Workshop on New Trends in Dark Matter

Parametrization of dark matter self-interactions

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In collaboration with

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I. Introduction

We only see dark matter from the sky.

Reheating $\xrightarrow{\text{B-number}}$ Nucleosynthesis $\xrightarrow{\text{R-M equality}}$ CMB radiation

↓

Dark astrophysics $\xleftarrow{\Lambda\text{-dominated}}$ Dark ages ← Large structure

kpc - Mpc

What can we learn from the sky?

There are astrophysical **puzzles** at scales below cluster sizes. One example is:

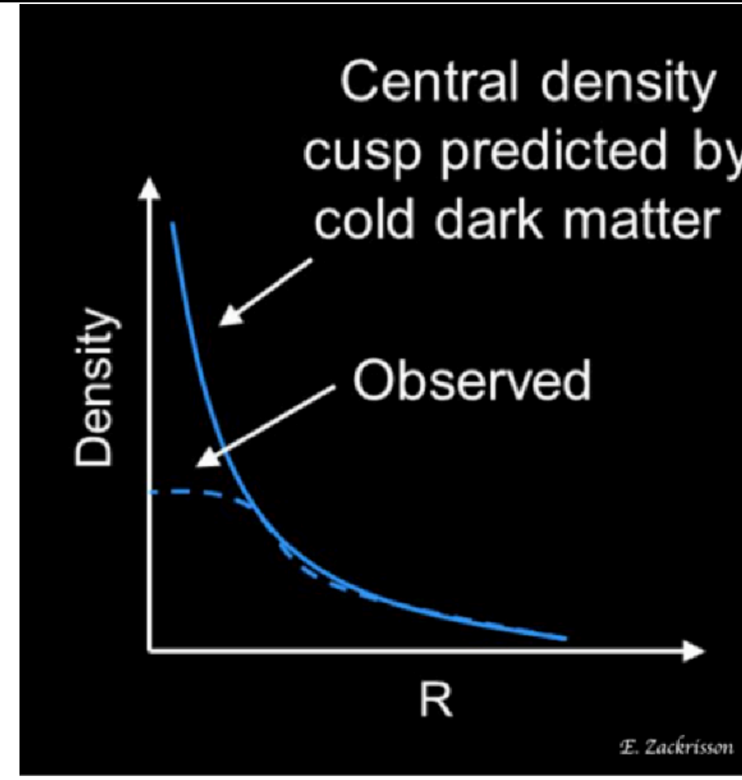
Mass deficit in small dark matter halos.

Self-interacting dark matter (SIDM)

Observational evidence for self-interacting cold dark matter

D.N. Spergel and P J. Steinhardt [astro-ph/9909386]

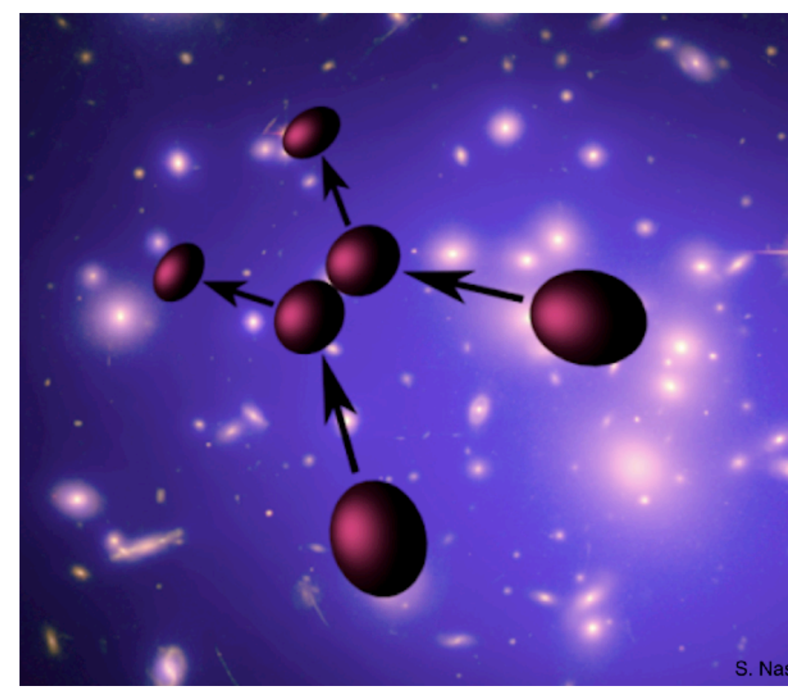
Infalling dark matter is scattered before reaching the center of the galaxy so that the orbit distribution is isotropic rather than radial. These collisions increase the entropy of the dark matter phase space distribution and lead to a dark matter halo profile with a shallower density profile.



- **Stronger self-scattering needed for (dwarf-sized) halos**

$$\frac{\sigma_{\rm SI}}{m_{\rm DM}} \sim 0.5 - 10 {\rm cm}^2/{\rm g}$$

at dwarf scales of DM velocity ~ 10 km/s



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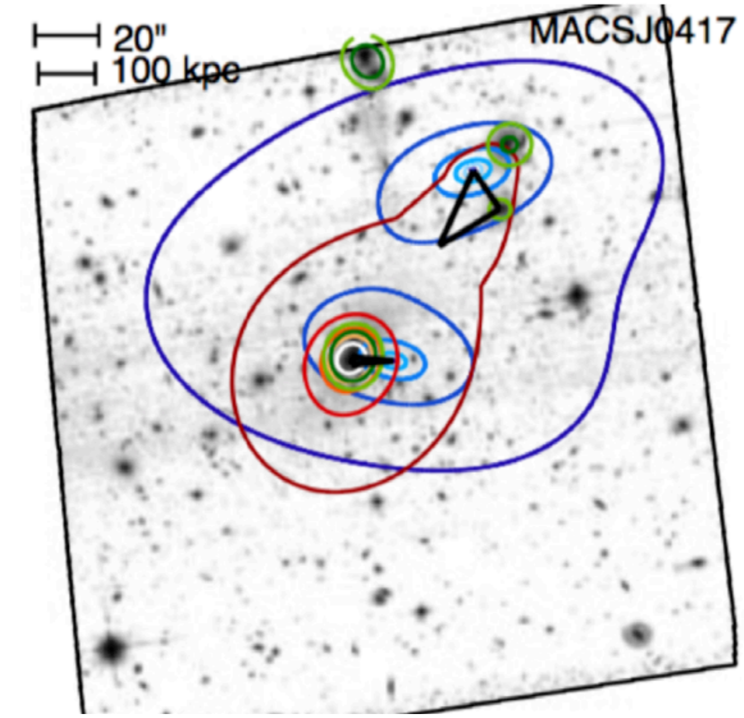
[D. Wittman et al. 2018]



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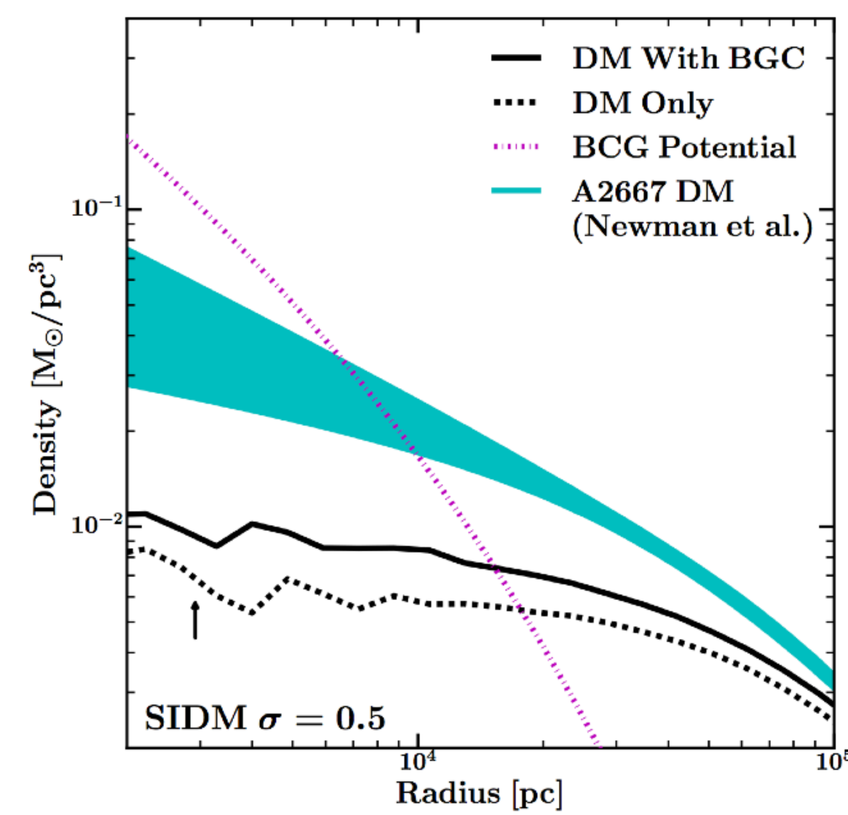
at dwarf scales of DM velocity ~ 10 km/s

- **Bounds on self-scattering from clusters/halo profiles.**

[S.W.Randall et al. 2008, M.Kaplinghat et al. 2015, K. Bondarenko et al. 2016,….]

