Killing Horizon

Hyper surface:

\[ \Sigma: \quad S(x) = \text{constant} \]

Normal vector

\[ n^\mu = f(x) \ g^{\mu\nu} \nabla_\nu S \]

We select \( f(x) \) s.t. it is normalized appropriately.

\[ n^2 = -1 \quad \rightarrow \quad \text{spacelike } \Sigma \]

\[ n^2 = 1 \quad \rightarrow \quad \text{timelike } \Sigma \]

\[ n^2 = 0 \quad \rightarrow \quad \text{null } \Sigma \]

Focus on null case, \( n^2 = 0 \), which have many interesting properties:

1) It is a normal \( \Sigma \) & tangent

2) \( g^{\alpha\beta} = -n^\alpha n^\beta - N^\alpha N^\beta + \epsilon^{AB} e^A_\alpha e^B_\beta \)

Where \( N^\alpha \) is an aux. vector

\[ N_\alpha n^\alpha = -1, \quad N^2 = 0, \quad N_\alpha e^\alpha_\beta = 0 \quad \text{on } \Sigma \]

\[ dS_{\mu\nu} = 2 \ n_\alpha N_\beta \sqrt{g} \ d^3x \]
Definition: a null hypersurface $\Sigma$ is a killing horizon of a killing vector $X$ if $X$ is normal to $\Sigma$ on $\Sigma$.

Why do we care?

1) Event Horizon for stationary BH is killing Horizon.

E.g. Schwarzschild

$$k = \nabla_t \Rightarrow k^2 \bigg|_{\nabla^t} = 0$$

Killing (r=2m)

Note: a killing horizon is not nec. an event horizon.

Homework

Consider:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Boost in the $x$-direction are generated by

$$X = x \nabla_t + t \nabla_x$$

Consider the null hypersurface $\Sigma_{t \pm}: x \pm t = 0$

Show that $\Sigma_{t \pm}$ is a killing horizon.

To every killing horizon we can associate a quantity called surface gravity. The definition is

$$K^\nu \nabla_\nu X^\nu = K \cdot X^\nu \quad \text{on } \Sigma$$

$K$ is constant on $\Sigma$.

Recall 0th law of BH mechanics.
To show it (Homework)

1) Show that \[ \kappa [\mu \nabla_\nu X_\sigma] = 0 \] on \( \Sigma \) (Frobenius thm)

2) Show that
\[
\kappa^2 = -\frac{1}{2} (\nabla_\mu X_\nu) (\nabla^\mu X^\nu) \bigg|_\Sigma
\]

3) Show that
\[ \kappa^\mu \nabla_\mu (\kappa^2) = 0 \] on \( \Sigma \)

4) For Bifurcate horizon, show
\[ \kappa^\mu \nabla_\mu (\kappa^2) = 0 \] on \( \Sigma \)

\[ \bullet \text{: Bifurcate horizon } \bullet \text{ at } X = 0 \]

**Note:** \( \kappa \) depends on the normalization of \( X \)

E.g. \( X = a r_t \) convention is that \( a \) is picked s.t.

\[
X^2 \rightarrow -1 \quad \text{for asymptotic flat}
\]

\[
X^2 \rightarrow -r^2 \quad \text{for asymptotic AdS.}
\]

**Homework:**

1) Evaluate \( \kappa \) for Schw (easy)
   \[ X = \vartheta_t \rightarrow \kappa = \frac{1}{4M} \]

2) Evaluate \( \kappa \) for Kerr (difficult)
   \[ X = \vartheta_t + \sqrt{2} a \varphi \]
   with \( \sqrt{2} a = \frac{\alpha}{\sqrt{t^2 + a^2}} \)
   Show that \[ \kappa = \frac{r_t - M}{\sqrt{r_t^2 + a^2}} \]
We will assume (for simplicity)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \]
\[ \rightarrow R = 0 \]
\[ \rightarrow R_{\mu\nu} = 0 \]

Consider mass as a Komar integral

\[
M = -\frac{1}{8\pi} \oint_{S^2} \nabla^* k^0 \, dS_{\alpha\rho}
\]

\[ \text{Stokes} \]
\[ = -\frac{1}{4\pi} \oint_{\Sigma} \nabla_\rho (\nabla^* k^\rho) \, dS_{\alpha} - \frac{1}{8\pi} \oint_{\Sigma} \nabla^* k^0 \, dS_{\alpha\rho}
\]

\[ \text{Einstein} \]
\[ = -\frac{1}{4\pi} \oint_{\Sigma} R^{\alpha\rho} k_\rho \, dS_{\alpha} - \frac{1}{8\pi} \oint_{\Sigma} \nabla^* k^0 \, dS_{\alpha\rho}
\]

Consider angular momentum as a Komar integral.

\[
J = \frac{1}{16\pi} \oint_{S^2} \nabla^* m^\alpha \, dS_{\alpha\rho}
\]

\[ m = \Theta \phi \]

\[ = \frac{1}{16\pi} \oint_{\Sigma} \nabla^* m^\alpha \, dS_{\alpha\rho}
\]

Now lets consider a combination s.t. \( X = K + 524 \) m appears in the integrand.

\[
M - 2524 J = -\frac{1}{8\pi} \oint_{\Sigma} \nabla^* X^\alpha \, dS_{\alpha\rho}
\]
\[
\delta M = \frac{\kappa}{8\pi} \delta A_+ + \Sigma_+ \delta J + \Phi_+ \delta Q
\]
Proof: \( M, J \) determine BH (uniqueness thm)

\[ M = M(A^H, J) \]

\[ M(l^2 A^H, l^2 J) = l M(A^H, J) \]

\[ \Delta^H + \frac{\partial M}{\partial A^H} + J \frac{\partial M}{\partial J} = \frac{1}{2} M \quad \text{(see thm below)} \]

\[ = \frac{\nu}{8\pi} A^H + 5\nu J \]

\( \implies \)

\[ SM = \frac{\nu}{8\pi} 8A^H + 5\nu 8J \]

\[ A^H = 4\pi (r^2 + a^2) \]

\[ J = Ma \]

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**Euler's Homogeneous Function Theorem**

Let \( f(x, y) \) be a homogeneous function of order \( n \)

\[ f(tx, ty) = t^n f(x, y) \]

Differentiate wrt \( t \)

\[ n t^{n-1} f(x, y) = x \frac{\partial f}{\partial (x)} + y \frac{\partial f}{\partial (y)} \]

Set \( t = 1 \)

\[ x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f \]