Entanglement in QM and QFT

4/5 - The “Extensive Mutual Information” model

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It from Qubit
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Outline

1. General structure of EE and universal terms
2. The “extensive mutual information” model
Some references

- The expression for the EE universal term in the case of smooth entangling regions in $d = 4$ is due to Solodukhin: https://arxiv.org/abs/0802.3117.
- The unitarity bounds are due to Hoffman and Maldacena: https://arxiv.org/abs/0803.1467.
- The connection between the disk entanglement entropy and the three-sphere partition function is due to Casini, Huerta and Myers: https://arxiv.org/abs/1102.0440.
- A detailed account on the EE for entangling regions containing singularities and references can be found in: https://arxiv.org/abs/1904.11495.
- The relation between the corner coefficient $\sigma$ and the stress tensor two point function $C_T$ was conjectured in my paper with Myers and Witczak-Krempa: https://arxiv.org/abs/1505.04804 based on free-field and holographic calculations, and proved later for general CFTs by Faulkner, Leigh and Parrikar in: https://arxiv.org/abs/1511.05179.
General structure of EE and universal terms
Given CFT$_d$ and smooth entangling region $V$, EE takes the generic form

$$S_{\text{EE}}^{(d)} = b_{d-2} \frac{H^{d-2}}{\delta^{d-2}} + b_{d-4} \frac{H^{d-4}}{\delta^{d-4}} +$$

$$\cdots + \begin{cases} 
  b_1 \frac{H}{\delta} + (-1)^{\frac{d-1}{2}} s^\text{univ}, & \text{(odd } d) \\
  b_2 \frac{H^2}{\delta^2} + (-1)^{\frac{d-2}{2}} s^\text{univ} \log \left( \frac{H}{\delta} \right) + b_0, & \text{(even } d) 
\end{cases}$$

$H$ is some characteristic length of $V$ and $\delta$ a UV regulator.
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\(H\) is some characteristic length of \(V\) and \(\delta\) a UV regulator.

non-universal and local; universal and local;
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General structure of EE and universal terms

General structure of EE for CFTs

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$$S_{EE}^{(2)} = \frac{c}{3} \log \left( \frac{H}{\delta} \right) + b_0,$$

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Coefficients $b_{d-2}, \ldots, b_1$ are “non-universal”: they are not well-defined in the continuum. They are “local” in the sense that they come from short-range correlations across $\partial V$. 
Universal terms: even dimensions

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$s^{\text{univ}}$ are “universal”: they are well-defined in the continuum and capture meaningful information about the CFT.

In even dimensions, the universal term is logarithmic and $s^{\text{univ}}$ is given by a linear combination of local integrals over $\partial V$ weighted by theory-dependent coefficients which can be shown to coincide with the trace-anomaly charges,

$$\langle T_\mu^\mu \rangle = -2(-)^{d/2} A \mathcal{X}_d + \sum_n B_n I_n.$$
Universal terms: even dimensions

For instance, in $d = 4$:

$$
\langle T_\mu^{\mu} \rangle = -\frac{a}{16\pi^2} \mathcal{X}_4 + \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}
$$
Universal terms: even dimensions

For instance, in \( d = 4 \):

\[
\langle T_\mu^\mu \rangle = -\frac{a}{16\pi^2} \chi_4 + \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}
\]

and

\[
S^{(4)}_{\text{EE}} = b_2 \frac{H^2}{\delta^2} - \left[ \frac{a}{2\pi} \int_{\partial V} \mathcal{R} + \frac{c}{2\pi} \int_{\partial V} \left( \text{tr} k^2 - \frac{1}{2} k^2 \right) \right] \log \left( \frac{H}{\delta} \right) + b_0.
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and

$$
S^{(4)}_{\text{EE}} = b_2 \frac{H^2}{\delta^2} \left[ \frac{a}{2\pi} \int_{\partial V} R + \frac{c}{2\pi} \int_{\partial V} \left( \text{tr}k^2 - \frac{1}{2} k^2 \right) \right] \log \left( \frac{H}{\delta} \right) + b_0.
$$

Dependence on details of entangling-surface geometry and CFT considered appear highly “disentangled” from each other.