$$\frac{\sigma_{\rm SI}}{m_{\rm DM}} \leq 0.2 - 1 {\rm cm}^2/{\rm g}$$

at cluster scales of DM velocity ~ 1000km/s



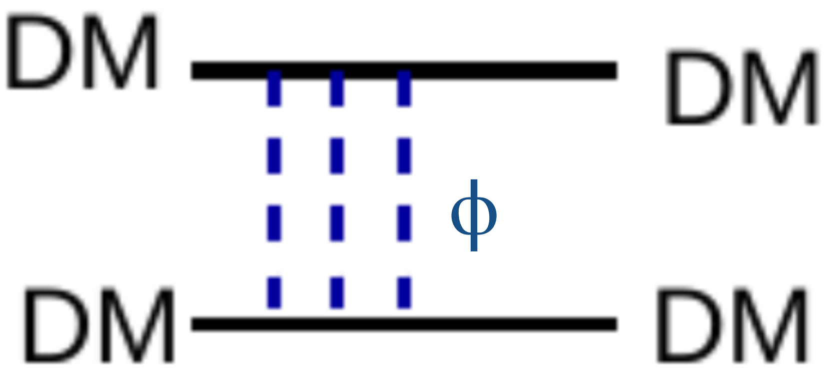
[O. D. Elbert et al. 2016]

II. Velocity-dependence

in

DM Self-interactions

SIDM via light mediators [D.N. Spergel&P. J. Steinhardt 1999, J. Feng, M. Kaplinghat&H.-B. Yu 2009, …]

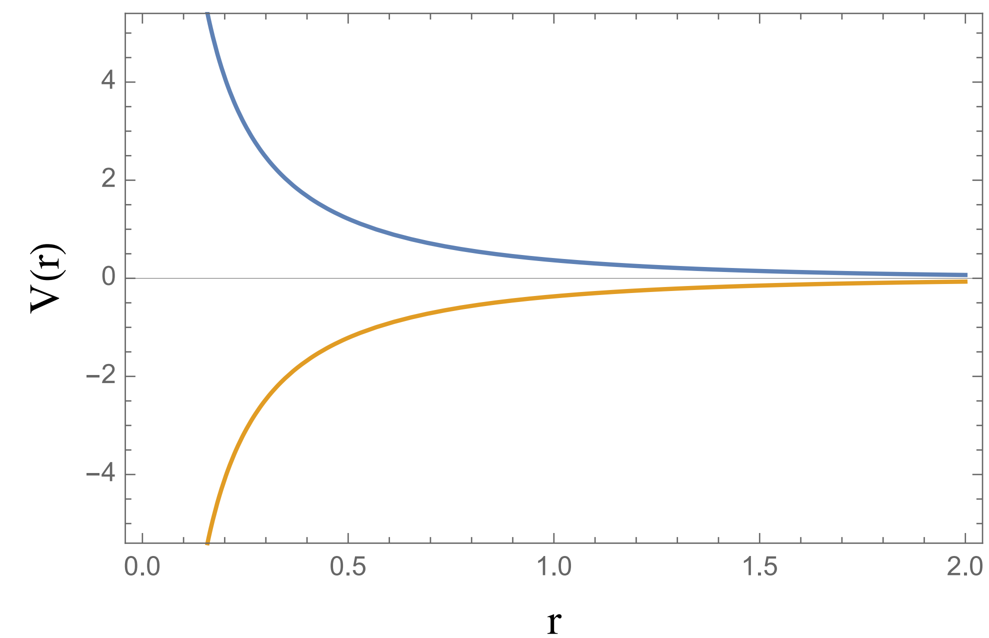


t/u-channel elastic scattering

a light ϕ: enhanced cross section at low velocities

Described by a **Yukawa** potential at non-relativistic limit:

$$V(r) = \pm \frac{\alpha_\chi}{r} e^{-m_\phi r}$$



Non-trivial calculations may be needed:

1. **Born** regime: $m_\phi \gg \alpha_\chi m_\chi$

   (1st-order Born approximation)

2. (Non-perturbative) **Resonances** (for attractive force): $m_\phi \leq \alpha_\chi m_\chi \;\&\; m_\phi \geq m_\chi v$

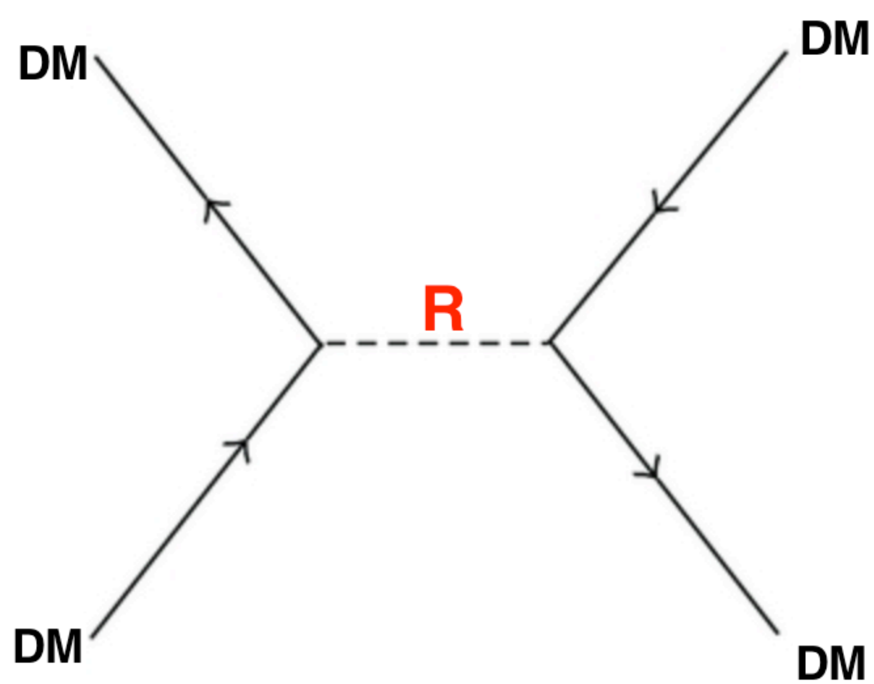
   (**bound state** formation)

3. (Non-perturbative) **Classical** regime: $m_\phi \leq \alpha_\chi m_\chi \;\&\; m_\phi \leq m_\chi v$

   potential range longer than **DM dB wavelength**

SIDM via a Breit-Wigner resonance [XC, C. Garcia-Cely, H. Murayama 2018]

Self-scattering via $\mathrm{DM} + \mathrm{DM} \to R^{(*)} \to \mathrm{DM} + \mathrm{DM}$



$$\text{If } v_R^2 \equiv \frac{m_R - 2m_{\mathrm{DM}}}{m_{\mathrm{DM}}} \ll 1$$

Large enhancement
&
velocity-dependence

$$\sigma \;=\; \sigma_0 + \frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v)^2/4}{(E(v)-E(v_R))^2 + \Gamma(v)^2/4}$$

averaged over velocity distribution

t/u - channel

$E(v) = \frac{1}{2}\frac{m}{2}v^2$ and $S = \frac{2J_R+1}{(2J_{\mathrm{DM}}+1)^2}$

$\Gamma(v) = m_R \gamma v^{2L+1}$

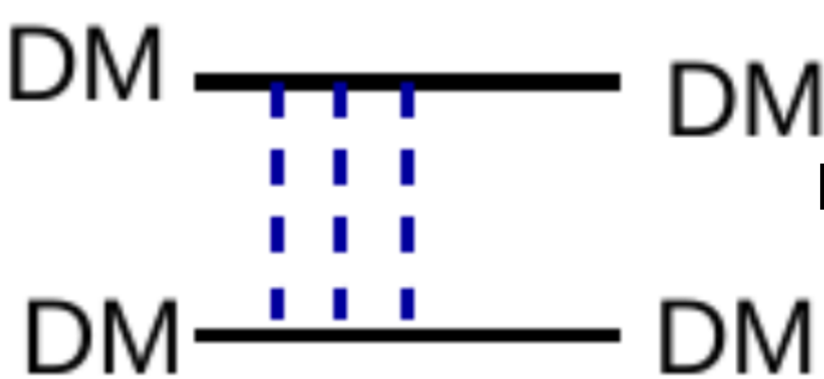
To generalize various interactions (via finite-range potentials)

Back to definition of **phase shift** from **quantum scattering theory**: [assuming real phase shift]

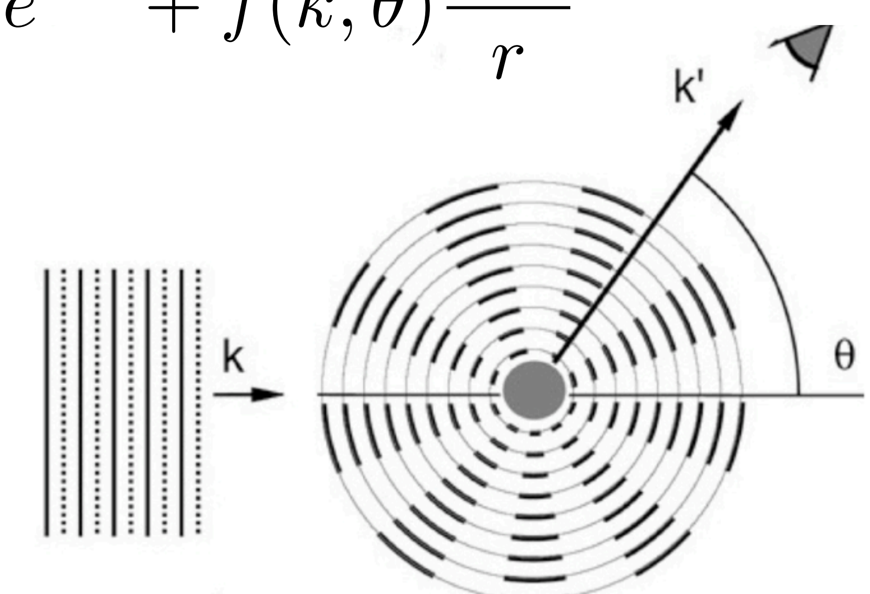
$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \qquad \text{with} \quad f(k,\theta) = \sum_{\ell=0}^{\infty}(2l+1) f_\ell(k) P_\ell(\cos\theta) \ ,$$

$$\text{with} \quad f_\ell(k) \equiv \frac{e^{2i\delta_\ell(k)}-1}{2ik} = \frac{1}{k\left(\cot\delta_\ell(k) - i\right)}$$

$$\psi(r) \simeq e^{i\mathbf{kr}} + f(k,\theta)\frac{e^{ikr}}{r}$$



Non-relativistic scattering



Effective-range theories (ERT) [Schwinger, Blatt & Jackson, Bethe, 1940s]

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$$\text{with} \quad f_\ell(k) \equiv \frac{e^{2i\delta_\ell(k)}-1}{2ik} = \frac{1}{k\,(\cot\delta_\ell(k)-i)}$$

Boundary conditions of **finite-range** potential suggest [Schwinger, Blatt & Jackson, Bethe, 1940s]

$$\lim_{k\to 0}\frac{\tan\delta_l}{k^{2l+1}} = const.$$

allowing an expansion of phase shift at low velocities of DM:

$$k^{2\ell+1}\cot\delta_\ell = -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2\,r_{e,\ell}^{2\ell-1}}k^2 + \mathcal{O}(k^4)$$

scattering length   effective range

And in general, s-wave parameters (ℓ=0) dominate at low velocities.

