Komar Integrals

First attempt to define a conserved quantity. To understand & motivate let's first look at Maxwell:

\[ \nabla_\nu F^{\mu\nu} = 4\pi J^\mu \quad \left( \nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \right) \]

\[ \nabla_\nu J^\mu = 0 \quad \text{conserved current} \]

Electric charge is then defined as:

\[ Q = - \oint_{\Sigma} d^3x \sqrt{h^n} \eta \mu J^\mu \]

\[ \eta : \text{induced metric on } \Sigma \]
\[ \eta : \text{normal vector} \]
\[ \eta^2 = -1 \]

\[ = - \frac{1}{4\pi} \oint_{\Sigma} d^3x \sqrt{h^n} \eta \mu \nabla_\nu F^{\mu\nu} \]

Stokes theorem on current manifolds

\[ \oint_{\Sigma} \nabla \cdot \vec{E} \, dV = \oint_{\partial \Sigma} \vec{E} \cdot d\vec{n} \]

\[ \partial \Sigma: \text{body of } \Sigma \]
\[ \eta: \text{normal to } \partial \Sigma \]
Conservation: \( Q(\Sigma_1) = Q(\Sigma_2) \) indep of time!

Let's try to implement this in GR. The e.o.m

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad \nabla_\mu T^{\mu\nu} = 0
\]

\[ T_{\mu\nu} = T_{\nu\mu} \]

Looks similar to Maxwell, we just need to build a current. Consider

\[
J^\mu = \beta_\nu T^{\mu\nu}
\]

\( \beta \): Killing vector

Check conservation:

\[
\Rightarrow \nabla_\mu J^\mu = \nabla_\mu (\beta_\nu T^{\mu\nu}) = \nabla_\mu \beta_\nu T^{\mu\nu} + \beta_\nu \nabla_\mu T^{\mu\nu}
\]

\[
= \frac{1}{2} (\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu) T^{\mu\nu}
\]

\[
= 0 \quad \text{K.V.}
\]

Define a conserved quantity

\[
E = \int d^3x \sqrt{h} \eta_\mu J^\mu
\]

\[
= \int d^3x \sqrt{h} \eta_\mu \beta_\nu T^{\mu\nu}
\]

But if \( T^{\mu\nu} = 0 \) then \( E = 0!! \) Boring!

Second try: Consider instead

\[
J^\mu = \beta_\nu R^{\mu\nu}
\]

\( \beta \): Killing vector

Check that it is conserved

\[
\nabla_\mu J^\mu = \nabla_\mu (\beta_\nu R^{\mu\nu}) = 0
\]

(?)
To show (*) use: 
\[ \{\nabla_\mu, \nabla_\nu\} \tilde{q}^\lambda = R^\lambda \alpha_\nu \tilde{q}^\alpha \]
\[ \nabla_\mu \tilde{q} + \nabla_\nu \tilde{q}_\lambda = 0 \]

\[ \Rightarrow R_{\mu \nu} \tilde{q}^\lambda = \nabla_\mu \nabla_\nu \tilde{q}^\lambda \]  (*)

\[ \Rightarrow \nabla_\nu (R_{\mu \nu} \tilde{q}^\lambda) = \nabla_\nu \nabla_\mu \nabla_\nu \tilde{q}^\lambda = 0 \]

From (*) we see that

\[ \tilde{q}^\mu = \tilde{q}_\nu \rightleftharpoons R_{\mu \nu} = \nabla_\nu \nabla_\mu \tilde{q}_\nu \]

Furthermore we can define

\[ K_{\mu \nu} = \nabla_\mu \tilde{q}_\nu - \nabla_\nu \tilde{q}_\mu \]

\[ \tilde{q}^\mu = \frac{1}{2} \nabla_\mu \left( K_{\mu \nu} \tilde{q}^\nu \right) \sim \text{ resembles } \nabla_\mu T^\mu_{\nu} = 4\pi \tilde{J}_\nu \]

Next, define the conserved charge

\[ Q(\tilde{q}) = - \frac{1}{4\pi G} \int_\Sigma d^3x \sqrt{\eta} \eta_{\mu \nu} \tilde{q}^\mu \]

\[ = - \frac{1}{4\pi G} \int_\Sigma d^3x \sqrt{\eta} \eta_{\mu \nu} \nabla_\nu (\nabla_\mu \tilde{q}^\nu) \]

\[ = - \frac{1}{4\pi G} \int_\Sigma d^3x \sqrt{\omega} \eta_{\mu \nu} \tilde{q}_\mu \nabla_\nu \tilde{q}^\nu \rightleftharpoons \text{ KOMAR INTEGRAL} \]

**Homework: "Easy"**

Schw metric \( \tilde{q} = 2\pi \) , Evaluate \( Q(\tilde{q}) \)

\[ \Sigma: t = \text{constant} \quad n^t = - \left( 1 - \frac{2M}{r} \right)^{-\frac{1}{2}} \quad n^2 = -1 \]

\[ \partial \Sigma: r = \text{constant} \quad q^r = \left( 1 - \frac{2M}{r} \right)^{\frac{1}{2}} \quad q^2 = 1 \]

\[ ds^2_{\partial \Sigma} = r^2 (dt^2 + \sin^2 \theta d\phi^2) \]
For Kerr, $k = \partial_t$, $m = \partial \phi$, check that:

$$Q(\partial \phi) = -2Ma = -2J$$

$$J = -\frac{1}{2}Q(\partial \phi) : \text{Angular Momentum}$$

$$\Rightarrow \text{adjusted to match class. limit}$$

$$Q(\partial_t) = M$$

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**Useful Notation**

$$Q(\gamma) = -\frac{1}{8\pi G} \int_{\Sigma} dS_{\alpha \rho} \nabla^{\alpha} \gamma^{\rho}$$

$$dS_{\alpha \rho} = 2n_{\alpha} q_{\rho} \sqrt{\sigma} d^2x$$

$$n \cdot q = 0 \quad n^2 = -1 \quad q^2 = 1$$

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**Killing Horizon**

⇒ Hyper surface:

$$\Sigma : \mathcal{S}(x) = \text{constant}$$

normal vector

$$n^{\mu} = f(x) g^{\mu \nu} \nabla_{\nu} \mathcal{S}$$

we select $f(x)$ s.t. it is normalized appropriately.
\( n^2 = -1 \quad \rightarrow \quad \text{spacelike } \Sigma \)

\( n^2 = 1 \quad \rightarrow \quad \text{timelike } \Sigma \)

\( n^2 = 0 \quad \rightarrow \quad \text{null } \Sigma \)