For unitary CFTs, $a$ and $c$ constrained to the range:

$$
\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}.
$$
Universal terms: even dimensions

\( a \) and \( c \) can be isolated by considering entangling surfaces corresponding to spheres and cylinders, respectively,

\[
S_{\text{EE}}^{(4)}|_{\text{sphere}} \supset -4a \log(R/\delta), \quad S_{\text{EE}}^{(4)}|_{\text{cylinder}} \supset -\frac{c}{2} \frac{L}{R} \log(R/\delta),
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where \( R \) is the radius of the sphere or the cylinder, respectively, and \( L \) the length of the former.
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\[ S_{\text{EE}}^{(4)}|_{\text{sphere}} \supset -4a \log(R/\delta), \quad S_{\text{EE}}^{(4)}|_{\text{cylinder}} \supset -\frac{c}{2R} \log(R/\delta), \]

where $R$ is the radius of the sphere or the cylinder, respectively, and $L$ the length of the former.

For comparison, in $d = 6$ there are three “B-type” charges, $B_1, B_2, B_3$, besides the “A-type” one.
Universal terms: odd dimensions

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$$S_{EE}^{(3)} = b_1 \frac{H}{\delta} - F.$$
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In $F$, dependence on geometric details of $V$ and dependence on the details of the CFT are no longer disentangled from each other. In $d = 5, 7, \ldots$ similar story: for $s^{\text{univ}}$ for $\partial V = S^{d-2}$ equals free energy on $S^d$. 
General structure of EE for CFTs

non-universal and local; universal and local;
universal and non-local; non-universal and local+non-local

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\]

\[ \cdots + \begin{cases} 
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No one has ever paid much attention to the constant coefficient \( b_0 \) appearing for even-dimensional theories. Just like \( b_{d-2}, \ldots, b_1 \), this is a non-universal piece. However, all this pollution has a local origin and \( b_0 \) also contains a universal non-local part which does not depend on the regulator details...
Singular regions

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$$S_{EE}^{(3)}|_{\text{corner}} = b_1 \frac{H}{\delta} - a^{(3)}(\Omega) \log \left( \frac{H}{\delta} \right) + \tilde{b}_0,$$

where $a^{(3)}(\Omega)$ is a cutoff-independent function of the opening angle.
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The dependence of $a^{(3)}(\Omega)$ on the opening angle changes (apparently rather drastically) from one CFT to another.
Some general properties:

\[ a^{(3)}(\pi + \Omega) = a^{(3)}(\pi - \Omega), \quad a^{(3)}(\Omega) \geq 0, \]
\[ \partial_\Omega a^{(3)}(\Omega) \leq 0, \quad \partial^2_\Omega a^{(3)}(\Omega) \geq -\frac{\partial_\Omega a^{(3)}(\Omega)}{\sin \Omega} \quad \text{for} \quad \Omega \in [0, \pi]. \]
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In the very-sharp and almost-smooth limits, the function behaves as

\[ a^{(3)}(\Omega \simeq 0) = \frac{k}{\Omega} + O(\Omega), \quad a^{(3)}(\Omega \simeq \pi) = \sigma \cdot (\Omega - \pi)^2 + O(\Omega - \pi)^4. \]
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\( k \) is a constant which coincides with the universal coefficient corresponding to a slab region for general theories.
Singular regions

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Leading coefficient in almost-smooth regime, \( \sigma \), is related to the stress-energy tensor two-point function coefficient* \( C_T \) through

\[ \sigma = \frac{\pi^2}{24} C_T, \]

for general CFTs.
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*For general CFTs in \( d \) dimensions, the stress-tensor correlator behaves as \( \langle T_{ab}(x) T_{cd}(0) \rangle = C_T I_{ab,cd}(x)/|x|^{2d} \) where \( I_{ab,cd}(x) \) is a fixed tensorial structure, and the only theory-dependence appears through \( C_T \).

[In \( d = 4 \), \( C_T \) is proportional to the trace-anomaly coefficient \( c \)]
Singular regions

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Lower bound on $a^{(3)}(\Omega)$ valid for general CFTs

$$a^{(3)}(\Omega) \geq a_{\text{min}}(\Omega), \quad \text{where} \quad a_{\text{min}}(\Omega) \equiv \frac{\pi^2 C_T}{3} \log \left[ \frac{1}{\sin(\Omega/2)} \right]$$
Singular regions

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Similar logarithmic enhancement of universal term

\[
S_{EE}^{(4)}|_{\text{cone}} = b_2 \frac{H^2}{\delta^2} - a^{(4)}(\Omega) \log^2 \left( \frac{H}{\delta} \right) + \bar{b}_0 \log \left( \frac{H}{\delta} \right) + b_0,
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\]

but now

\[
a^{(4)}(\Omega) = \frac{c}{4} \cdot \frac{\cos^2 \Omega}{\sin \Omega}
\]

for all CFTs.

