Lecture IV Cosmological perturbations after inflation

Recap:

- metric perturbations

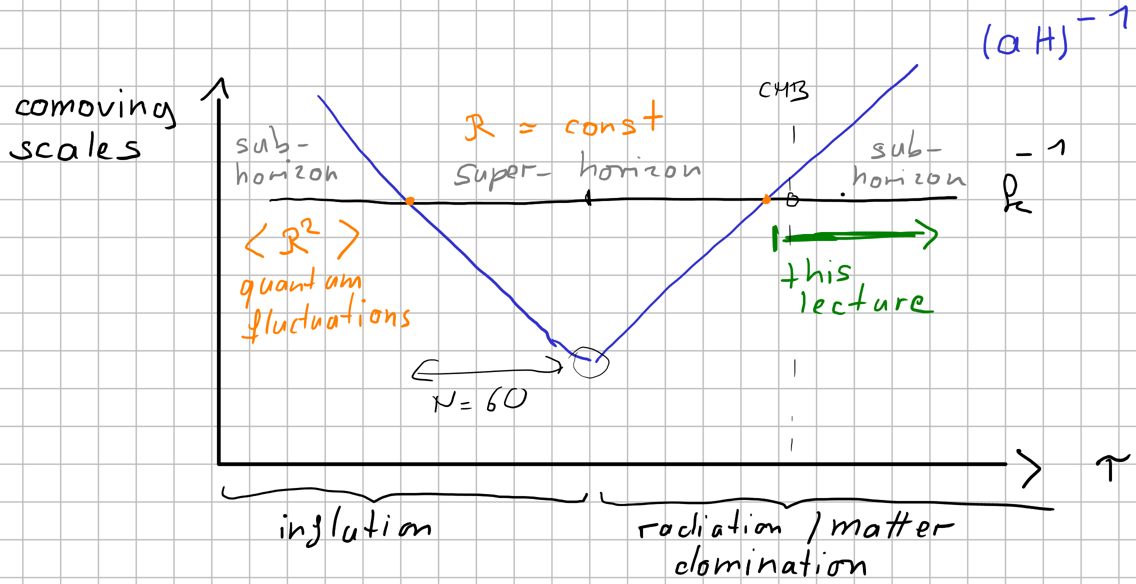
$$ds^2 = -(1+2\psi)\,dt^2 + 2a\,(\partial_i B - S_i)\,dt\,dx^i$$
$$+\, a^2\left[(1-2\Phi)\delta_{ij} + 2\partial_{ij}F + 2\partial_{(i}F_{j)} + h_{ij}\right]dx^i dx^j$$

$\to$ 4 scalars (Φ, ψ, B, F), 1 tensor (h_{ij}) (+ 2 vectors)

↳ • 2 of these = gauge dofs
  • $\mathcal{R}$ gauge invariant

- power spectra of scalar & tensor fluctuations generated during inflation

$$\Delta_{\mathcal{R}}^2 = (2\pi)^{-2}\frac{H_*^4}{\dot\phi^2}\ ,\qquad \Delta_t^2 = \frac{2}{M_p^2}\frac{H_*^2}{\pi^2}$$

- 

0) Computing (CMB) observables for a given $V(\phi)$

- Convenient parametrization of an approx. scale invariant spectrum:

amplitude : $A_s \equiv \Delta_R^2(k_{ref})$

spectral tilt: $n_s - 1 = \left.\frac{d \ln \Delta_R^2}{d \ln k}\right|_{k=k_{ref}}$

running : $\alpha_s = \left.\frac{d n_s}{d \ln k}\right|_{k=k_{ref}}$

$\vdots$

i.e. $$\Delta_R^2(k) = A_s \left(\frac{k}{k_{ref}}\right)^{n_s - 1 + \frac{1}{2}\alpha_s \ln\frac{k}{k_{ref}} + \dots}$$

- express in terms of slow-roll parameters

$3H^2 M_p^2 = V$

$$\Delta_R^2 = \frac{H_*^2}{(2\pi)^2}\frac{H_*^2}{\dot\phi_*^2} = \frac{H_*^2}{(2\pi)^2}\frac{1}{2\varepsilon} \approx \left.\frac{V(\phi)}{24\pi^2 \varepsilon_V}\right|_{\phi=\phi_{CMB}}$$

$$n_s - 1 = \frac{d \ln \Delta_R^2}{d \ln k} = \frac{d \ln \Delta_R^2}{d\phi}\frac{d\phi}{dN}\frac{dN}{d \ln k}$$

$k(N) = a_v H = k_0 e^{N-N_0}$

$\frac{\varepsilon_v}{V}\left(\frac{V'}{\varepsilon_v} - \frac{V\varepsilon_v'}{\varepsilon^2}\right)$

$= \frac{1}{\sqrt{2\varepsilon_v}}(-6\varepsilon_v + 2\eta_v)$

$dN = -H\,dt = -H/\dot\phi\, d\phi$

$= -\frac{1}{\sqrt{2\varepsilon_v}}d\phi$

$\approx (1 + \varepsilon_v)$

$$= -6\varepsilon_v + 2\eta_v \Big|_{\phi=\phi_{CMB}}$$

$$r = \frac{\Delta_t^2}{\Delta_R^2} = \frac{8\dot\phi_*^2}{M_p^2 H_*^2} = 16\varepsilon \simeq 16\varepsilon_V \Big|_{\phi=\phi_{CMB}}$$

1) Cosmological perturbation theory after inflation

dofs: $g_{\mu\nu}(t,\vec{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x})$

