Lecture II: Slow-roll inflation & cosmological perturbation theory

Recap:
- Horizon & Flatness problems
  → solved if equation of state \( w = \frac{p}{\rho} < -\frac{1}{3} \) in the past.

1) Slow-roll inflation
- Consider a single real scalar field \( \phi \):

\[
S = \int \left[ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] - S_{\text{det}} + S_p
\]

\[
\Gamma^{(1)} \equiv -\frac{2}{\Gamma^{(3)}} \implies \frac{\delta S_{\phi}}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g^{\mu\nu} \left( \frac{1}{2} \partial_\gamma \phi \partial_\gamma \phi + V(\phi) \right)
\]

\[
= \text{diag} \left( \rho, -p, -p, -p \right)
\]

Assume \( \phi(\vec{r}, t) = \phi(t) \) homogeneous, \( \partial_t \phi = 0 \)

\[
\implies \rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V(\phi)
\]

\[
\omega_{\phi} = \frac{p_{\phi}}{\rho_{\phi}} = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}
\]

if \( V(\phi) \gg \frac{1}{2} \dot{\phi}^2 \) \( \implies \omega \to -1 \) \( < -\frac{1}{3} \) \( \square \)

Equation of motion:

\[
\frac{\partial S}{\partial \phi} - \partial_\mu \frac{\partial S}{\partial \partial_\mu \phi} = \frac{\Lambda}{\Gamma^{(3)}} \partial_\mu \left( \Gamma^{(3)} \partial^\mu \phi \right) + V_{,\phi} = 0
\]

\( \phi \) homogeneous, FRW metric:

\[
\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = 0
\]

with \( H^2 = \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \) \( \left( \frac{\rho}{a^2} \ll \rho \right) \)
\( F(t) \Rightarrow H = \frac{\dot{a}}{a} = \sqrt{\frac{4}{3} \dot{\Sigma}}^{\frac{1}{2}} \), \quad \dot{\Sigma} = 0 \stackrel{-3(1+w)}{\longrightarrow} \text{const}

\Rightarrow \text{for } w = -1: \dot{a}(t) \propto e^{Ht}

\Rightarrow \tau = -\frac{1}{aH} \quad (-\infty < \tau < 0)

\Rightarrow \text{exponentially expanding Universe, dominated by potential energy of scalar field}

2) **Slow-roll approximation**

- \( \epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2} \frac{\dot{\phi}^2}{H^2} < 1 \iff \text{accelerated expansion} \iff \text{inflation} \)

- \( \eta = -\frac{\ddot{\phi}}{H\dot{\phi}} < 1 \iff \text{change in } \epsilon \text{ small} \iff \text{sustained slow-roll inflation} \)

**Slow-roll inflation**: \( \epsilon, |\eta| < 1 \)

**Often more convenient to work with potential slow-roll parameters**: \( \epsilon_V = \frac{H^2}{2} \left( \frac{V}{V} \right)^2, \quad \eta = H^2 \frac{V}{V} \)

\( \epsilon_V, |\eta| < 1 \Rightarrow \epsilon \sim \epsilon_V, \eta \sim \eta_V - \epsilon_V \)

- **Slow-roll approximation** (\( \epsilon \eta \ll 1 \)):
  \( 3H \dot{\phi} + V, \dot{\phi} = 0 \), \( H^2 = \frac{A}{2} V = \text{const} \), \( \dot{a}(t) \propto e^{Ht} \)

- **Introduce a new time coordinate, \( t' \) holds**:
  \( \frac{dN}{dt} = -H \frac{\dot{a}}{a} \int \frac{V, \phi}{V} d\phi \)

\( N(\phi) = \ln \frac{a_{end}}{a} = \int \frac{H}{V} \frac{1}{2} d\phi + \int \frac{1}{2c^2} d\phi \)

**Solving horizon & flatness problem requires** \( N \geq 60 \)
3) A worked example: \( V(\phi) = \frac{1}{2} m^2 \phi^2 \)

\[
\varepsilon_v = \frac{V(\phi)}{V(1)} = 2 \left( \frac{M_p^2}{\phi} \right)^{\frac{2}{3}} \leq 1 \quad \rightarrow \quad \phi_{\text{end}} = \sqrt[3]{\frac{12}{2}} M_p
\]