Only dependence on the theory under consideration appears through the charge \(c\)
Contrast between odd- and even-dimensional cases persists for higher $d$. 

\[
S(d)(\text{hyper})\text{cone} \supset (-1)^{d-2} a(d)(\Omega) \log \left( \frac{H}{\delta} \right),
\]

where $a(d)(\Omega)$ differs for each CFT.

Still some degree of universality in almost-smooth limit $\Leftrightarrow C_{TT} \frac{16}{32}$.
**Singular regions**

Contrast between odd- and even-dimensional cases persists for higher $d$. For the latter

$$S_{EE}^{(d)(\text{even})}_{(\text{hyper})\text{cone}} \supset (-1)^{\frac{d-2}{2}} a^{(d)}(\Omega) \log^2 (H/\delta)$$

where

$$a^{(d)}(\Omega) = \frac{\cos^2 \Omega}{\sin \Omega} \sum_{j=0}^{\frac{d-4}{2}} \left[ \gamma_j^{(d)} \cos(2j\Omega) \right]$$

Again functional dependence on $\Omega$ completely fixed for any CFT up to $(d/2-1)$ coefficients $\gamma_j^{(d)}$ related to trace-anomaly charges.
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Still some degree of universality in almost-smooth limit $\Leftrightarrow C_T$. 
Singular regions

One can consider other types of singular regions, with their own peculiarities and features. These include wedges, cones with non-circular sections, curved corners and cones, polyhedral corners, etc.
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- Polyhedral corner of opening angles $\theta_1, \theta_2, \ldots, \theta_j$

\[
S_{\text{EE}}^{(4)}\big|_{\text{polyh.}} = \frac{b_2}{\delta^2}\frac{H^2}{\delta} - w_1 \frac{H}{\delta} + v(\theta_1, \theta_2, \cdots, \theta_j) \log \left( \frac{H}{\delta} \right) + b_0
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  \]

- Infinite wedge of opening angle $\Omega$

  \[
  S_{\text{EE}}^{(4)}|_{\text{wedge}} = b_2 \frac{H^2}{\delta^2} - f(\Omega) \frac{H}{\delta} + b_0
  \]
The “extensive mutual information” model
A model has an extensive mutual information if the “tripartite information” vanishes

$$I_3(A; B, C) \equiv I(A, B) + I(A, C) - I(A, B \cup C) = 0 \quad \forall \quad A, B, C$$
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[In general, it is possible both \(I_3(A; B, C) \geq 0\) and \(I_3(A; B, C) \leq 0\).]

Imposing \(I_3(A; B, C) = 0 + \) some physically reasonable requirements such as causality and Poincaré invariance, strongly restricts the form of EE and mutual information in general \(d\).
The “extensive mutual information” model

The result defines the EMI model. Its EE is given by

\[ S_{\text{EE}}^{\text{EMI}} = \kappa(d) \int_{\partial A} d^{d-2} \sigma_1 \int_{\partial A} d^{d-2} \sigma_2 \frac{n^i(x_1) n^j(x_2) \delta_{ij}}{|x_1 - x_2|^{2(d-2)}} \]

where \( n^i(x_1) \) is the unit normal vector to the boundary of \( A \), \( \partial A \), at the point \( x_1 \) and \( \kappa(d) \) is a positive parameter.
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It is an open problem to find out whether this is the EE of an actual CFT for \( d \geq 3 \).
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Regardless of this, the model respects all general principles of EE and is a useful tool for understanding various features. Computationally, even simpler than Ryu-Takayanagi formula.
EE of a disk in $d = 3$
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Consider disk region of radius $R$. 

Due to the symmetry of the problem, we can fix $x_2 = (R, 0)$, and then $\vec{n}(x_2) = (1, 0)$, which makes one of them trivial.

On the other hand, we have $x_1 = (R\cos \theta_1, R\sin \theta_1)$, $\vec{n}(x_1) = (\cos \theta_1, \sin \theta_1)$.

Also, $d\sigma_1 = R\,d\theta_1$, $d\sigma_2 = R\,d\theta_2$.

From this, one finds 

$$|x_1 - x_2|^2 = 2R^2(1 - \cos \theta_1) = 4R^2\sin^2(\theta_1/2).$$

Then, we have 

$$S_{EMI}^{EE} = \kappa(3) \int_{2\pi}^{0} R\,d\theta_2 \int_{R}^{0} \cos \theta_1 4R^2\sin^2(\theta_1/2).$$

Second integral diverges when $|x_1 - x_2|^2 \to 0$, so we need to regulate it: 

1) allow only for angles larger than $\delta/R$; 
2) replace $|x_1 - x_2|^2 \to |x_1 - x_2|^2 + \delta^2$. 