Useful parametrization of SIDM

A **s-wave (ℓ=0) dominated** cross section is parametrized by:

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1+k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

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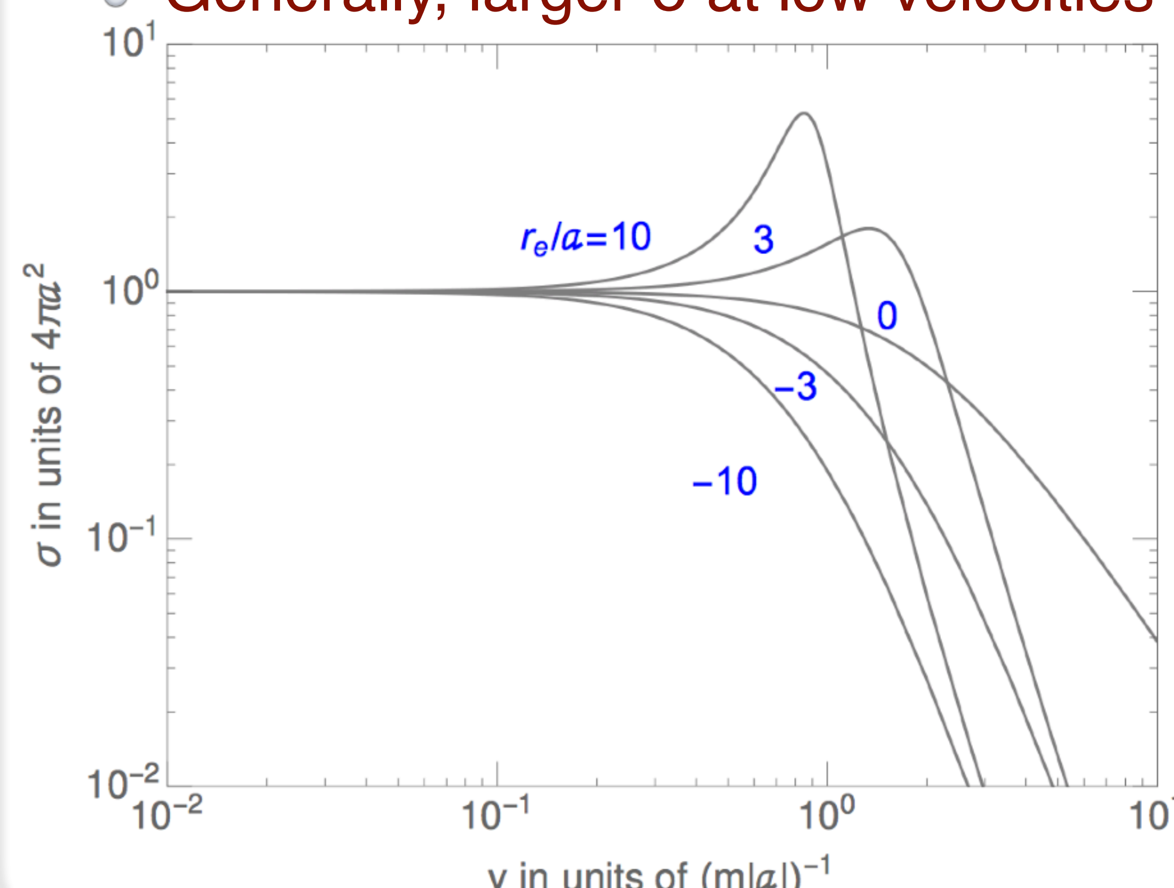
- For hard scattering $(1/k \gg |a| \gg |r_e|)$:

$$\sigma_0 \to 4\pi a^2 \quad \text{(contact interaction)}$$

- For very large scattering $(a \to \infty)$:

$$\sigma_0 \to \frac{4\pi}{k^2} \quad \text{(unitarity boundary)}$$

- Generally, larger σ at low velocities



III. To verify effective-range theories (ERT) in DM models

From DM models to s-wave ERT

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

DM Models with simple potentials:

$$V(r) = -\frac{\alpha}{r}e^{-m_\phi r}$$

1. **Contact interaction** (i.e. effective operator, quartic interaction).
2. **Born regime** for Yukawa potential: $a = -\frac{m\alpha}{m_\phi^2}\,, \quad \text{and} \quad r_e = \frac{4}{m\alpha}$
3.

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3. **Or more general** Yukawa potential (including bound-states, but not classical regimes):

Analytically: approximated with Hulthén potential

$$a = \frac{\psi^{(0)}(1+\eta) + \psi^{(0)}(1-\eta) + 2\gamma}{\delta},$$

$$\begin{aligned} r_e = \frac{2a}{3} - &\frac{1}{3\delta\eta\left[\psi^{(0)}(1+\eta) + \psi^{(0)}(1-\eta) + 2\gamma\right]^2} \\ &\times \left\{3\left[\psi^{(1)}(1+\eta) - \psi^{(1)}(1-\eta)\right]\right. \\ &\left.+ \eta\left[\psi^{(2)}(1+\eta) + \psi^{(2)}(1-\eta) + 16\zeta(3)\right]\right\} \end{aligned}$$

where $\eta = \sqrt{\alpha m/\delta}$, $\psi^{(n)}(z)$ are the polygamma functions of order n and $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant.

From DM models to s-wave ERT

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

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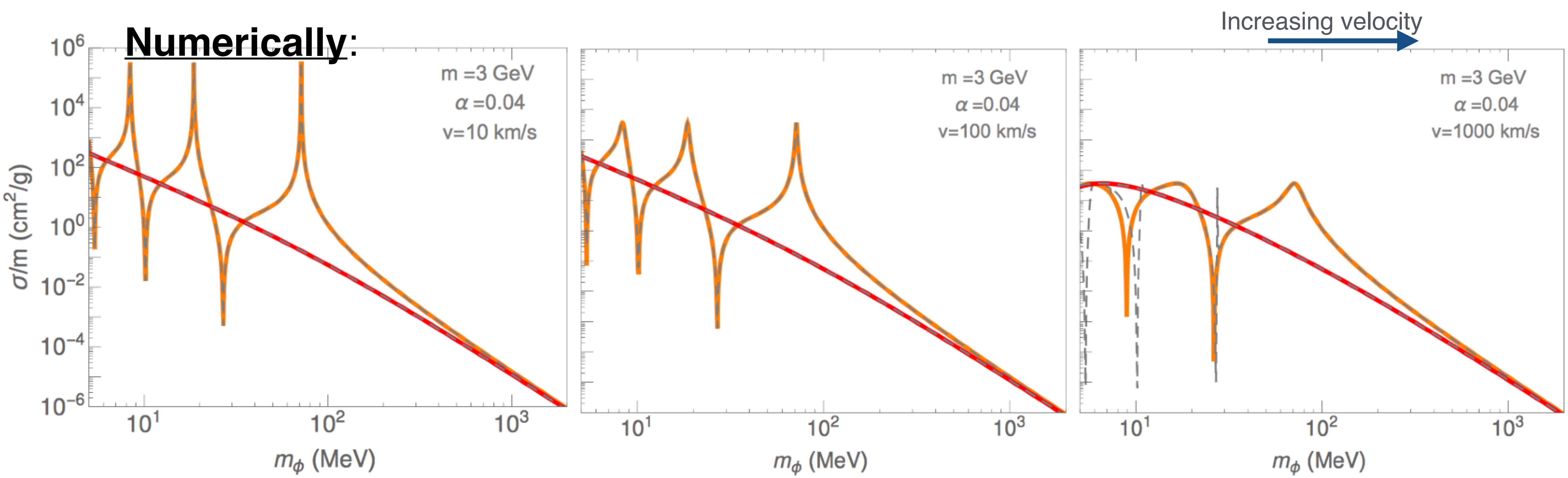
Numerically:



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$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

DM Models with simple potentials:

1. **Contact interaction** (i.e. effective operator, quartic interaction).
2. **Born regime** for Yukawa potential:
3. **Or more general** Yukawa potential (including bound-states, but not classical regimes):
4. Similarly, also describe **Breit-Wigner resonances**

One-to-One correspondence

$$\frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v)^2/4}{(E(v) - E(v_R))^2 + \Gamma(v)^2/4}$$

