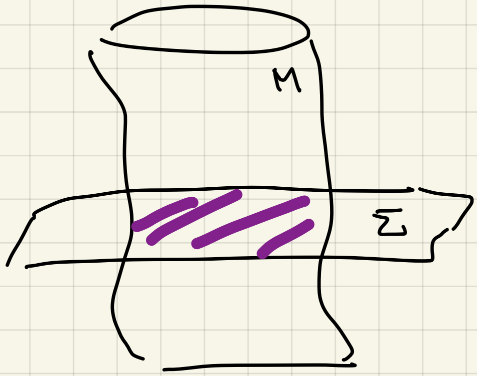
BLACK HOLE THERMODYNAMICS

LECTURE 3

Killing Horizon



⇒ Hypersurface:

$$\Sigma: \quad S(x) = \text{constant}$$

normal vector

$$n^\mu = f(x)\, g^{\mu\nu} \nabla_\nu S$$

we select $f(x)$ s.t. it is normalized appropriately.

$n^2 = -1 \longrightarrow$ spacelike Σ

$n^2 = 1 \longrightarrow$ timelike Σ

$n^2 = 0 \longrightarrow$ null Σ

Focus on null case, $n^2 = 0$, which have many interesting properties:

1) n it is a normal Σ & tangent

2) $$g^{\alpha\beta} = -n^\alpha N^\beta - n^\beta N^\alpha + \sigma^{AB} e^\alpha_A e^\beta_B$$

where N^β is an aux. vector

$$N_\mu n^\mu = -1 \qquad N^2 = 0 \qquad N_\alpha e^\alpha_A = 0 \qquad \text{on } \Sigma$$

$$dS_{\alpha\beta} = 2\, n_{[\alpha} N_{\beta]} \sqrt{\sigma}\, d^2x$$

Definition: a null hypersurface Σ is a killing horizon of a killing vector χ if χ is normal to Σ on Σ

Why do we care?

1) Event Horizon for stationary BH is killing Horizon.

E.g. Schw

$$\underbrace{K = \partial_t}_{\text{Killing}} \Rightarrow K^2\Big|_{\substack{\mathcal{H}^+ \\ (r=2m)}} = 0$$

Note: a killing horizon is not nec. an Event Horizon.

Homework

Consider $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Boost in the $x-$direction are generated by

$$\chi = x\,\partial_t + t\,\partial_x$$

Consider the null hypersurface $\Sigma_\pm : x \pm t = 0$

Show that $\Sigma_\pm$ is a killing horizon.

To every killing horizon we can associate a quantity called SURFACE GRAVITY. The definition is

$$\chi^\mu \nabla_\mu \chi^\nu \equiv \kappa\, \chi^\nu \quad \text{on } \Sigma$$

$\kappa \to$ surface gravity.

$\Rightarrow$ $\boxed{\kappa \text{ is constant on } \Sigma}$ Recall 0th law of BH mechanics.

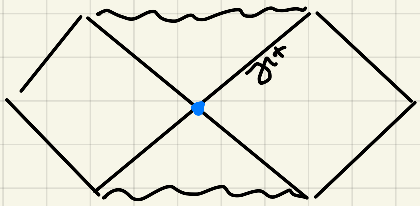
To show it (Homework)

1) Show that $\chi_{[\mu}\nabla_{\nu}\chi_{\sigma]} = 0$ on Σ (Frobenius thm)

2) Show that
$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\Big|_\Sigma$$

3) Show that
$$\chi^\mu \nabla_\mu (\kappa^2) = 0 \quad \text{on } \Sigma$$

4) For Bifurcate horizon, show
$$e^\mu_A \nabla_\mu (\kappa^2) = 0 \quad \text{on } \Sigma$$



• : Bifurcate horizon @ $\chi = 0$

Note: κ depends on the normalization of χ

E.g. $\chi = a\,\partial_t$ convention is that a is picked s.t.

$\chi^2 \xrightarrow[r\to\infty]{} -1$ for asymp flat

$\chi^2 \xrightarrow[r\to\infty]{} -r^2$ for asymp AdS.

Homework:

1) Evaluate κ for Schw (easy)

$\chi = \partial_t \quad \rightarrow \quad \kappa = \frac{1}{4M}$

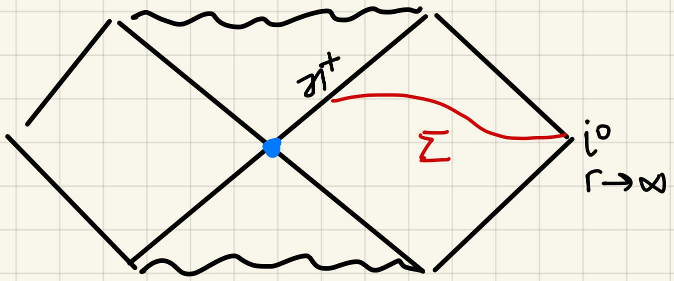
2) Evaluate κ for Kerr (difficult)