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EE of a disk in $d = 3$

Consider disk region of radius $R$. Due to the symmetry of the problem, we can fix $x_2 = (R, 0)$, and then $\vec{n}(x_2) = (1, 0)$, which makes one of them trivial.
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$$S_{EE}^{\text{EMI}} = \kappa(3) \int_0^{2\pi} Rd\theta_2 \int R d\theta_1 \frac{\cos \theta_1}{4R^2 \sin^2(\theta/2)}.$$
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Second integral diverges when $|x_1 - x_2|^2 \to 0$, so we need to regulate it:
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Second integral diverges when $|x_1 - x_2|^2 \to 0$, so we need to regulate it:
1) allow only for angles larger than $\delta/R$;
The “extensive mutual information” model

**EE of a disk in $d = 3$**

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Second integral diverges when $|x_1 - x_2|^2 \to 0$, so we need to regulate it:

1) allow only for angles larger than $\delta/R$;
2) replace $|x_1 - x_2|^2 \to |x_1 - x_2|^2 + \delta^2$. 
EE of a disk in $d = 3$

First regulator:

$$S^{\text{EMI}}_{\text{EE}} = \frac{\pi \kappa(3)}{2} \cdot 2 \int_{\delta/R}^{\pi} \frac{\cos \theta_1 d\theta_1}{\sin^2(\theta/2)} = 4\pi \kappa(3) \frac{R}{\delta} - 2\pi^2 \kappa(3)$$

Second regulator:

$$S^{\text{EMI}}_{\text{EE}} = 2\pi R^2 \kappa(3) \int_0^{2\pi} \frac{\cos \theta_1 d\theta_1}{[4R^2 \sin^2(\theta/2) + \delta^2]} = 2\pi^2 \kappa(3) \frac{R}{\delta} - 2\pi^2 \kappa(3)$$
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Universal piece unchanged, $F = 2\pi^2 \kappa(3)$, whereas $b_1$ depends on regulator.
The “extensive mutual information” model

EE across sphere in $d = 4$
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Radius-$R$ spherical entangling surface.
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Radius-$R$ spherical entangling surface. All points equivalent on sphere surface, so we can fix $x_2 = (0, 0, R)$ and $\vec{n}(x_2) = (0, 0, 1)$. 
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\[
S_{EE}^{\text{EMI}} = \kappa(4) \cdot 4\pi R^2 \cdot 2\pi R^2 \int d\theta_1 \frac{\sin \theta_1 \cos \theta_1}{16R^4 \sin^4(\theta_1/2)}.
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Regulators:
1) allow only for angles larger than $\delta/R$;
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**EE across sphere in** $d = 4$

First regulator:

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S_{EE}^{\text{EMI}} = \frac{\pi^2 \kappa(4)}{2} \int_{\delta/R}^{\pi} d\theta_1 \frac{\sin \theta_1 \cos \theta_1}{\sin^4(\theta/2)}
\]

\[
= 4\pi^2 \kappa(4) \frac{R^2}{\delta^2} - 4\pi^2 \kappa(4) \log (R/\delta) - \frac{\pi^2 \kappa(4)}{3} + 4 \log 2
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S_{EE}^{\text{EMI}} = 8\pi^2 R^4 \kappa(4) \int_0^\pi d\theta_1 \frac{\sin \theta_1 \cos \theta_1}{[16R^4 \sin^4(\theta/2) + \delta^4]}
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\]

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= 2\pi^3 \kappa(4) \frac{R^2}{\delta^2} - 4\pi^2 \kappa(4) \log (R/\delta) - \pi^2 \kappa(4) \left[ 1 + 4 \log 2 \right].
\]

Would-be trace-anomaly coefficient \( a_{\text{EMI}} = \pi^2 \kappa(4) \). Non-universal piece \( b_2 \) depends on regulator. \( b_0 \) has a “piece” which remains the same and one which varies.*
EE across cylinder in $d = 4$

Cylinder of radius $R$. 
EE across cylinder in $d = 4$

Cylinder of radius $R$. In cylindrical coordinates we can write $x_1 = (R \cos \phi_1, R \sin \phi_1, z_1)$, $\vec{n}(x_1) = (\cos \phi_1, \sin \phi_1, 0)$, and $x_2 = (R, 0, z_2)$, $\vec{n}(x_2) = (1, 0, 0)$, where we already took advantage of the circular symmetry of the surface.
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\[
S_{EE}^{EMI} = \kappa(4) \cdot 2\pi \int Rdz_2 \int Rdz_1 \int d\phi_1 \frac{\cos \phi_1}{[4R^2 \sin^2(\phi_1/2) + (z_1 - z_2)^2]^2}.
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$$S_{EE}^{EMI} = \kappa(4) \cdot 2\pi \int R dz_2 \int R dz_1 \int d\phi_1 \frac{\cos \phi_1}{[4R^2 \sin^2(\phi_1/2) + (z_1 - z_2)^2]^2}.$$

