BLACK HOLE THERMODYNAMICS

LECTURE 2

PLAN

- Komar Integrals
- Killing Horizons
- Smarr Formula (?)

Komar Integrals

First attempt to define a conserved quantity.

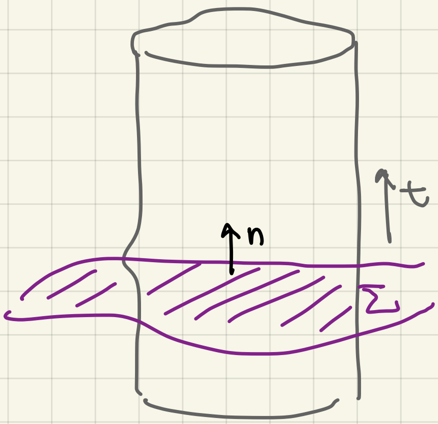
To understand & motivate let's first look at Maxwell

$$\nabla_\nu F^{\mu\nu} = 4\pi J_e^\mu \qquad \left(\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\varepsilon_0} \right)$$

$$\Downarrow$$

$$\nabla_\mu J^\mu_e = 0 \quad \text{conserved current}$$

Electric charge is then defined as.



$$Q = -\int_\Sigma d^3x \sqrt{h}\; n_\mu J^\mu_e$$

h : induced metric on Σ
n : normal vector
$n^2 = -1$

$$= -\frac{1}{4\pi}\int_\Sigma d^3x \sqrt{h}\; n_\mu \nabla_\nu F^{\mu\nu}$$

Stokes thm on curve manifolds

$$= -\frac{1}{4\pi}\int_{\partial\Sigma} \underbrace{d^2x \sqrt{\sigma}}_{d\theta\, d\phi\, \sin^2\theta}\; n_\mu \underbrace{q_\nu}_{q=\partial r} F^{\mu\nu}$$

$$\int \vec{\nabla}\cdot\vec{E}\, dV = \oint \vec{E}\cdot d\vec{A}$$

$\partial\Sigma$: bndy of Σ

σ : ind metric on $\partial\Sigma$
q : normal to $\partial\Sigma$

Conservation: $\quad Q(\Sigma_1) = Q(\Sigma_2)$ indep of time!

Let's try to implement this in GR. The e.o.m

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi\, T_{\mu\nu} \qquad \nabla_\mu T^{\mu\nu} = 0$$
$$T_{\mu\nu} = T_{\nu\mu}$$

Looks similar to maxwell, we just need to build a current. Consider

$$J^\mu = \xi_\nu T^{\mu\nu} \qquad \xi: \text{killing vector}$$

Check conservation:

$$\Rightarrow \nabla_\mu J^\mu = \nabla_\mu (\xi_\nu T^{\mu\nu}) = \nabla_\mu \xi_\nu\, T^{\mu\nu} + \xi_\nu \underbrace{\nabla_\mu T^{\mu\nu}}_{=0}$$

$$= \frac{1}{2} \underbrace{(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)}_{=0 \ \text{K.V}} T^{\mu\nu}$$

$$\Rightarrow \nabla_\mu J^\mu = 0$$

Define a conserved quantity

$$E = \int_\Sigma d^3x \sqrt{h}\, n_\mu J^\mu$$
$$= \int_\Sigma d^3x \sqrt{h}\, n_\mu \xi_\nu T^{\mu\nu}$$

But if $T^{\mu\nu} = 0$ then $E = 0$!! Boring!