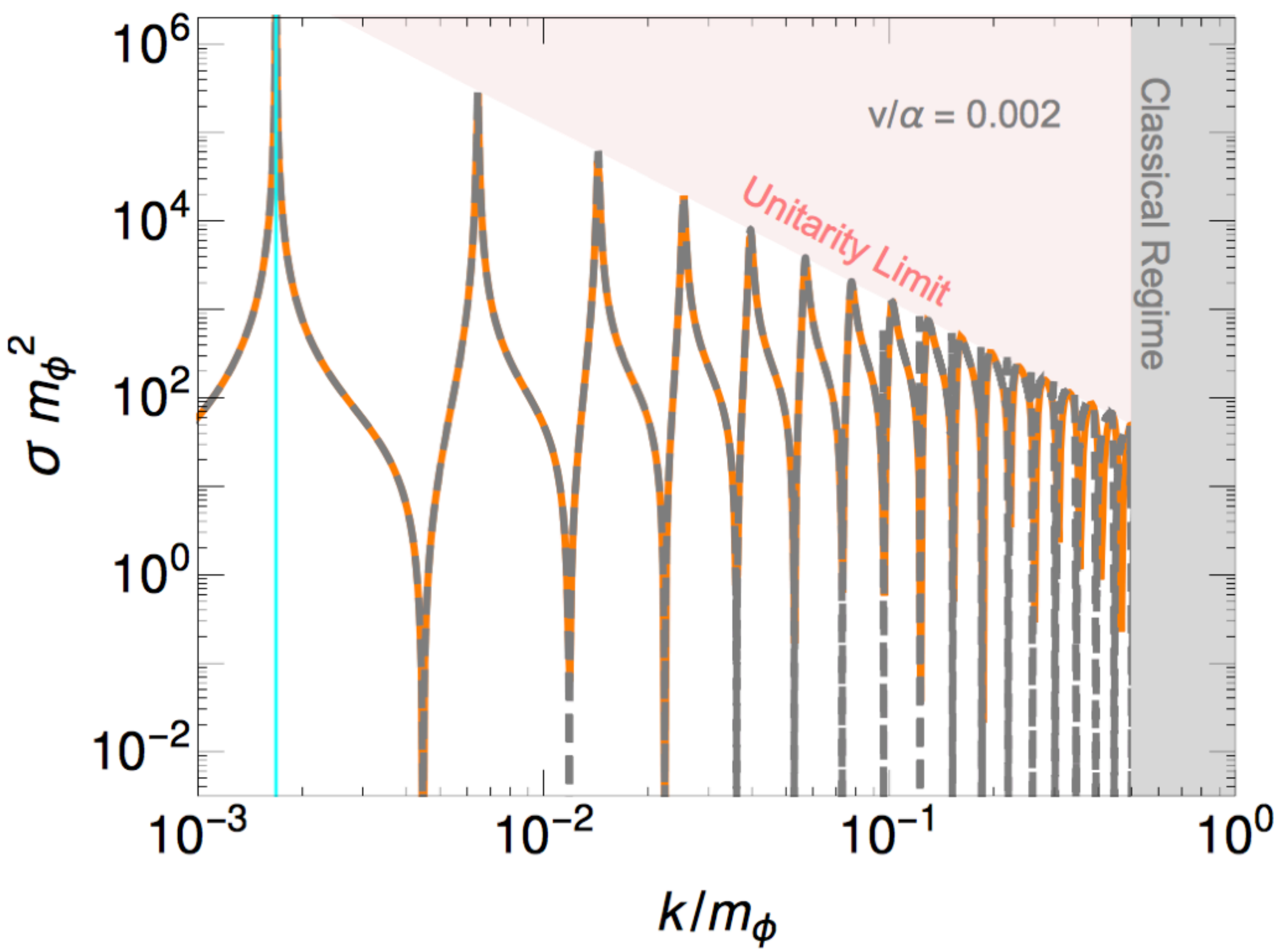
Generalizations

- **For long-range forces** (QED-like) → Modified ERT

  e.g. **classical regime** of Yukawa potentials:

$$m_\phi \leq \alpha_\chi m_\chi \;\&\; m_\phi \leq m_\chi v$$

$$V(r) = -\frac{\alpha}{r} e^{-m_\phi r}$$



Generalizations

- **For long-range forces** (QED-like) → Modified ERT

- **More complicated potentials**

e.g. **hard well/wall + Yukawa potential**

described by **constant piece + ERT piece**

$$e^{2i\delta_0(k)} = e^{2ikR} e^{2i\delta_a(k)} = e^{2ikR} \left(\frac{-\frac{1}{\mathfrak{a}} + \frac{1}{2}\mathfrak{r}_e k^2 + ik}{-\frac{1}{\mathfrak{a}} + \frac{1}{2}\mathfrak{r}_e k^2 - ik} \right),$$

which leads to the scattering amplitude

$$f_0(k) = \frac{e^{2i\delta_0(k)} - 1}{2ik} = \frac{e^{2ikR} - 1}{2ik} + \frac{e^{2ikR}}{-\frac{1}{\mathfrak{a}} + \frac{1}{2}\mathfrak{r}_e k^2 - ik}$$

(Also apply to anti-resonances)

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(Also apply to anti-resonances)

- **Sub-leading imaginary phase shift**

e.g. **with DM pair annihilations**

$$\delta_l = \mathrm{Re}\delta_l + \mathrm{Im}\delta_l$$

leading to **complex ERT parameters:**

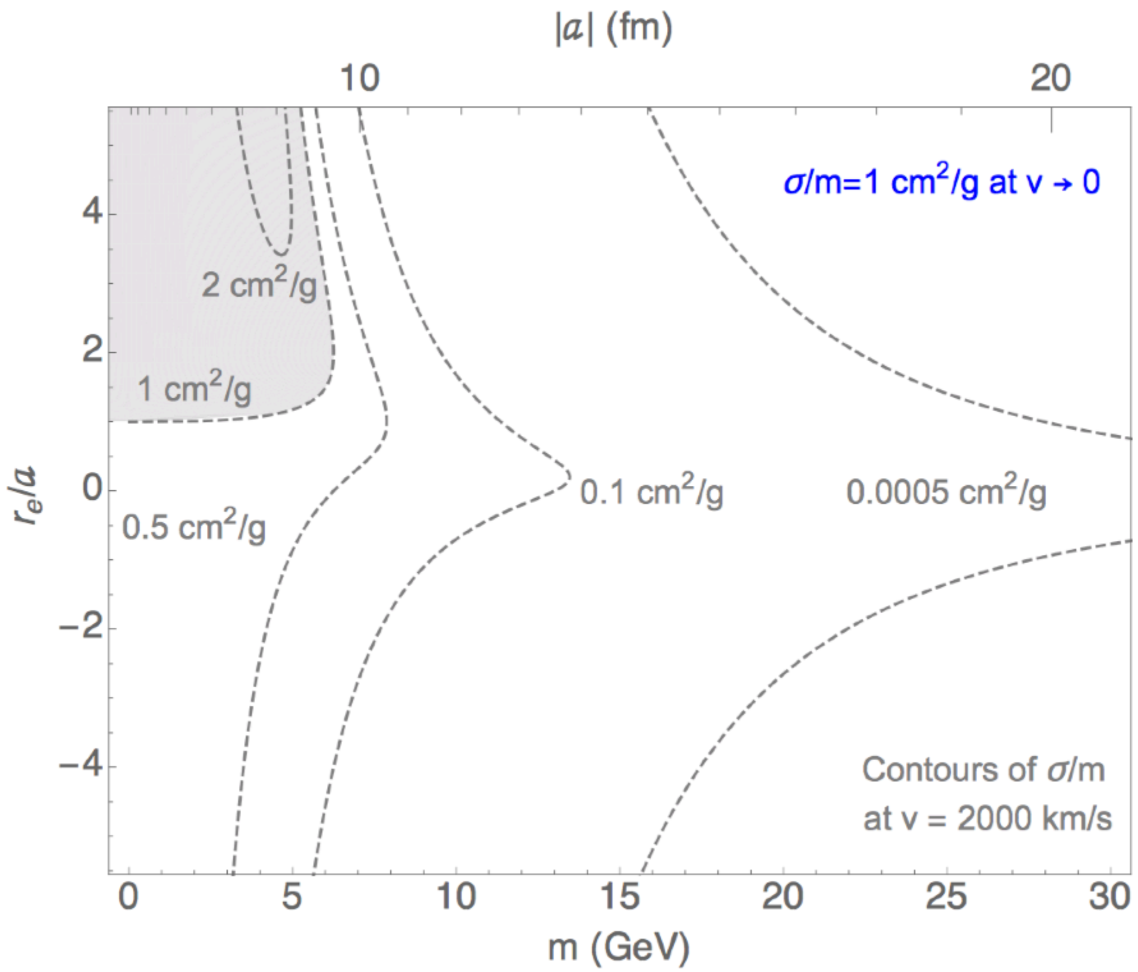
$$\sigma_{an,0}(k) = \frac{4\pi}{k^2} \frac{1 - |e^{2i\delta_0}|^2}{4} \simeq \frac{\sigma_0(k)}{k} \frac{|\mathrm{Im}a|}{(\mathrm{Re}a)^2}$$

Connections between self-scattering and annihilation? *[e.g. Eric Braaten&H.-W. Hammer 2013, K. Blum, R. Sato&T. R. Slatyer,2016].*