$T_{\mu\nu}(t,\vec{x}) = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(t,\vec{x})$ energy-momentum tensor

Consider scalar perturbations in Newtonian gauge, $k=0$

$ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Phi)dx^i dx_i \rightarrow 2$ dofs

$\delta T_{\mu\nu} \rightarrow 4$ dofs (per sector)

- $T^0{}_0 = \bar{\rho} + \delta\rho \rightarrow \delta \equiv \frac{\delta\rho}{\bar{\rho}}$ density perturbation
- $T^i{}_i = -(\bar{p} + \delta p) \rightarrow \delta p$ pressure perturbation
- $(\partial_i \partial^j - \frac{1}{3}\delta_i^j \Delta)\delta T^i{}_j = (\bar{\rho} + \bar{p})\,\sigma \rightarrow \sigma =$ anisotropic stress potential
- $\partial_i T^i{}_0 = (\bar{\rho} + \bar{v})\,\partial_i v^i \rightarrow \Theta = \partial_i v^i$ velocity divergence

$\Rightarrow \{\delta_x, p_x, \Theta_x, \sigma_x\}$ with $x = \begin{cases} \text{CDM} \\ \nu \\ \text{baryons} \\ \text{photons} \end{cases}$

$\mathcal{R} = \Psi - \frac{1}{3}\frac{\delta\rho}{\bar{\rho}+\bar{p}}$

perfect fluid (baryons, photons, [CDM])

$\left.\begin{array}{l} \text{isotropic} \rightarrow \sigma_x = 0 \\ \text{adiabatic} \rightarrow \delta p_x = \underbrace{c_{a,x}^2}_{\text{adiabatic sound speed}} \delta\rho_x \end{array}\right\} \rightarrow$ only 2 dof

2) Equations of motion

- energy momentum conservation

$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^\mu{}_\nu - \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha$

$\rightarrow$ 2 scalar equations

a) continuity equation

$dt = a\,d\eta$

$0 < \eta < \infty$

$$\boxed{\partial_\eta \delta_x = -3aH\left(\frac{\delta p_x}{\bar{\rho}_x} - \frac{\bar{p}_x}{\bar{\rho}_x}\delta_x\right) - \left(1+\frac{\bar{p}_x}{\bar{\rho}_x}\right)(\theta_x - 3\phi')}$$

b) Euler equation

$\to \sigma$

$$\boxed{\partial_\eta v_x^i = -\left(aH + \frac{\bar{p}_x'}{\bar{\rho}_x+\bar{p}_x}\right)v_x^i - \frac{1}{\bar{\rho}_x+\bar{p}_x}\left(\partial^i \delta p_x - \partial_j \Pi^{ij}\right) - \partial^i \psi}$$

$\to$ 2 eqs + Einstein equs

$H = \frac{\dot{a}}{a}$

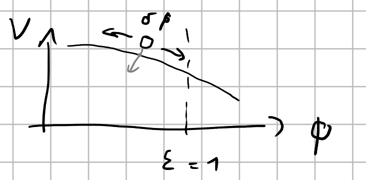
- $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$

$(00) \to \nabla^2 \Phi - 3(aH)(\Phi' + aH\psi') = 4\pi G a^2 \delta\rho$ Poisson eq

$(ij) \to \nabla^2(\Phi - \psi) = -8\pi G a^2(\bar{\rho} + \bar{p})\,\sigma \simeq 0$ for no ν

$\hookrightarrow \psi = \phi$

$(ii) \to \Phi'' + 3(aH)\Phi' + 2(\partial_t(aH) + (aH)^2)\Phi \simeq 4\pi G a^2 \delta p_{tot}$



3) Adiabatic initial conditions

- super-horizon modes
- inflation = local single clock perturbation

$$\rho_x(\eta, \vec{x}) = \bar{\rho}_x(\eta + \delta\eta(\vec{x}))$$
$$= \bar{\rho}_x(\eta) + \underbrace{\bar{\rho}_x'(\eta)\,\delta\eta(\vec{x})}_{\equiv\, \delta\rho_x(\eta,\vec{x})}$$

F1 + F2: $\bar{\rho}_x' = -3\frac{a'}{a}(\bar{\rho}_x + \bar{p}_x)$

$$\Rightarrow \frac{\delta\rho_x}{\bar{\rho}_x + \bar{p}_x} = -3\frac{a'}{a}\frac{\delta\rho_x}{\bar{\rho}_x} = -3\frac{a'}{a}\,\delta\eta(\vec{x}) \quad \forall_x$$

$$w_x = \frac{\bar{p}_x}{\bar{\rho}_x} \quad \Rightarrow \quad \boxed{\frac{\delta_x}{1+w_x} = \frac{\delta_y}{1+w_y} \quad \forall x, y}$$

$$\gamma, \nu \rightarrow w_x = 1/3 \quad , \quad b, CDM \rightarrow w_x = 0$$

$$\Rightarrow \boxed{\delta_b = \delta_{CDM} = \frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma}$$

"Adiabatic initial conditions"

? statistical properties of some observable $A(\eta, \vec{x})$

$$\langle A(\eta, \vec{k})\, A^*(\eta, \vec{k}')\rangle = \delta(\vec{k}-\vec{k}')\, \underbrace{T_A^2(\eta, k)}_{\substack{\text{transfer function:} \\ \mathcal{R} \rightarrow A \\ \eta_* \rightarrow \eta}}\, \underbrace{P_{\mathcal{R}}(k)}_{\substack{\text{inflation} \\ \eta = \eta_*}}$$