We can set $z_1 = 0$ and regulate $\int_{-L/2}^{L/2} dz_1 = L$. 
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$$S_{EE}^{\text{EMI}} = \kappa(4) \cdot 2\pi \int R dz_2 \int R dz_1 \int d\phi_1 \frac{\cos \phi_1}{\left[4R^2 \sin^2(\phi_1/2) + (z_1 - z_2)^2\right]^2}.$$ 

We can set $z_1 = 0$ and regulate $\int_{-L/2}^{L/2} dz_1 = L$. In the resulting expression, we can perform the integral over $z_2$, which requires no regulation,

$$S_{EE}^{\text{EMI}} = 2\pi R^2 \cdot L \cdot \kappa(4) \cdot \int d\phi_1 \frac{\pi \cos \phi_1}{16R^3 \sin^3(\phi_1/2)}.$$
EE across cylinder in $d = 4$

First regulator:

$$S_{EE}^{EMI} = \pi^2 \kappa(4) \frac{RL}{\delta^2} - \frac{3\pi^2 \kappa(4)}{4} \frac{L}{R} \log \left( \frac{R}{\delta} \right) - \frac{\pi^2 \kappa(4)}{2} \left[ \frac{1}{12} + 3 \log 2 \right] \frac{L}{R}.$$

Second regulator:

$$S_{EE}^{EMI} = \pi^3 \kappa(4) \frac{RL}{\delta^2} - \frac{3\pi^2 \kappa(4)}{4} \frac{L}{R} \log \left( \frac{R}{\delta} \right) - \frac{\pi^2 \kappa(4)}{2} \left[ -\frac{1}{2} + \frac{9}{2} \log 2 \right] \frac{L}{R}.$$
EE across cylinder in $d = 4$

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Would-be trace-anomaly coefficient $c_{\text{EMI}} = 3\pi^2 \kappa(4)/2$. For the EMI model $a_{\text{EMI}}/c_{\text{EMI}} = 2/3$
EE across cylinder in \( d = 4 \)

First regulator:

\[
S_{EE}^{\text{EMI}} = \pi^2 \kappa(4) \frac{R L}{\delta^2} - \frac{3\pi^2 \kappa(4)}{4} \frac{L}{R} \log \left( \frac{R}{\delta} \right) - \frac{\pi^2 \kappa(4)}{2} \left[ \frac{1}{12} + 3 \log 2 \right] \frac{L}{R}.
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Would-be trace-anomaly coefficient \( c_{\text{EMI}} = 3\pi^2 \kappa(4)/2 \). For the EMI model \( a_{\text{EMI}}/c_{\text{EMI}} = 2/3 \) \( \leftarrow \) satisfies unitarity bounds.
EE across cylinder in $d = 4$

First regulator:

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Second regulator:

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EE for a corner in $d = 3$
EE for a corner in $d = 3$

Entangling surface is defined by the lines $Y(X) = 0$ and $Y(X) = X \cdot \tan \Omega$ with $X \geq 0$. 
EE for a corner in $d = 3$

Entangling surface is defined by the lines $Y(X) = 0$ and $Y(X) = X \cdot \tan \Omega$ with $X \geq 0$. In this case there are two contributions, one from considering 1 and 2 on the same line, and another one from considering 1 on the $Y(X) = 0$ line and 2 on the $Y(X) = X \cdot \tan \Omega$ one.
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$$s_I = \kappa(3) \int dX_1 \int dX_2 \frac{1}{(X_1 - X_2)^2}.$$
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$$s_1 = \kappa(3) \int dX_1 \int dX_2 \frac{1}{(X_1 - X_2)^2}.$$ 

Second contribution: $\vec{n}(X_1) = (0, -1)$, $\vec{n}(X_2) = (-\sin \Omega, \cos \Omega)$, $d\sigma_1 = dX_1$, $d\sigma_2 = dX_2 / \cos \Omega$,

$$s_{II} = -\kappa(3) \int dX_1 \int dX_2 \frac{1}{(X_1 - X_2)^2 + \tan^2 \Omega X_2^2}.$$
EE for a corner in $d = 3$