Second try: Consider instead

$$g^\mu = \xi_\nu R^{\mu\nu} \qquad \xi: \text{killing vector}$$

Check that it is conserved

$$\nabla_\mu g^\mu = \nabla_\mu (\xi_\nu R^{\mu\nu}) \overset{!}{\underset{(*)}{=}} 0$$

To show (*) use : $\qquad [\nabla_\mu, \nabla_\nu] \xi^\lambda = R^\lambda{}_{\alpha\mu\nu} \xi^\alpha$

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

$$\Rightarrow \quad R_{\mu\nu} \xi^\mu = \nabla_\mu \nabla_\nu \xi^\mu \quad (*)$$

$$\Rightarrow \quad \nabla^\nu (R_{\mu\nu} \xi^\mu) = \nabla^\nu \nabla_\mu \nabla_\nu \xi^\mu \overset{!}{=} 0$$

From (*) we see that

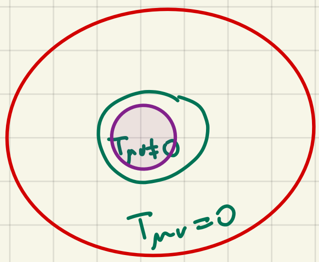
$$J^\mu = \xi_\nu R^{\mu\nu} = \nabla_\nu \nabla^\mu \xi^\nu$$

Furthermore we can define

$$K_{\mu\nu} = \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$$

$$J^\mu = \frac{1}{2} \nabla_\mu K^{\mu\nu} \quad \rightsquigarrow \text{resembles} \quad \nabla_\mu F^{\mu\nu} = 4\pi J_e^\nu$$

Next, define the conserved charge



$$Q(\xi) = -\frac{1}{4\pi G} \int_\Sigma d^3x \sqrt{h}\, n_\mu J^\mu$$

$$= -\frac{1}{4\pi G} \int_\Sigma d^3x \sqrt{h}\, n_\mu \nabla_\nu (\nabla^\mu \xi^\nu)$$

$$= -\frac{1}{4\pi G} \int_{\partial\Sigma} d^2x \sqrt{\sigma}\, n_\mu q_\nu \nabla^\mu \xi^\nu \quad \Rightarrow \text{KOMAR INTEGRAL}$$

Homework: "Easy"

Schw metric $\quad \xi = \partial_t$, Evaluate $Q(\partial_t)$

Σ: $t = \text{constant} \qquad n^t = -\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad n^2 = -1$

$\partial\Sigma$: $r = \text{constant} \qquad q^r = \left(1 - \frac{2M}{r}\right)^{1/2}, \quad q^2 = 1$

$$ds_{\partial\Sigma}^2 = r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$Q(\partial_t) = M$$

Homework: "difficult"

For Kerr, $k = \partial_t$ $m = \partial\phi$, check that.

$$Q(\partial\phi) = -2Ma = -2J$$

$$J = \underbrace{-\frac{1}{2}}_{\hookrightarrow \text{ adjusted to match class. limit}} Q(\partial\phi) \quad : \text{Angular Momentum}$$

$$Q(\partial_t) = M$$

Useful Notation

$$Q(\xi) = -\frac{1}{8\pi G}\int_{\partial\Sigma} dS_{\alpha\beta}\, \nabla^\alpha \xi^\beta$$

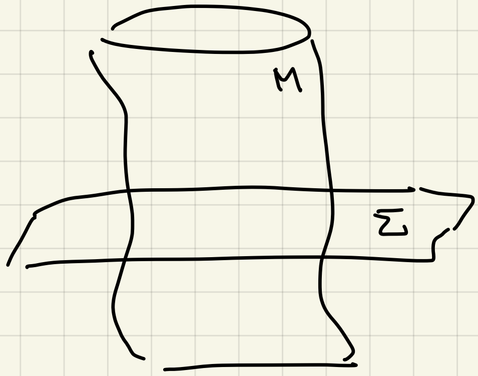
$$dS_{\alpha\beta} = 2\, n_{[\alpha} q_{\beta]} \sqrt{\sigma}\, d^2x$$

$$n\cdot q = 0 \qquad n^2 = -1 \quad q^2 = 1$$

Killing Horizon

$\Rightarrow$ Hypersurface:

$$\Sigma: \quad S(x) = \text{constant}$$



normal vector

$$n^\mu = f(x)\, g^{\mu\nu} \nabla_\nu S$$

we select $f(x)$ s.t. it is normalized appropriately

$n^2 = -1 \longrightarrow$ spacelike Σ

$n^2 = 1 \longrightarrow$ timelike Σ

$n^2 = 0 \longrightarrow$ null Σ