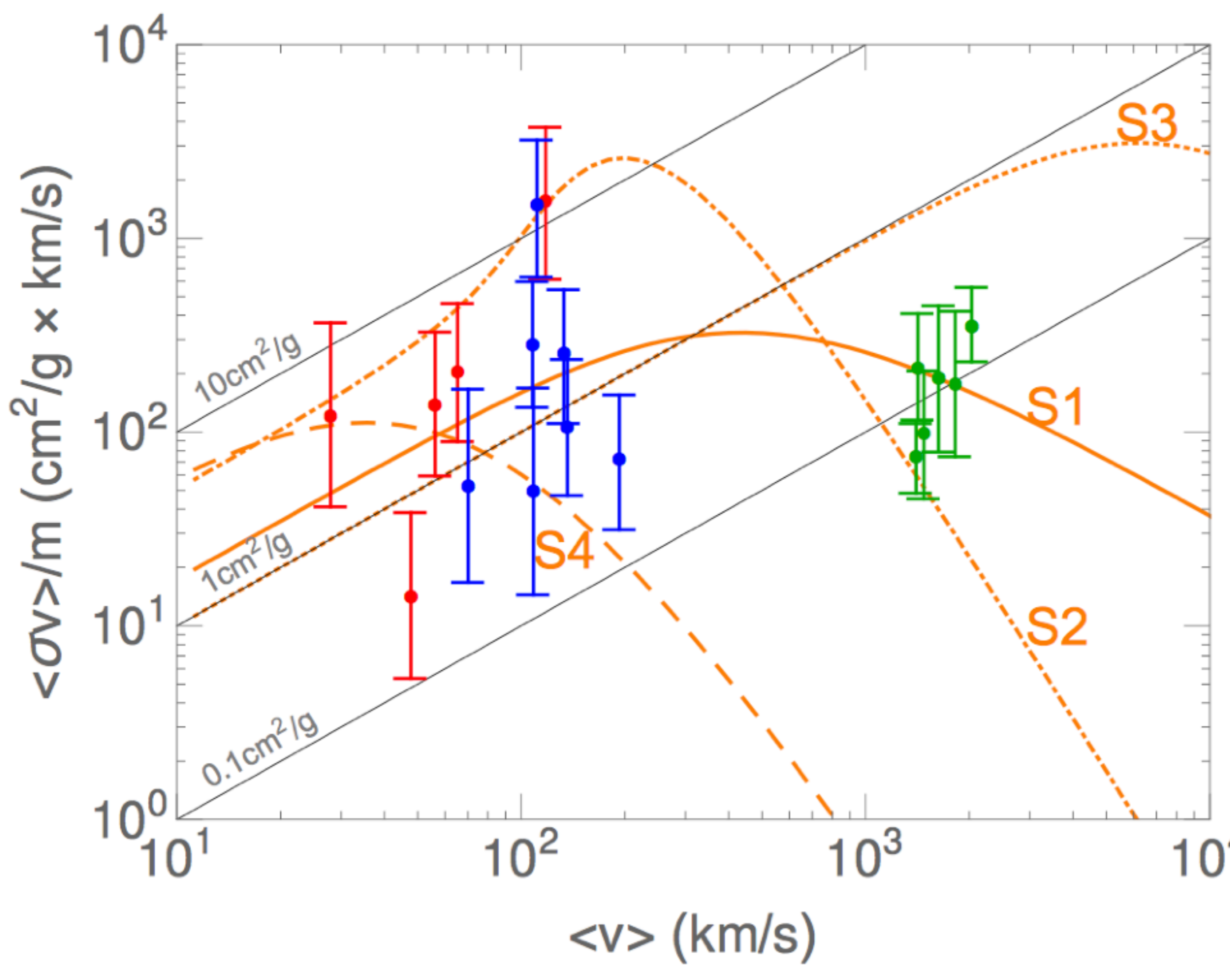
III. From ERT to DM properties

ERT for observables

A. If fix the self-scattering cross section to the preferred value at dwarf-scale:



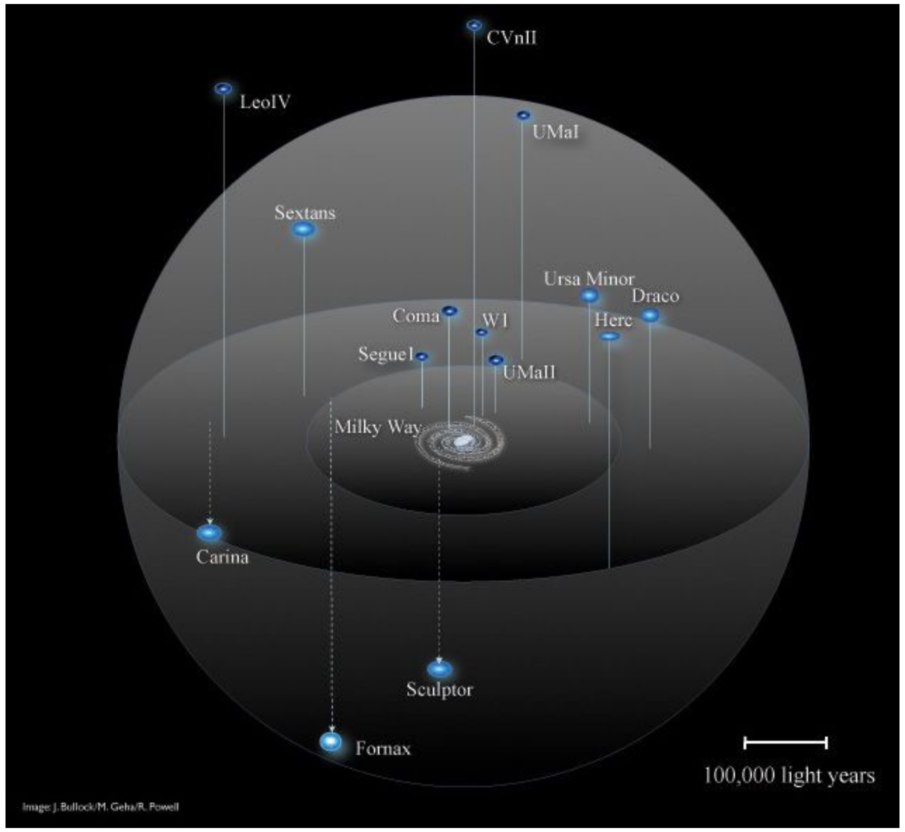
B. If fit inferred values of self-scattering cross section at various velocities.



- So far constraints/hints from observations are **too weak**.
- Observations can only tell the relative sign of **scattering length** and **effective range**.

ERT for simulations

Mass deficit problems are mostly found in **satellite halos** (e.g. in Milky Way):



Cosmological simulations with a consistent description of such **cross section involving various scales of DM relative velocities** can be crucial, e.g. to understand evolution of satellite halos moving inside host large halo [A. Banerjee et al. 2019].

Effective range theories only require **two parameters**, together with **the DM mass**:

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

Futuristic: ERT parameters to DM properties

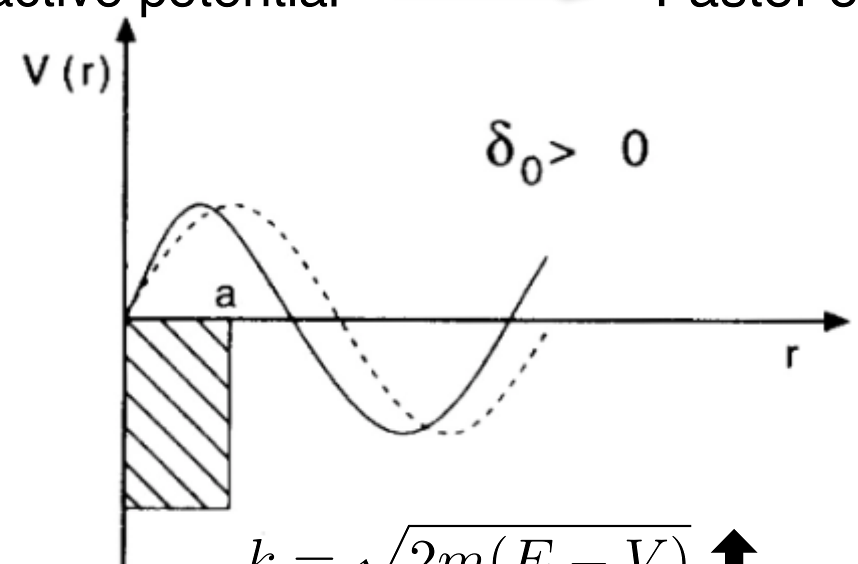
If we measure **velocity dependence**, what it tells about the **particle properties of DM**?

- **For weak potential** (Born approximation applies)

