Lecture II : Slow-roll inflation & cosmological perturbation theory

Recap:

- Horizon & Flatness problems
  - $\to$ solved if equation of state $\omega = \frac{p}{\rho} < -1/3$ in the past.

1) Slow-roll inflation

- Consider a single real scalar field ϕ,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) = S_{EH} + S_\phi$$

$\delta\sqrt{-g}/\delta g^{\mu\nu}$

$$\to T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - \underset{(-,+,+,+)}{g_{\mu\nu}} \Big(\underbrace{\tfrac{1}{2} \partial^\sigma \phi \partial_\sigma \phi}_{-\partial_0\phi\partial_0\phi + \partial_i\phi\partial_i\phi} + V(\phi) \Big)$$

$$= \mathrm{diag}(\rho, -p, -p, -p)$$

assume $\phi(\vec{x}, t) = \phi(t)$ homogeneous, $\partial_i \phi = 0$

$$\hookrightarrow \underline{\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)}, \qquad \underline{p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)}$$

$$\to \omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{1/2\, \dot{\phi}^2 - V(\phi)}{1/2\, \dot{\phi} + V(\phi)}$$

$$\text{if } V(\phi) \gg 1/2\, \dot{\phi}^2 \Rightarrow \underline{\omega \to -1} < -1/3$$

equation of motion:

exercise

$$\frac{\partial S}{\partial \phi} - \partial_\mu \frac{\partial S}{\partial \phi_{,\mu}} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \partial^\mu \phi \right) + V_{,\phi} = 0$$

ϕ homogeneous, FRW metric:

$$\boxed{\begin{aligned} &\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \\ &\quad \text{with } H^2 = 1/3 \left(\tfrac{1}{2} \dot{\phi}^2 + V(\phi) \right) \end{aligned}} \qquad \left(\frac{k}{a^2} \ll \rho \right)$$

$(F1) \rightarrow H = \frac{\dot{a}}{a} = \sqrt{\frac{1}{3}\rho}\,, \quad \rho = a^{-3(1+\omega)} \xrightarrow{\omega=-1} \text{const}$

$\rightarrow$ for $\omega = -1$: $\underline{a(t) \propto e^{Ht}}$

$\rightarrow \underline{\tau = -\frac{1}{aH}} \qquad (-\infty < \tau < 0)$

$\rightarrow$ exponentially expanding Universe, dominated by potential energy of scalar field

2) Slow-roll approximation

- $\varepsilon = -\frac{\dot{H}}{H^2} \overset{F1+F2}{=} \frac{1}{2}\frac{\dot{\phi}^2}{H^2} < 1 \iff$ accelerated expansion $\equiv$ inflation

  $\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} < 1 \iff$ change in ε small $\hookrightarrow$ sustained slow-roll inflation

  slow-roll parameters

  Slow-roll inflation : $\varepsilon, |\eta| \ll 1$

  Often more convenient to work with "potential slow-roll parameters" :

  $$\varepsilon_V = \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \quad \eta = M_P^2 \frac{V_{,\phi\phi}}{V}$$

  $$\varepsilon_V, |\eta_V| \ll 1 \Rightarrow \varepsilon \approx \varepsilon_V, \quad \eta \approx \eta_V - \varepsilon_V$$

- Slow-roll approximation ($\varepsilon, \eta \ll 1$) :

  $$\underline{3H\dot{\phi} + V_{,\phi} = 0\,, \quad H^2 = \frac{1}{3}V \simeq \text{const}\,, \quad a(t) \propto e^{Ht}}$$

- Introduce a new time-coordinate, 'e-folds' :

  $$dN = -H\,dt = \frac{da}{a}$$

  $$N(\phi) = \ln\frac{a_{end}}{a} = \int_t^{t_{end}} H\,dt = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}}\,d\phi \approx \int_{\phi_{end}}^{\phi} \frac{1}{\sqrt{2\varepsilon}}\,d\phi$$

  (t_{end}: $\varepsilon = 1$)

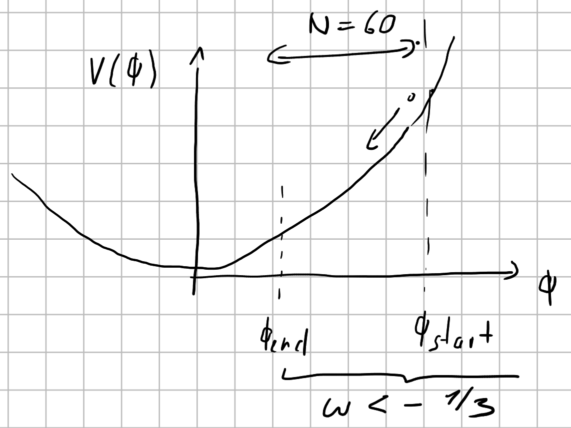
  Solving horizon & flatness problem requires $N \gtrsim 60$