Possible regulator:

$$\int dX_1 \rightarrow \int_{\delta}^{H} dX_1, \quad \int dX_2 \rightarrow \left[ \int_{0}^{x_1-\delta} dX_2 + \int_{x_1+\delta}^{\infty} dX_2 \right].$$
EE for a corner in $d = 3$

Possible regulator:

$$
\int dX_1 \to \int^H_\delta dX_1, \quad \int dX_2 \to \left[ \int_0^{x_1-\delta} dX_2 + \int_{x_1+\delta}^\infty dX_2 \right].
$$

Final result:

$$
S_{\text{EE}}^{\text{EMI}} = \frac{4\kappa(3)H}{\delta} - a_{\text{EMI}}^{(3)}(\Omega) \log(H/\delta) + O(\delta^0),
$$

where

$$
a_{\text{EMI}}^{(3)}(\Omega) = 2\kappa(3)[1 + (\pi - \Omega) \cot \Omega].
$$
The “extensive mutual information” model

**EE for a corner in \( d = 3 \)**

Possible regulator:

\[
\int dX_1 \to \int_\delta^H dX_1, \quad \int dX_2 \to \left[ \int_0^{x_1 - \delta} dX_2 + \int_{x_1 + \delta}^\infty dX_2 \right].
\]

Final result:

\[
S_{\text{EE}}^{\text{EMI}} = \frac{4\kappa(3)H}{\delta} - a_{\text{EMI}}^{(3)}(\Omega) \log(H/\delta) + \mathcal{O}(\delta^0),
\]

where

\[
a_{\text{EMI}}^{(3)}(\Omega) = 2\kappa(3)[1 + (\pi - \Omega) \cot \Omega].
\]

This satisfies all general properties for a decent EE corner function. In particular, in the very-sharp and almost-smooth limits,

\[
a_{\text{EMI}}^{(3)}(\Omega \simeq 0) = \frac{k}{\Omega} + \mathcal{O}(\Omega), \quad a_{\text{EMI}}^{(3)}(\Omega \simeq \pi) = \sigma \cdot (\Omega - \pi)^2 + \mathcal{O}(\Omega - \pi)^4.
\]

\[
k_{\text{EMI}} = 2\pi \kappa(3) \quad \text{and} \quad \sigma_{\text{EMI}} = 2\kappa(3)/3.
\]
EE for a corner in $d = 3$

Possible regulator:

$$\int dX_1 \rightarrow \int_\delta^H dX_1, \quad \int dX_2 \rightarrow \left[ \int_0^{x_1-\delta} dX_2 + \int_{x_1+\delta}^\infty dX_2 \right].$$

Final result:

$$S_{EE}^{\text{EMI}} = \frac{4\kappa(3)H}{\delta} - a^{(3)}_{\text{EMI}}(\Omega) \log(H/\delta) + O(\delta^0),$$

where

$$a^{(3)}_{\text{EMI}}(\Omega) = 2\kappa(3)[1 + (\pi - \Omega) \cot \Omega].$$

This satisfies all general properties for a decent EE corner function. In particular, in the very-sharp and almost-smooth limits,

$$a^{(3)}_{\text{EMI}}(\Omega \simeq 0) = \frac{k}{\Omega} + O(\Omega), \quad a^{(3)}_{\text{EMI}}(\Omega \simeq \pi) = \sigma \cdot (\Omega - \pi)^2 + O(\Omega - \pi)^4.$$

$k_{\text{EMI}} = 2\pi\kappa(3)$ and $\sigma_{\text{EMI}} = 2\kappa(3)/3$. Value of the would-be stress-tensor two-point function charge $C_T^{\text{EMI}} = 16\kappa(3)/\pi^2$. 
EE across a cone in $d = 4$

Parametrize cone in cylindrical coordinates by $z = \rho / \tan \Omega$.
EE across a cone in $d = 4$

Parametrize cone in cylindrical coordinates by $z = \rho / \tan \Omega$. Induced metric on cone: $ds^2 = d\rho^2 / \sin^2 \Omega + \rho^2 d\phi$, so $d^2 \sigma_1 = [\rho_1 / \sin \Omega] d\rho_1 d\phi_1$ and analogously for 2.
Parametrize cone in cylindrical coordinates by $z = \rho/\tan \Omega$. Induced metric on cone: $ds^2_h = d\rho^2/\sin^2 \Omega + \rho^2 d\phi$, so $d^2\sigma_1 = [\rho_1/\sin \Omega]d\rho_1 d\phi_1$ and analogously for 2. Given the symmetry of the problem, we can set $\phi_2 = 0$ everywhere and multiply the remainder integrals by an overall $2\pi$. 
EE across a cone in $d = 4$