$\chi = \partial_t + \Omega_H \partial_\phi$

with $\Omega_H = \frac{a}{r_+^2 + a^2}$

Show that $\kappa = \frac{r_+ - M}{r_+^2 + a^2}$

SMARR LAW



We will assume (for simplicity)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad (T_{\mu\nu} = 0)$$

$\rightarrow R = 0$

$\rightarrow R_{\mu\nu} = 0$

Consider mass as a komar integral

$$M = -\frac{1}{8\pi} \oint_{S^2} \nabla^\alpha K^\beta \, dS_{\alpha\beta} \qquad K = \partial_t$$

stokes

$$= -\frac{1}{4\pi} \int_\Sigma \nabla_\beta (\nabla^\alpha K^\beta) \, dS_\alpha - \frac{1}{8\pi} \oint_{\mathcal{I}^+} \nabla^\alpha K^\beta \, dS_{\alpha\beta}$$

$$= -\frac{1}{4\pi} \int_\Sigma \underbrace{R^{\alpha\beta}}_{=0} K_\beta \, dS_\alpha - \frac{1}{8\pi} \oint_{\mathcal{I}^+} \nabla^\alpha K^\beta \, dS_{\alpha\beta}$$

Einstein eqn

$$= -\frac{1}{8\pi} \oint_{\mathcal{I}^+} \nabla^\alpha K^\beta \, dS_{\alpha\beta}$$

Consider angular momentum as a komar integral.

$$J = \frac{1}{16\pi} \oint_{S^2} \nabla^\alpha m^\beta \, dS_{\alpha\beta} \qquad m = \partial\phi$$

$$= \frac{1}{16\pi} \oint_{\mathcal{I}^+} \nabla^\alpha m^\beta \, dS_{\alpha\beta}$$

Now lets consider a combination s.t. $\chi = K + \Omega_H m$ appears in the integrand.

$$M - 2\Omega_H J = -\frac{1}{8\pi} \oint_{\mathcal{I}^+} \nabla^\alpha \chi^\beta \, dS_{\alpha\beta}$$

$$d S_{\alpha\beta} = 2\, \chi_{[\alpha} N_{\beta]} \sqrt{\sigma}\, d^2x$$

$$M - 2\Omega_H J = -\frac{1}{4\pi} \oint_{\mathcal{H}^+} \underbrace{\nabla^\alpha \chi^\beta \; \chi_\alpha}_{\chi^\alpha \nabla_\alpha \chi^\beta = \kappa \chi^\beta} N_\beta \sqrt{\sigma}\, d^2x$$

$$= -\frac{1}{4\pi} \oint_{\mathcal{H}^+} \kappa\, \underbrace{\chi^\beta N_\beta}_{\Rightarrow\ \chi \cdot N = -1} \sqrt{\sigma}\, d^2x$$

$$= \frac{1}{4\pi} \oint_{\mathcal{H}^+} \kappa \sqrt{\sigma}\, d^2x$$

$$= \frac{\kappa}{4\pi} \underbrace{\oint_{\mathcal{H}^+} \sqrt{\sigma}\, d^2x}_{A_H}$$

$$\Rightarrow \boxed{M = \frac{\kappa}{4\pi} A_H + 2\Omega_H J}$$ Smarr relation

" $+ \Phi_H Q$ " if electrically charged.

First law of BH mechanics :

$$\delta M = \frac{\kappa}{8\pi} \delta A_H + \Omega_H \delta J + \Phi_H \delta Q$$

" If a stationary black hole of mass M, charge Q and ang. mom J with future event horizon of surface gravity κ_+, angular veloc Ω_+, and electric potential Φ_+ is perturbed such that it settles down to another BH with mass $M + \delta M$, charge $Q + \delta Q$, and ang. mom. $J + \delta J$ then

$$\delta M = \frac{\kappa}{8\pi} \delta A_H + \Omega_H \delta J + \Phi_H \delta Q \text{ "}$$

$$\Phi_H = \chi^\mu A_\mu \Big|_{r_+}^{\infty}$$

Proof: Assume M, J determine BH (uniqueness thm)

$$M = M(A_H, J)$$

$$M(\ell^2 A_H, \ell^2 J) = \ell M(A_H, J)$$

$$A_H \frac{\partial M}{\partial A_H} + J \frac{\partial M}{\partial J} = \frac{1}{2} M \quad \text{(see thm below)}$$

$$= \frac{\kappa}{8\pi} A_H + \Omega_H J$$

$$\Rightarrow \frac{\partial M}{\partial A_H} = \frac{\kappa}{8\pi}, \quad \frac{\partial M}{\partial J} = \Omega_H$$

$$\Rightarrow \boxed{\delta M = \frac{\kappa}{8\pi} \delta A_H + \Omega_H \delta J}$$

$$A_H = 4\pi (r_+^2 + a^2)$$

$$J = Ma$$

Euler's Homogeneous funct theorem

Let $f(x,y)$ be a homogeneous function of order n

$$f(tx, ty) = t^n f(x,y)$$

Differentiate wrt t

$$n t^{n-1} f(x,y) = x \frac{\partial f}{\partial (tx)} + y \frac{\partial f}{\partial (ty)}$$

set $t = 1$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f$$