3) A worked example : $V(\phi) = \frac{1}{2} m^2 \phi^2$

$$\varepsilon_V = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 = 2\left(\frac{M_p}{\phi}\right)^2 \overset{!}{\lesssim} 1 \quad \longrightarrow \quad \phi_{end} \overset{\varepsilon = 1}{=} \sqrt{2}\, M_p$$

inflation for $\phi > \phi_{end}$

$$N(\phi) = \frac{1}{M_p}\int_{\phi_{end}}^{\phi} \frac{1}{\sqrt{2\varepsilon}}\, d\phi \simeq \frac{\phi^2}{4M_p^2} - \frac{1}{2} \overset{!}{=} 60 \quad \longrightarrow \quad \phi_{start} \simeq 15\, M_p$$



Simple case of massive scalar field does the job!

4) Cosmological perturbation theory

Consider small perturbations around homogeneous background,

$$X(t,\vec{x}) = \bar{X}(t) + \delta X(t,\vec{x}) \quad , \quad \delta X \ll \bar{X} \quad , \quad X = \phi, g_{\mu\nu}$$

→ linear order in δX

- symmetries

general coordinate transformation $\xrightarrow[\text{FRW background}]{1}$ $SO(3)$ global spatial rotations $\xrightarrow[\text{Fourier transf. direction } \vec{k} \text{ singled out}]{2}$ $SO(2)$ helicity eigenstates $X_{\vec{k}} \xrightarrow{\theta} e^{im\theta} X_{\vec{k}}$

$m=0$ → scalar
$m=\pm 1$ → vector
$m=\pm 2$ → tensor

→ SVT decomposition at linear order.

- example : $g_{\mu\nu}$

$$1: \left(\begin{array}{c|c} g_{00} & g_{0j} \\ \hline g_{i0} & g_{ij} \end{array}\right)$$

$$2: \quad g_{00} = -(1 + 2\Phi) \quad \rightarrow \quad 1 \text{ scalar dof } (\Phi)$$

$$g_{i0} = g_{0i} = 0 + \underbrace{2a}_{\text{convention}} (\partial_i B - S_i) \; , \quad \underbrace{\partial^i S_i = 0}_{\text{no hidden scalar}}$$

→ 1 scalar (B), 1 vector (S_i)

$$g_{ij} = a^2 \left[(1 - 2\Psi)\delta_{ij} + 2\partial_{ij}F + 2\underbrace{\partial_{(i}F_{j)}}_{\text{symmetrize}} + h_{ij} \right]$$

$$\partial_i F^i = 0 \; , \quad h^i_i = \partial^i h_{ij} = 0 \quad \Big\} \text{ no hidden scalars, or vectors}$$

→ 2 scalars (Ψ, F), 1 vector (F_j), 1 tensor (h_{ij})

⇒ in total

4 scalars :	Φ, Ψ, B, F	→	lecture 3
2 vectors :	S_i , F_i	→	decay
1 tensor :	h_{ij}	→	lecture 4

note on representation theory:

$$g_{ij} = \underset{\text{SO(3) reps}}{5 \oplus 1} \longrightarrow \underset{\text{SO(2) helicity eigenvalues}}{-2, -1, 0, 1, 2 \oplus 0}$$

symmetric, → 6 dofs

scalar (0), vector (−1, 0, 1), tensor (−2, −1, 0, 1, 2); scalar (0)

- gauge invariance

physical observables must be invariant under infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu \quad , \quad \varepsilon^\mu = (\varepsilon^0, \partial_i e + f_i), \quad \partial_i f^i = 0$$

↳ contains 2 scalars, 1 vector

→ 2 out of 5 scalars $(\underbrace{\delta g_{\mu\nu}}_{4}, \underbrace{\delta\phi}_{+1})$ can be gauged away

$\hat{=}$ choice of equal time hypersurface:

$$\underbrace{\delta X(t, \vec{x})}_{\text{gauge dependent}} = \underbrace{X(t, \vec{x})}_{\text{locally unambig. defined}} - \underbrace{\bar{X}(t)}_{\text{depends on choice of equal time hypersurface}}$$

⇒ a) define gauge invariant scalar, e.g.

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}} \delta\phi \qquad \text{(during slow roll inflation)}$$

"comoving curvature perturbation"

= spatial curvature of constant-ϕ hypersurface

or b) choose a gauge

→ intermediate steps gauge dependent but physical observables are not

e.g. Newtonian gauge: $B = F = 0$

note: tensor sector has no gauge freedom.