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EE across a cone in $d = 4$

Parametrize cone in cylindrical coordinates by $z = \rho / \tan \Omega$. Induced metric on cone: $ds_h^2 = d\rho^2 / \sin^2 \Omega + \rho^2 d\phi$, so $d^2 \sigma_1 = [\rho_1 / \sin \Omega] d\rho_1 d\phi_1$ and analogously for 2. Given the symmetry of the problem, we can set $\phi_2 = 0$ everywhere and multiply the remainder integrals by an overall $2\pi$. The unit normal vector to the cone surface is given $\vec{n} = \vec{u}_\rho \cos \Omega - \vec{u}_z \sin \Omega$, where $\vec{u}_\rho = \cos \phi \vec{u}_x + \sin \phi \vec{u}_y$. Using this, it is straightforward to find $\vec{n}(x_1) \cdot \vec{n}(x_2) = \cos^2 \Omega \cos \phi_1 + \sin^2 \Omega$. Similarly, we find $|x_1 - x_2|^4 = \left[ \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \phi_1 + (\rho_1 - \rho_2)^2 / \tan^2 \Omega \right]^2$. Then, we are left with the integrals

$$S_{\text{EE}}^{\text{EMI}} = \frac{2\pi \kappa(4)}{\sin^2 \Omega} \int \rho_1 d\rho_1 \int \rho_2 d\rho_2 \int_0^{2\pi} d\phi_1 \frac{[\cos^2 \Omega \cos \phi_1 + \sin^2 \Omega]}{[a - b \cos \phi_1]^2},$$

where $a \equiv \rho_1^2 + \rho_2^2 + (\rho_1 - \rho_2)^2 / \tan^2 \Omega$, $b \equiv 2\rho_1\rho_2$. 
EE across a cone in $d = 4$

Performing the angular integrals, we are left with

$$S_{EE}^{\text{EMI}} = \frac{4\pi^2 \kappa(4)}{\sin^2 \Omega} \left[ \cos^2 \Omega \ s_I + \sin^2 \Omega \ s_{II} \right],$$

where

$$s_I = \int \int \frac{b \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2, \quad s_{II} = \int \int \frac{a \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2.$$
EE across a cone in $d = 4$

Performing the angular integrals, we are left with

$$S^{EMI}_{EE} = \frac{4\pi^2 \kappa(4)}{\sin^2 \Omega} \left[ \cos^2 \Omega s_I + \sin^2 \Omega s_{II} \right],$$

where

$$s_I = \int \int \frac{b \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2, \quad s_{II} = \int \int \frac{a \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2.$$

Possible regulator:

$$\int d\rho_1 \rightarrow \int_{\delta}^{H} d\rho_1, \quad \int d\rho_2 \rightarrow \left[ \int_{0}^{\rho_1 - \delta} d\rho_2 + \int_{\rho_1 + \delta}^{\infty} d\rho_2 \right].$$
**EE across a cone in \( d = 4 \)**

Performing the angular integrals, we are left with

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S_{\text{EE}}^{\text{EMI}} = \frac{4\pi^2 \kappa(4)}{\sin^2 \Omega} \left[ \cos^2 \Omega s_I + \sin^2 \Omega s_{II} \right],
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where

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s_I = \int \int \frac{b \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2, \quad s_{II} = \int \int \frac{a \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2.
\]

Possible regulator:

\[
\int d\rho_1 \rightarrow \int_\delta^H d\rho_1, \quad \int d\rho_2 \rightarrow \left[ \int_0^{\rho_1 - \delta} d\rho_2 + \int_\rho_1^{\rho_1 + \delta} \right].
\]

Putting pieces together, we are left with

\[
S_{\text{EE}}^{\text{EMI}} \supset -\frac{3\pi^2 \kappa(4)}{8} \cdot \frac{\cos^2 \Omega}{\sin \Omega} \log^2 \left( \frac{H}{\delta} \right).
\]

Angular dependence is the expected one.
EE across a cone in $d = 4$

Performing the angular integrals, we are left with

$$S_{\text{EE}}^{\text{EMI}} = \frac{4\pi^2 \kappa(4)}{\sin^2 \Omega} \left[ \cos^2 \Omega s_I + \sin^2 \Omega s_{II} \right],$$

where

$$s_I = \int \int \frac{b \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2, \quad s_{II} = \int \int \frac{a \rho_1 \rho_2}{(a^2 - b^2)^{3/2}} d\rho_1 d\rho_2.$$