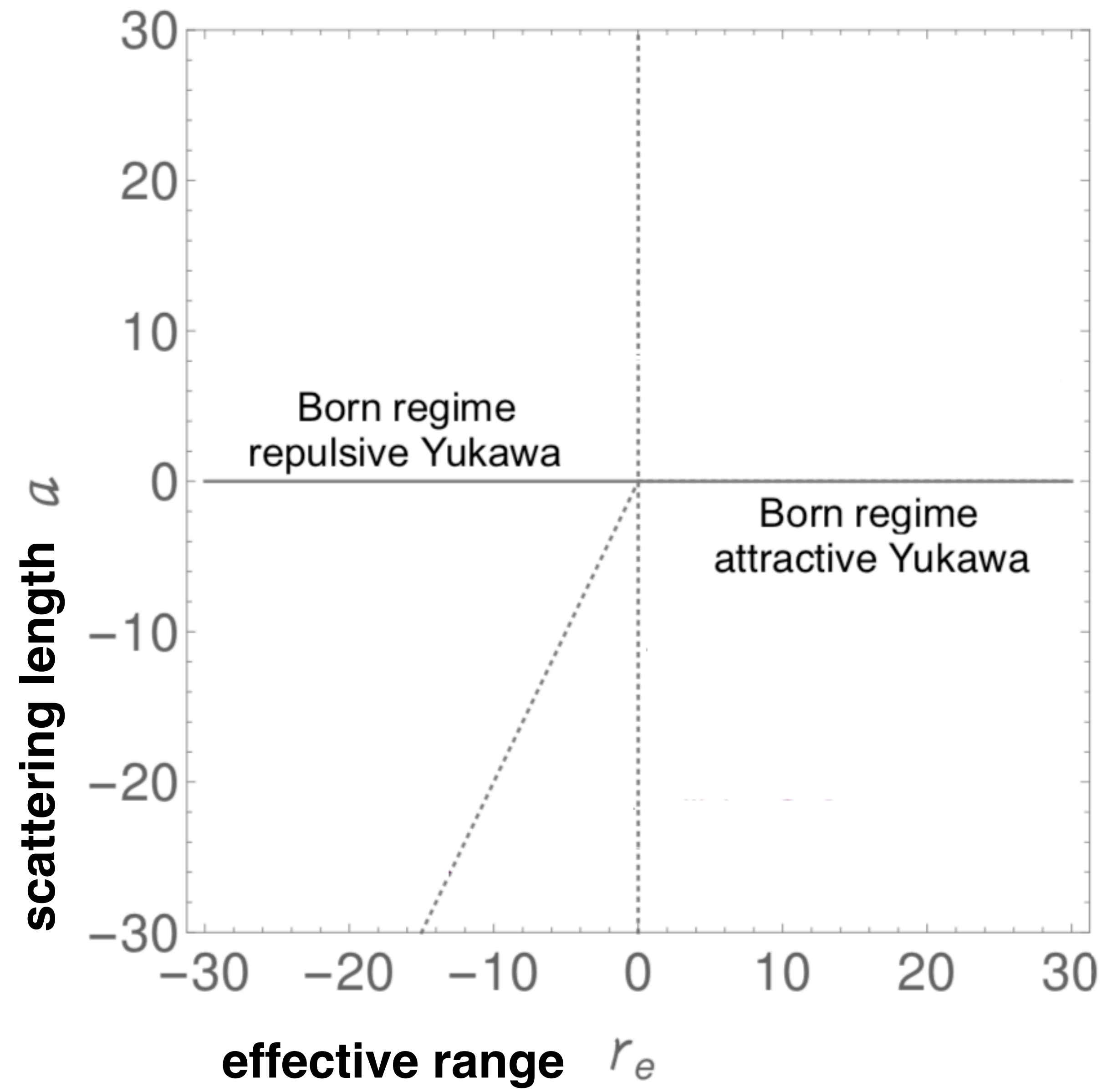
$$a = -\lim_{k\to 0}\frac{\tan\delta_0(k)}{k}$$

Attractive potential: $a < 0, \quad r_e > 0$

Inside attractive potential → Faster oscillation



$k = \sqrt{2m(E-V)}$ ⬆



Repulsive potential: $a > 0, \quad r_e < 0$

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If we measure **velocity dependence**, what it tells about the **particle properties of DM**?

- **For strong potential** (pole-dominated)

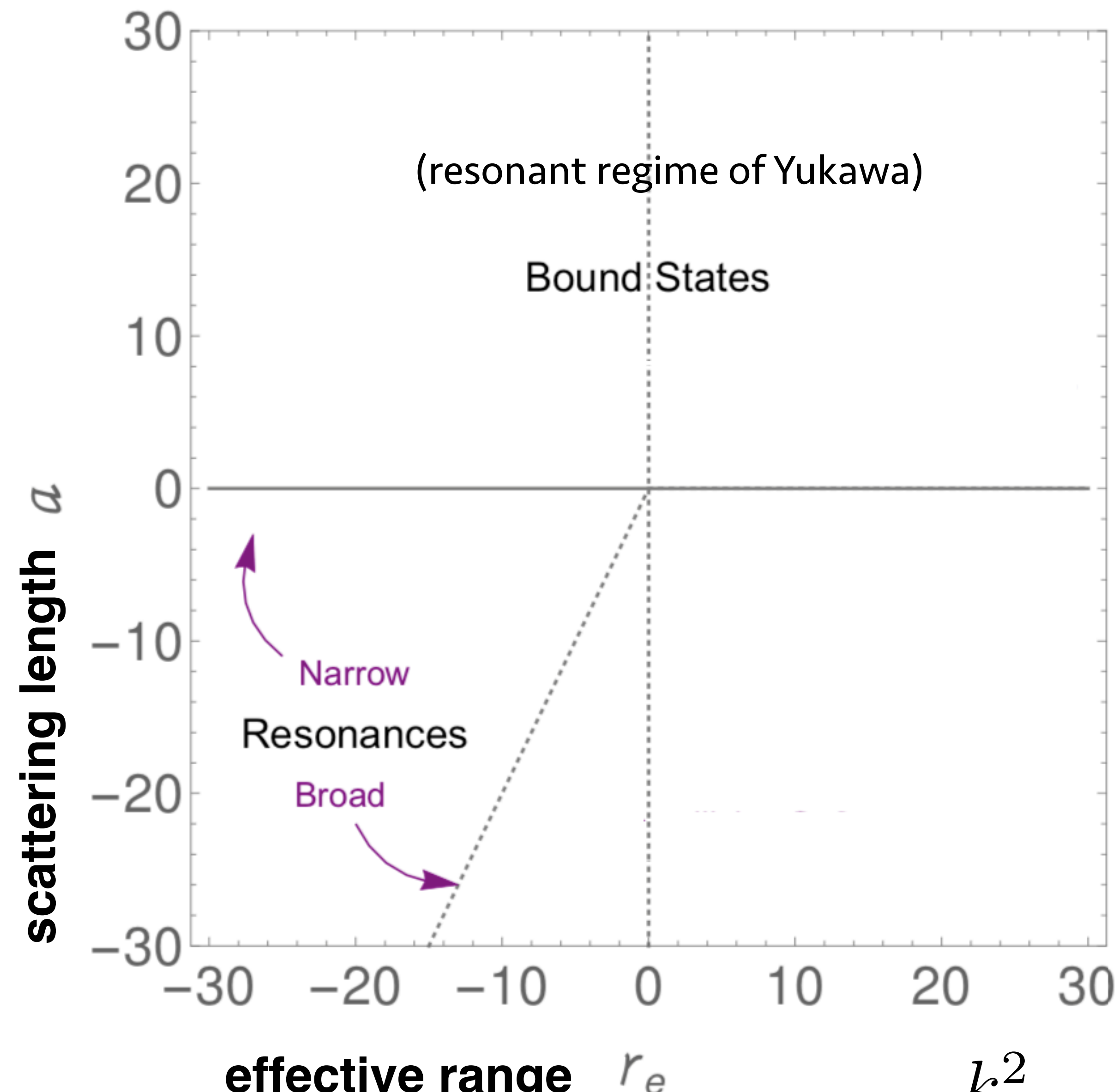
$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

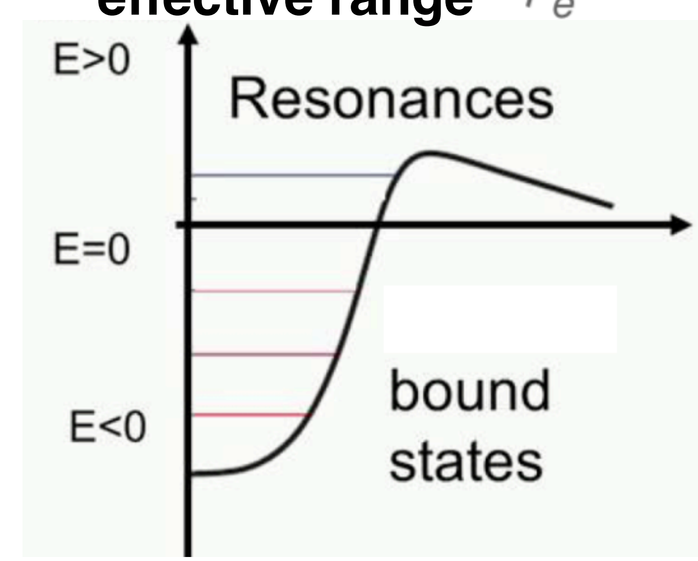
Usually dominated by **k-pole with smaller |Re[k]|**:

