Lecture III : Scalar Perturbations

Recap:

- Slowly rolling homogeneous scalar field $\rightarrow w \simeq -1$
  $\rightarrow$ solves flatness & horizon problems
- SVT - decomposition:
  $\delta g_{\mu\nu} \rightarrow$ 2 scalars, 2 vectors, 1 tensor
  $\delta\phi \rightarrow$ 1 scalar
  coordinate transformation $\rightarrow$ 2 scalars, 1 vector
- gauge dependence

$\hat{=}$ choice of equal time hypersurface:

$$\underbrace{\delta X(t,\vec{x})}_{\text{gauge dependent}} = \underbrace{X(t,\vec{x})}_{\text{locally unambig. defined}} - \underbrace{\bar{X}(t)}_{\text{depends on choice of equal time hypersurface}}$$

$\Rightarrow$ a) define gauge invariant scalar, e.g.

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}}\delta\phi \qquad \text{(during slow roll inflation)}$$

"comoving curvature perturbation"

= spatial curvature of constant-ϕ hypersurface

or b) choose a gauge

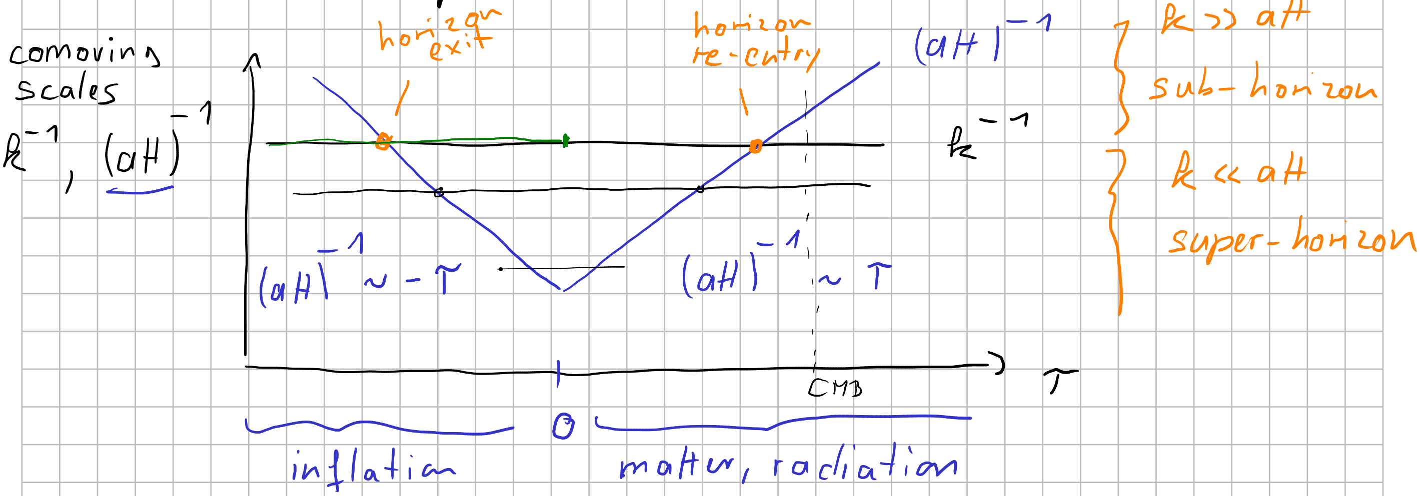
$\rightarrow$ intermediate steps gauge dependent
but physical observables are not

e.g. Newtonian gauge : $B = F = 0$

note: tensor sector has <u>no</u> gauge freedom.

0) scales and horizons

Consider perturbation with wavenumber k



↳ this lecture

↳ lecture 4, Fabian's & Yacine's lecture

Computational strategy:

1) expand action to 2nd order in perturbations
   ↳ derive eom
2) quantize & set initial conditions
3) compute power spectrum
   (= statistical properties of perturbations)

1) EoM

gauge choice: $\delta\phi = 0$, $g_{ij} = a^2\left[(1-2\mathcal{R})\delta_{ij} + h_{ij}\right]$

$$\left(S_\phi + S_{EH}\right)^{(2)} = \ldots = \frac{1}{2}\int d^4x\; a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]$$

$z \equiv \frac{a\dot\phi}{H}$

$$= \frac{1}{2}\int d\tau\, d^3x\, \Big(\underbrace{a^2 z^2 \dot{\mathcal{R}}^2}_{(v' - z'\mathcal{R})^2} - \underbrace{z^2(\partial_i \mathcal{R})^2}_{(\partial_i v)^2}\Big)$$

$v \equiv z\mathcal{R}$ Mukhanov variable

$$= \frac{1}{2}\int d\tau\, d^3x \Big[v'^2 - (\partial_i v)^2 \underbrace{- 2v'v\frac{z'}{z} - \left(v\frac{z'}{z}\right)^2}_{-\frac{d}{d\tau}\left[\frac{z'}{z}v^2\right] + v^2\frac{z''}{z}} \Big]$$

$$\overset{p.I.}{=} \frac{1}{2}\int d\tau\, d^3x \Big[v'^2 - (\partial_i v)^2 + \underbrace{\frac{z''}{z}}_{\text{"mass term"}} v^2 \Big]$$

