is rather related to the idea of distinguishability: when restricted to a region, fluctuations can be as large as to make it hard to distinguish the vacuum from another state if the energy (times distance to the boundary) of this other state is not big enough.

Summary of the second part

Instituto Balseiro

- Free fields allow for special methods which simplify the task of computing EE.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$. In its classical form, it is not very rigorous statement...

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$. In its classical form, it is not very rigorous statement... A quantum version of the bound follows from the positivity of relative entropy.

Summary of the second part

Instituto Balseiro

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$. In its classical form, it is not very rigorous statement... A quantum version of the bound follows from the positivity of relative entropy. For any pair of states, one can show that $\Delta S_{\rm EE} \leq \Delta \langle H \rangle$, where H is the modular Hamiltonian of the reference state.

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d=2$ and $d=3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d=2$ using EE. Similar theorems exist for $d=3,4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$. In its classical form, it is not very rigorous statement... A quantum version of the bound follows from the positivity of relative entropy. For any pair of states, one can show that $\Delta S_{\rm EE} \leq \Delta \langle H \rangle$, where H is the modular Hamiltonian of the reference state. If we choose that state to be the vacuum and the entangling region to be the Rindler wedge, we find a quantum version of the bound,

$$\Delta S_{\rm EE} \leq 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x) \rangle$$

Summary of the second part

- Free fields allow for special methods which simplify the task of computing EE. In particular, there are useful formulas for EE which are written in terms of correlators of the fundamental fields. Our lattice calculations agree with the expected general results for CFTs in $d = 2$ and $d = 3$.
- QFTs can be thought as the study of RG flows. As we move from the UV to the IR, we expect less and less degrees of freedom to survive. Finding quantities which realize and quantify this monotonicity is an important problem in QFT. We proved a c-theorem in $d = 2$ using EE. Similar theorems exist for $d = 3, 4$ theories.
- The Bekenstein bound is an old result which establishes that the entropy of an object of energy E and size R is bounded as $S \leq 2\pi ER$. In its classical form, it is not very rigorous statement... A quantum version of the bound follows from the positivity of relative entropy. For any pair of states, one can show that $\Delta S_{\rm EE} \leq \Delta \langle H \rangle$, where H is the modular Hamiltonian of the reference state. If we choose that state to be the vacuum and the entangling region to be the Rindler wedge, we find a quantum version of the bound,

$$\Delta S_{\rm EE} \leq 2\pi \int_{x_1>0} \mathrm{d}^{d-1}x x^1 \langle T_{00}(x) \rangle$$

  The quantum bound is interpreted as the fact that it becomes difficult to distinguish a given state from the vacuum if its energy is too small.