Inflation for \( \phi > \phi_{\text{end}} \)

\[
N(\phi) = \frac{1}{H_p} \int_{\phi_{\text{end}}}^{\phi} \frac{1}{\sqrt{2\varepsilon_v}} \, d\phi = \frac{\phi^2}{4M_p^2} - \frac{1}{2} \frac{1}{12} \cong 60 \quad \rightarrow \quad \phi_{\text{start}} = 15 M_p
\]

Simple case of massive scalar field does the job!
4) **Cosmological perturbation theory**

Consider small perturbations around homogeneous background,

\[ X(t, \vec{x}) = \bar{X}(t) + \delta X(t, \vec{x}) , \quad \delta X \ll \bar{X} , \quad X = \phi, g_{\mu\nu} \]

\[ \rightarrow \text{Linear order in } \delta X \]

- **Symmetries**

  \[
  \begin{array}{cccc}
  \text{general coordinate transformation} & \text{FRW background} & \text{global spatial rotations} & \text{holonomy eigenstates} \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  \text{coordinate} & \text{background} & \text{direction} & \text{X} \\
  \text{transformation} & \text{singled out} & \text{E} & \text{X} \\
  \rightarrow \text{SUT decomposition at linear order.} \\
  \end{array}
  \]

- **Example:** \( g_{\mu\nu} \)

  1. \[
  \begin{pmatrix}
  g_{\alpha\alpha} & g_{\alpha j} \\
  g_{j\alpha} & g_{jj}
  \end{pmatrix}
  \]

  2. \[
  g_{\alpha\alpha} = -(1 + 2\Phi) \quad \rightarrow \text{1 scalar, } \partial \Phi \text{ (} \Phi \text{)}
  \]

  \[
  g_{j\alpha} = 0 + 2a \left( \partial_i B - S_i \right) , \quad \partial_i S_i = 0
  \]

  \[
  \rightarrow \text{1 scalar } (B) , \quad 1 \text{ vector } (S_i)
  \]

  \[
  g_{ij} = a^2 \left[ (1 - 2\Psi) \delta_{ij} + 2 \partial_i B - \partial_j B + \partial_i F - \partial_j F \right]
  \]

  \[
  \partial_i F^i = 0 , \quad h^i = \partial_i h_{ij} = 0 \quad \{ \text{no hidden scalar, or vectors} \}
  \]

  \[
  \rightarrow 2 \text{ scalars } (\Phi, B) , \quad 1 \text{ vector } (F) , \quad 1 \text{ tensor } (h_{ij})
  \]

\[ \Rightarrow \text{in total 4 scalars: } \Phi, \Psi, B, F \quad \rightarrow \text{lecture 3} \]

  2 vectors: \( S_i, F_i \) \quad \rightarrow \text{decay}

  1 tensor: \( h_{ij} \) \quad \rightarrow \text{lecture 4} \]
Note on representation theory:

\[ g_{ij} = \mathbf{5} \oplus \mathbf{1} \rightarrow -2, -1, 0, 1, 2 \oplus 0 \]

- 3\( \mathbf{3} \) reps
- 3\( \mathbf{\bar{3}} \) reps
- \( \mathbf{3} \cdot \mathbf{\bar{3}} \) helicity eigenvalues

\[ \text{symmetric,} \qquad \rightarrow 6 \text{ scalars} \]

\[ \Rightarrow \text{vector,} \qquad \Rightarrow \text{tensor} \]

- scalar
- vector
- tensor

\[ \Rightarrow \text{contains 2 scalars, 1 vector} \]

\[ \Rightarrow \text{out of 5 scalars, } \phi, \psi \Rightarrow \text{can be gauged away} \]

\[ \Rightarrow \text{choice of equal time hypersurface:} \]

\[ \delta X(t, x) = X(t, x) - \overline{X}(t) \]

\[ \text{gauge dependent, locally ambiguous} \]

\[ \text{defined,} \quad \text{depends on choice of equal time hypersurface} \]

\[ \Rightarrow a) \text{ define gauge invariant scalar, e.g.:} \]

\[ R = \phi + \frac{H}{\dot{\phi}} \delta \phi \quad \text{(during slow roll inflation)} \]

"comoving curvature perturbation"

\[ = \text{spatial curvature of constant- } \phi \text{ hypersurface} \]

\[ \Rightarrow b) \text{ choose a gauge} \]

\[ \Rightarrow \text{intermediate steps gauge dependent} \]

\[ \Rightarrow \text{physical observables are not} \]

\[ \text{e.g. Newtonian gauge . } R = F = 0 \]

\[ \text{note: tensor sector has no gauge freedom.} \]