$$\boxed{f(k,\theta)\frac{e^{ikr}}{r}}$$

$$k_{\rm pole} = -i|k| + \gamma \quad \rightarrow \quad e^{-iEt-\Gamma t}$$

$$k_{\rm pole} = i|k| \quad \rightarrow \quad e^{-|k|r}$$





$$E \equiv \frac{k_{\rm pole}^2}{2m}$$

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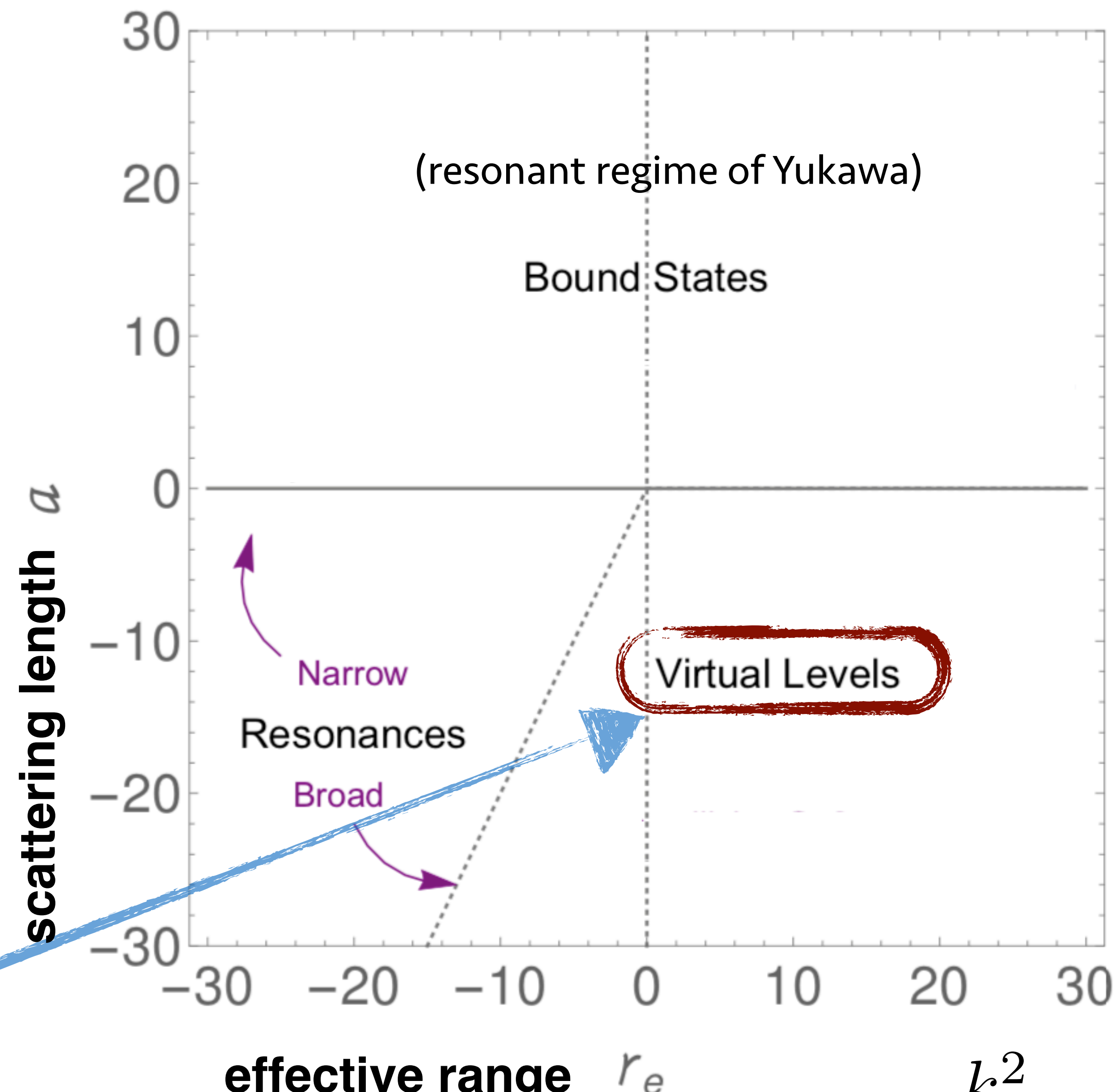
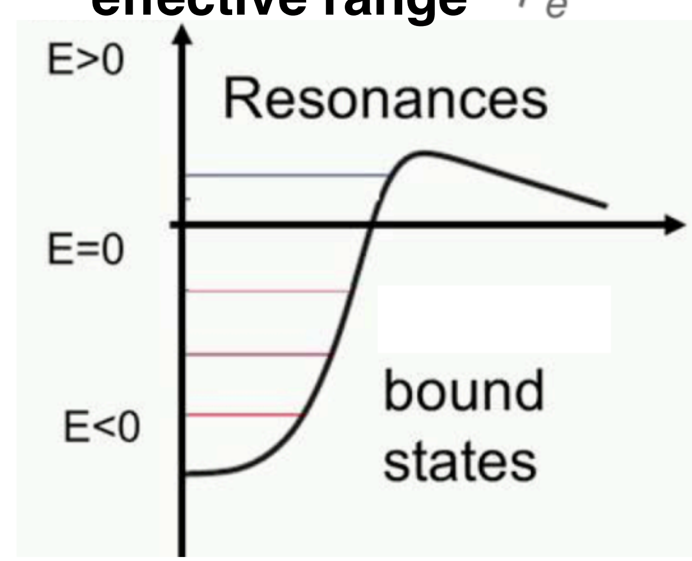
Usually dominated by **k-pole with smaller |Re[k]|**:

$$\boxed{\exists\, k_{\rm pole} = -i|k|\ ,\ \ \psi(r) \propto e^{|k|r}}$$

imaginary, intermediate state





$$E \equiv \frac{k_{\rm pole}^2}{2m}$$

IV. Conclusions

Conclusions:

- Halo mass deficit may be **hint of non-conventional DM;**
- For the SIDM solution, **velocity-dependence** may be necessary;
    - Interaction induced by a light mediator or s-wave resonance.
- **Consistent parametrization** is possible for well-motivated models;
    - In general scattering is enhanced at smaller DM velocities,
    - Complex phase shift to describe both scattering and annihilation?
- Detailed **cosmological simulations are crucial** (precise baryon effects, halo evolution with time, tidal effects, …).

Thanks!

The Schrödinger equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{\ell,k}}{dr}\right)+\left(k^2-\frac{\ell(\ell+1)}{r^2}-mV(r)\right)R_{\ell,k}=0\,,$$

Scattering length

$$t_{\ell,k}(r)=\frac{j_\ell(kr)\left(\frac{R'_{\ell,k}(r)}{R_{\ell,k}(r)}-\frac{\ell}{r}\right)+k\,j_{\ell+1}(kr)}{n_\ell(kr)\left(\frac{R'_{\ell,k}(r)}{R_{\ell,k}(r)}-\frac{\ell}{r}\right)+k\,n_{\ell+1}(kr)}\,. \qquad \text{(A3)}$$

Simple algebra shows that

$$\frac{dt_{\ell,k}(r)}{dr}=-k\,m\,r^2V(r)\left(j_\ell(kr)-t_{\ell,k}(r)n_\ell(kr)\right)^2\,. \qquad \text{(A4)}$$

The fact that $R_{\ell,k}\propto r^\ell$ and Eq. (A2) fix the boundary conditions of this differential equation to

$$t_{\ell,k}(0)=0 \quad \text{and} \quad t_{\ell,k}(r)\to\tan\delta_\ell \quad \text{at} \quad r\to\infty\,. \qquad \text{(A5)}$$

Notice that $j_\ell(kr)\propto k^\ell$ and $n_\ell(kr)\propto k^{-(\ell+1)}$ in the limit $k\to 0$, which together with Eq. (A4) imply that $\tan\delta_\ell\propto k^{2\ell+1}$ for small momenta. The corresponding coefficient of proportionality defines scattering length a_ℓ. More precisely,

$$a_\ell^{2\ell+1}\equiv-\lim_{k\to0}\frac{\tan\delta_\ell}{k^{2\ell+1}}\,. \qquad \text{(A6)}$$

Effective range

$$u_k(r)\frac{du_0(r)}{dr}-u_0(r)\frac{du_k(r)}{dr}\bigg|_0^r=k^2\int_0^r u_0(r')u_k(r')dr'\,. \qquad \text{(A14)}$$

Moreover, using the fact that $\psi_k(r)$ is the solution of the Schrödinger equation for $V(r)=0$, we find that

$$\psi_k(r')\frac{d\psi_0(r')}{dr}-\psi_0(r')\frac{d\psi_k(r')}{dr}\bigg|_0^r=k^2\int_0^r \psi_0(r')\psi_k(r')dr'\,. \qquad \text{(A15)}$$

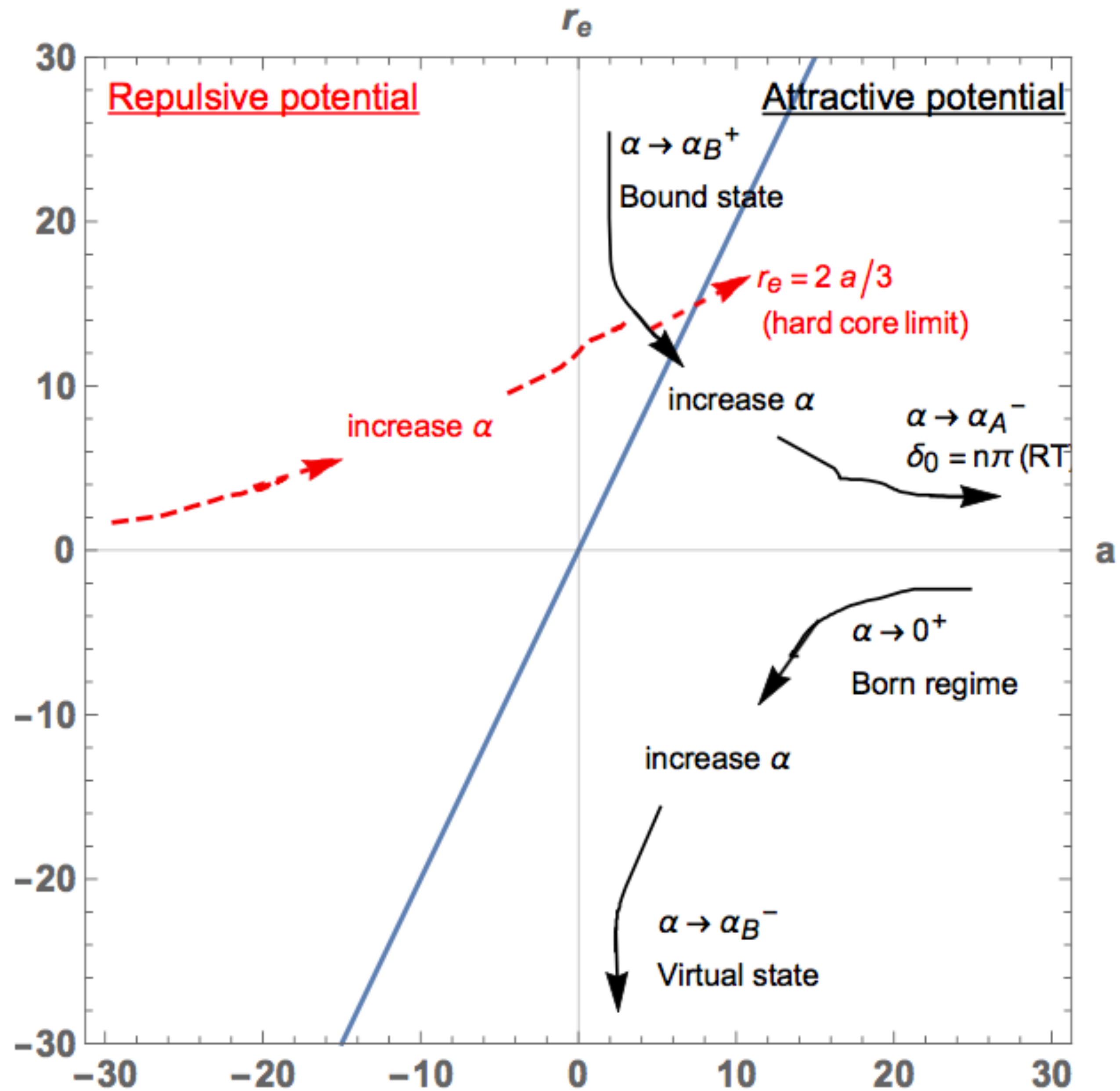
Notice that $\psi_0(r)=1-r/a_0$, where a_0 is the scattering length. Subtracting Eq. (A14) from Eq. (A15), taking $r\to\infty$ and using the fact that u_k and ψ_k approach to each other in that limit, we find that

$$\begin{aligned} k\cot\delta_0 &= -\frac{1}{a_0}+k^2\int_0^\infty(\psi_0\psi_k-u_0u_k)\,dr \\ &= -\frac{1}{a_0}+\frac{1}{2}r_{e,0}k^2+\mathcal{O}(k^4)\,. \end{aligned} \qquad \text{(A16)}$$