Possible regulator:

$$\int d\rho_1 \rightarrow \int_{\delta}^{H} d\rho_1, \quad \int d\rho_2 \rightarrow \left[ \int_{0}^{\rho_1 - \delta} d\rho_2 + \int_{\rho_1 + \delta}^{\infty} d\rho_2 \right].$$

Putting pieces together, we are left with

$$S_{\text{EE}}^{\text{EMI}} \supset -\frac{3\pi^2 \kappa(4)}{8} \cdot \frac{\cos^2 \Omega}{\sin \Omega} \log^2 (H/\delta).$$

Angular dependence is the expected one. Also, $c_{\text{EMI}} = 3\pi^2 \kappa(4)/2$, which matches the cylinder result.
**Bonus track: EE across a deformed sphere**

Parametrizing the deformed sphere as

\[ r(\Omega_{d-2}) = 1 + \varepsilon \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}} Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}), \]

where the \( Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}) \) are real (hyper)spherical harmonics, for general CFTs,
Parametrizing the deformed sphere as
\[ r(\Omega_{d-2}) = 1 + \varepsilon \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}} Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}), \]
where the \( Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}) \) are real (hyper)spherical harmonics, for general CFTs, the universal contribution \( s_{\text{univ}} \) takes the form
\[ s_{\text{univ}} = s_{0,\text{univ}} + \varepsilon^2 s_{2,\text{univ}} + O(\varepsilon^3), \]
where \( s_{0,\text{univ}} \) is the result for the round sphere in each case (e.g., \( s_{0,\text{univ}} = F \) in \( d = 3 \) and \( s_{0,\text{univ}} = 4a \) in \( d = 4 \)).
The “extensive mutual information” model

Bonus track: EE across a deformed sphere

Parametrizing the deformed sphere as

\[ r(\Omega_{d-2}) = 1 + \varepsilon \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}} Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}), \]

where the \( Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}) \) are real (hyper)spherical harmonics, for general CFTs, the universal contribution \( s^{\text{univ}} \) takes the form

\[ s^{\text{univ}} = s_0^{\text{univ}} + \varepsilon^2 s_2^{\text{univ}} + \mathcal{O}(\varepsilon^3), \]

where \( s_0^{\text{univ}} \) is the result for the round sphere in each case (e.g., \( s_0^{\text{univ}} = F \) in \( d = 3 \) and \( s_0^{\text{univ}} = 4a \) in \( d = 4 \)) and

\[ s_2^{\text{univ}} = C_T \frac{\pi^{d+2}}{2^{d-2}\Gamma(d+2)\Gamma(d/2)} \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}}^2 \frac{\Gamma(d+\ell-1)}{\Gamma(\ell-1)} \times \left\{ \begin{array}{ll} \frac{\pi}{2} & (d \text{ odd}) \\ 1 & (d \text{ even}) \end{array} \right. . \]

Sphere is a local extremum of the EE. Leading correction controlled by stress-tensor two-point function charge \( C_T \).
Bonus track: EE across a deformed sphere

Parametrizing the deformed sphere as

\[ r(\Omega_{d-2}) = 1 + \varepsilon \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}} Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}), \]

where the \( Y_{\ell,m_1,\ldots,m_{d-3}}(\Omega_{d-2}) \) are real (hyper)spherical harmonics, for general CFTs, the universal contribution \( s^{\text{univ}} \) takes the form

\[ s^{\text{univ}} = s_0^{\text{univ}} + \varepsilon^2 s_2^{\text{univ}} + O(\varepsilon^3), \]

where \( s_0^{\text{univ}} \) is the result for the round sphere in each case (e.g., \( s_0^{\text{univ}} = F \) in \( d = 3 \) and \( s_0^{\text{univ}} = 4a \) in \( d = 4 \)) and

\[ s_2^{\text{univ}} = C_T \frac{\pi^{d+2}}{2^{d-2} \Gamma(d+2) \Gamma(d/2)} \sum_{\ell,m_1,\ldots,m_{d-3}} a_{\ell,m_1,\ldots,m_{d-3}}^2 \frac{\Gamma(d + \ell - 1)}{\Gamma(\ell - 1)} \times \left\{ \begin{array}{ll} \pi/2 & (d \text{ odd}) \\ 1 & (d \text{ even}) \end{array} \right. . \]

Sphere is a local extremum of the EE. Leading correction controlled by stress-tensor two-point function charge \( C_T \).

No one has verified this result in the EMI...