Fourier expansion: $v(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, v_{\vec{k}}(\tau)\, e^{i\vec{k}\vec{x}}$

$$\boxed{v_k'' + \Big(k^2 - \underbrace{\frac{z''}{z}}_{\underset{\text{slow-roll}}{\approx} 2a^2H^2}\Big) v_k = 0} \quad (M)$$

Mukhanov equation

$\Rightarrow$ sub-horizon $(k \gg aH)$: free oscillator

super-horizon $(k \ll aH)$: z''/z important

2) Quantization and initial conditions

Quantization:

$$v \longrightarrow \hat{v} = \int \frac{d^3k}{(2\pi)^3} \left[v_k(\tau)\, \hat{a}_{\vec{k}}\, e^{i\vec{k}\vec{x}} + v_k^*(\tau)\, \hat{a}^+_{\vec{k}}\, e^{-i\vec{k}\vec{x}} \right]$$

with $[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3 \delta(\vec{k}-\vec{k}')$

HW $\hookrightarrow \frac{i}{\hbar}(v_k^* v_k' - v_k'^* v_k) = 1$ $\quad (*)$

fixes one out of two initial conditions

Initial conditions:

in the far past $k \gg aH \xrightarrow{(M)} v_k'' + k^2 v_k = 0$

$\hookrightarrow$ harmonic oscillator

$\hookrightarrow$ unique state which minimizes energy

$$\boxed{\lim_{\tau \to -\infty} v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}}$$

Bunch Davies vacuum

only positive frequency mode due $(*)$

$\rightarrow$ initial conditions fully fixed

Solve (M) with these i.c. in de-Sitter $(H = \text{const})$:

$$\tau = -\frac{1}{aH} \rightarrow v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0$$

$$\rightarrow \underline{v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)} \quad (**)$$

super-horizon, $\frac{k}{aH} = -k\tau \ll 1$

$$v_k'' - \frac{2}{\tau^2} v_k = 0 \rightarrow v_k = A\tau^2 + B/\tau$$

$$R \propto \frac{v_k}{a} = \underbrace{-AH\tau^3}_{\substack{\tau \to 0 \\ \text{decays}}} - \underbrace{BH}_{\text{constant}}$$

$\rightarrow$ no oscillations, "freeze-out"

3) Power spectrum $\langle R_{\vec{k}} \, R_{\vec{k}'} \rangle$

- statistical homogeneity & isotropy (SHI)

$$\langle R(t,\vec{x}) \, R(t,\vec{x}') \rangle \equiv \xi(t,\vec{x},\vec{x}') \overset{SHI}{=} \xi(t, \underbrace{|\vec{x}-\vec{x}'|}_{\text{homogeneity}})$$

isotropy

in Fourier space

$$\langle R_{\vec{k}}(t) \, , R^*_{\vec{k}'}(t) \rangle \overset{SHI}{=} (2\pi)^3 \underbrace{\delta(\vec{k}-\vec{k}')}_{\text{homogeneity}} \, P_R(\underbrace{k}_{\text{isotropy}})$$

we will also need

$$\int \frac{d^3k}{(2\pi)^3} P_R(k) = \int d\ln k \, \underbrace{\frac{k^3}{2\pi^2} P_R(k)}_{\equiv \Delta_R^2}$$

- power spectrum

$$R = \frac{v}{z} = \frac{vH}{a\dot{\phi}} \rightarrow \langle R_{\vec{k}} \, R_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}-\vec{k}') \left(\frac{H}{\dot{\phi}}\right)^2 \frac{|v_k(\tau)|^2}{a^2}$$

$$\left|\frac{v_k(\tau)}{a}\right|^2 \overset{(**)}{=} \frac{1}{a^2}\frac{1}{2k}\left(1 + \frac{1}{k^2\tau^2}\right)$$

$$= \frac{H^2}{2k^3}\left(1 + \underbrace{k^2\tau^2}_{\to 0 \text{ on super-horizon scales}}\right)$$

$$= \text{const}$$

$\Rightarrow \langle R^2 \rangle$ at horizon crossing, $k = a_* H_*$, with $\phi = \phi_*$, remains constant after horizon crossing

$$\Rightarrow \langle R_{\vec{k}} \, R_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \left(\frac{H_*}{\dot{\phi}_*}\right)^2 \frac{H_*^2}{2k^2} \quad \forall \tau \text{ with } aH \gg k$$

$$\Rightarrow \boxed{\Delta_R^2 = \frac{H_*^2}{(2\pi^2)} \frac{H_*^2}{\dot{\phi}^2}}$$

$H, \dot{\phi} \approx$ const

$\Rightarrow$ scale invariant