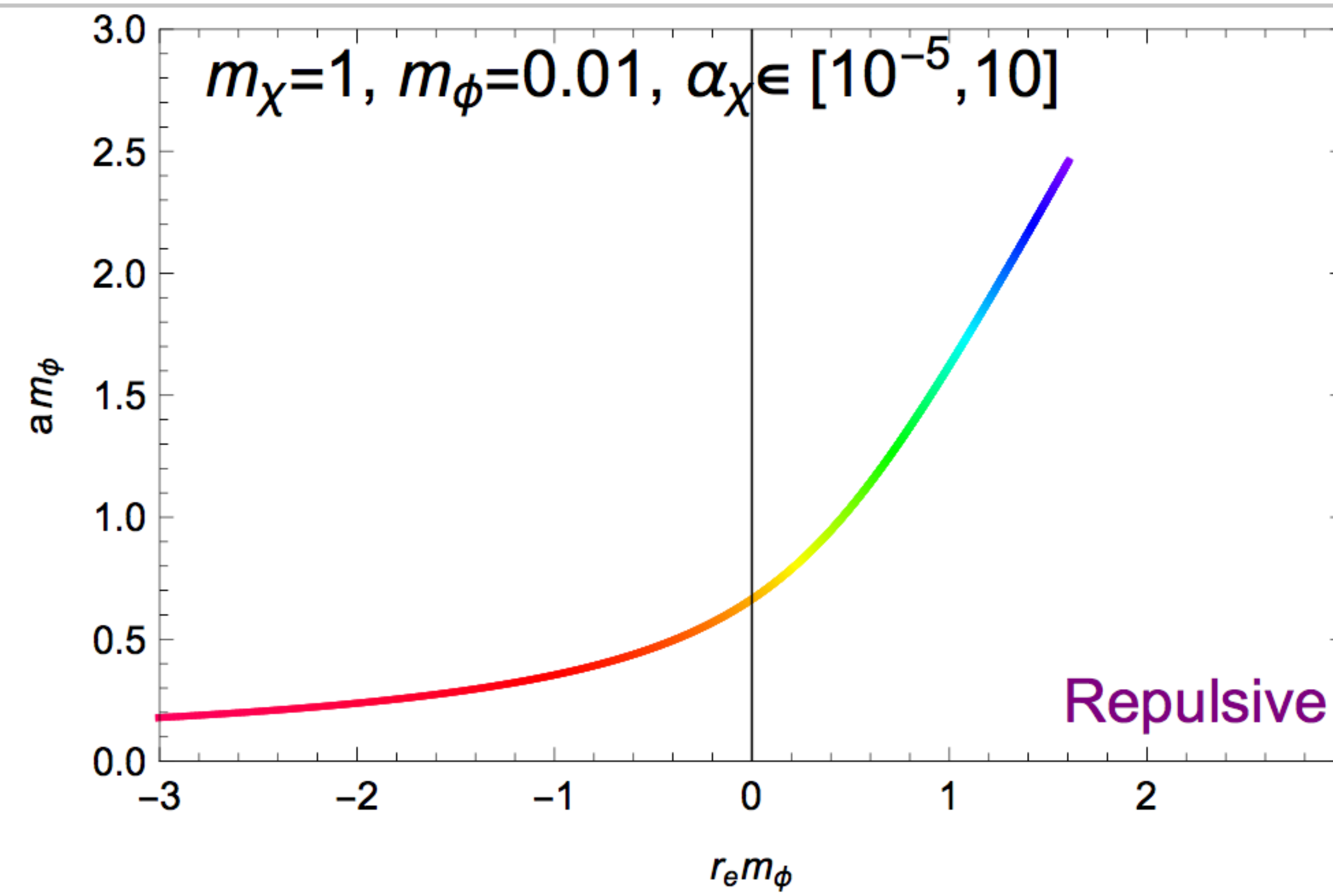
where

$$r_{e,0}=2\int_0^\infty\left(\psi_0^2-u_0^2\right)dr\,. \qquad \text{(A17)}$$

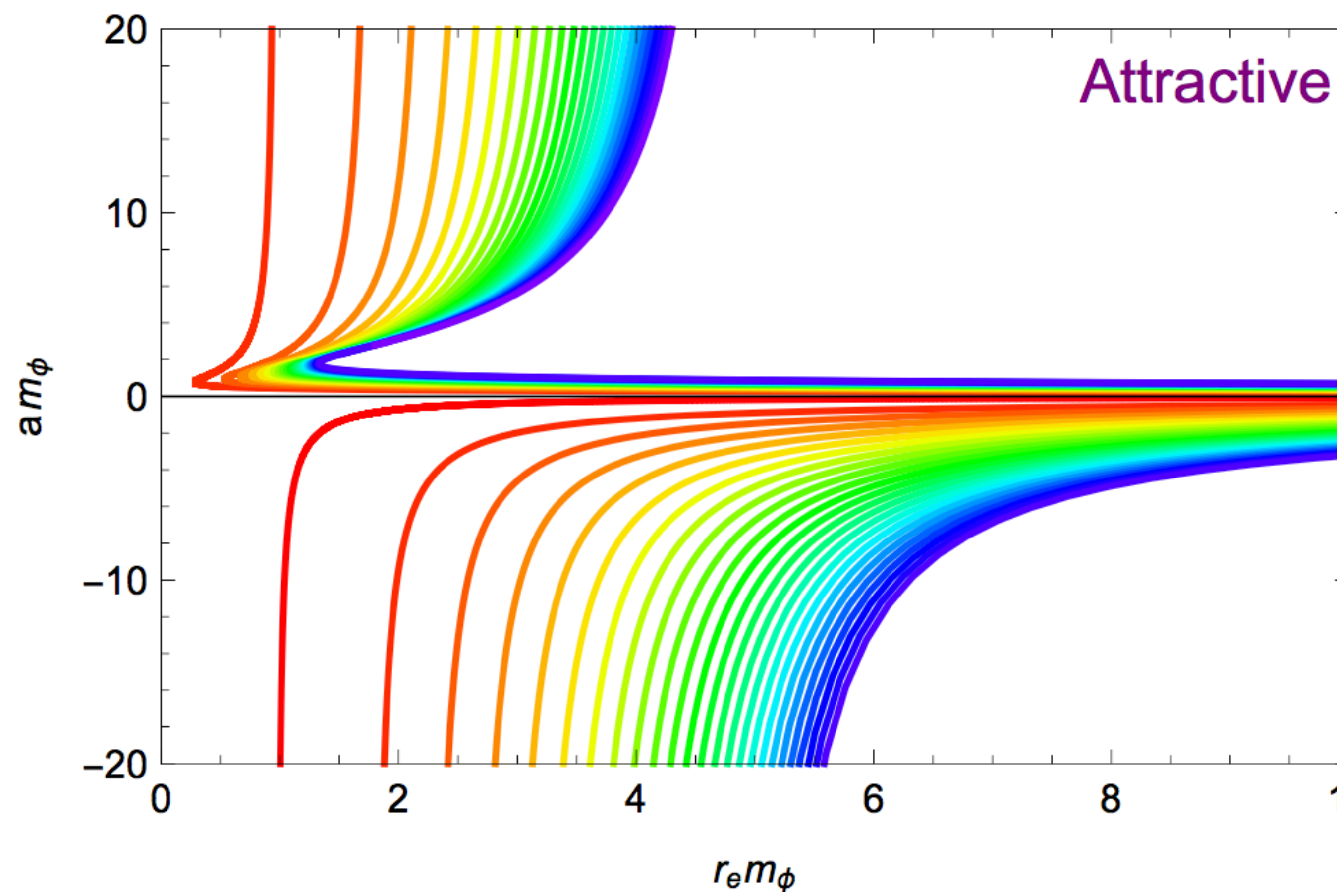
ArXiv: 1908.06067



Repulsive potential
Attractive potential
r_e
a
$\alpha \to \alpha_B^+$
Bound state
$r_e = 2a/3$
(hard core limit)
increase α
increase α
$\alpha \to \alpha_A^-$
$\delta_0 = n\pi$ (RT
$\alpha \to 0^+$
Born regime
increase α
$\alpha \to \alpha_B^-$
Virtual state



m_χ=1, m_ϕ=0.01, $\alpha_\chi \in [10^{-5}, 10]$
Repulsive
$a\,m_\phi$
$r_e m_\phi$



Attractive
$a\,m_\phi$
$r_e m_\phi$