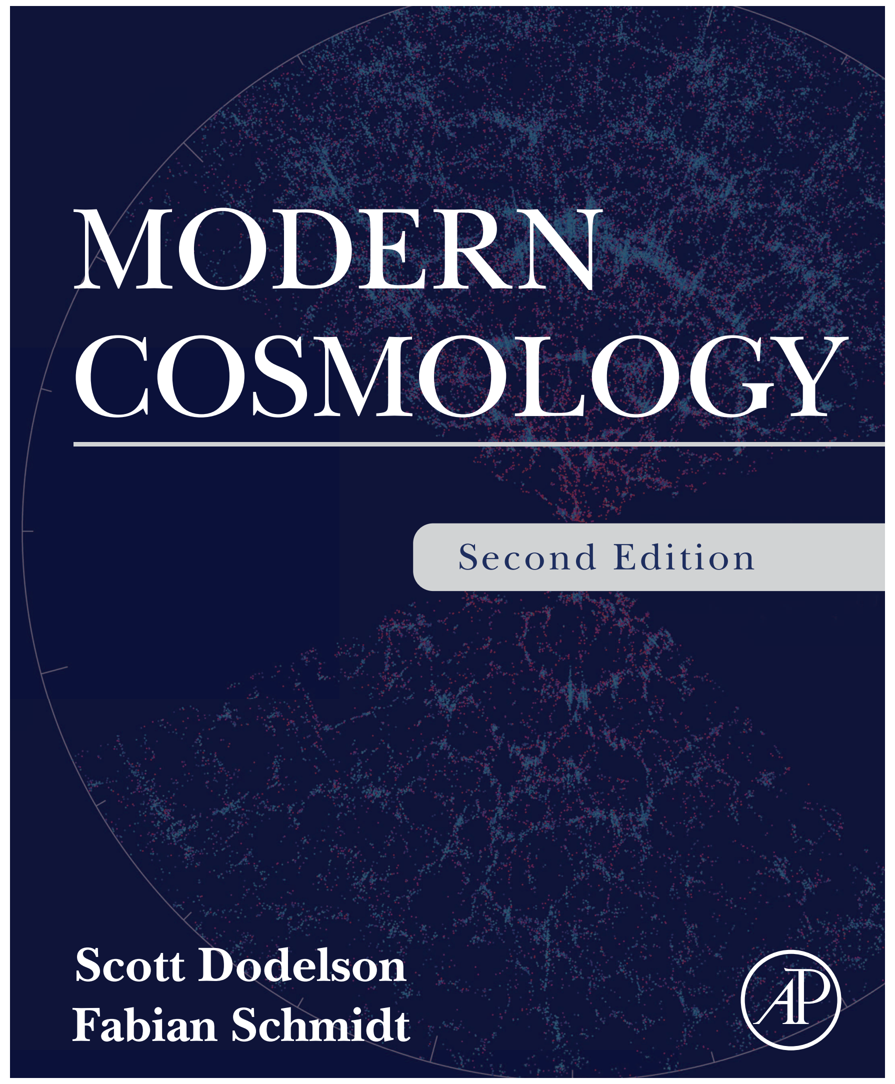
Exercises on Structure Formation

Note: new exercises might be added during the course of the week…

Including material from:



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12.0: take the moments of the Boltzmann equation to derive the fluid equations. Use:

$$\frac{1}{m}\left\langle p^i p^j \right\rangle_{f_\mathrm{m}} = \rho_\mathrm{m} u^i_\mathrm{m} u^j_\mathrm{m} + \sigma^{ij}_\mathrm{m}. \qquad \text{Eq (12.17)}$$

12.1 Show that the stress tensor σ^{ij}_m [Eq. (12.17)] vanishes for a "cold" distribution function of the form Eq. (12.9).

12.2 Use Eq. (12.23) to derive an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}_\mathrm{m}$ of the matter velocity. Show that no vorticity is generated if it is absent in the initial conditions. How does an initial vorticity evolve in time at linear order?

12.3 Fill in the missing steps of the transformation of the Euler–Poisson system into Fourier space, Eq. (12.31).

12.4 Use the equation for the linear growth factor Eq. (8.75) to prove Eq. (12.32). Note that this relation holds for any smooth dark energy model. Next, use this to transform Eq. (12.31) into Eqs. (12.33)–(12.34).

Equation numbers refer to *Modern Cosmology, second edition.*
However, the relevant ones should be indicated on the lecture slides as well

12.5 Use the solution in Eq. (12.40) to show that the NLO contribution in Eq. (12.42) is given by Eq. (12.48). Derive and use the relations

$$F_2(\boldsymbol{k}, -\boldsymbol{k}) = 0,$$
$$F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = F_n(-\boldsymbol{k}_1, \cdots, -\boldsymbol{k}_n). \tag{12.105}$$

Evaluate the terms numerically. For $P^{(13)}$, the expression of the kernel given in Makino et al. (1992) is useful. For $P^{(22)}$, care needs to be taken when $\boldsymbol{k} - \boldsymbol{p}$ becomes close to zero. Bertschinger and Jain (1994) provide a neat decomposition of the integral which is numerically robust.

12.6 Derive the leading contribution to the matter bispectrum, Eq. (12.51). How does this look in the diagram form of Fig. 12.3?

12.7 In Sect. 12.2, we developed perturbation theory based on the density field. An alternative, *Lagrangian* approach is based on the equations of motion for N-body "particles," Eq. (12.57). In this exercise, you will derive the lowest-order result, known as *Zel'dovich approximation*. The solution to Eq. (12.57) is a particle trajectory $\boldsymbol{x}(\eta)$. We write this as

$$\boldsymbol{x}(\eta) = \boldsymbol{q} + \boldsymbol{s}(\boldsymbol{q}, \eta), \tag{12.106}$$

where $\boldsymbol{q}$ is the initial position at $\eta = 0$, when all perturbations were negligible. Hence $\boldsymbol{s}(\boldsymbol{q}, 0) = 0$. Rewrite Eq. (12.57) as an equation for $\boldsymbol{s}$. Now expand to linear order in $\boldsymbol{s}$. Solve the equation by using the solution of the Poisson equation for Ψ at linear order. Your result should relate $\boldsymbol{s}^{(1)}(\boldsymbol{k}, \eta)$ to $\delta^{(1)}(\boldsymbol{k}, \eta)$. This result can be used to obtain the initial small displacements of particles to start an N-body simulation. We also need their initial momenta p_c^i. Derive these in terms of the displacement as well.

12.8 Derive an expression for the enclosed mass $M(< r)$ for the NFW profile Eq. (12.62). Replace r_s with the concentration c_Δ. Use this to derive R_Δ for a given mass M_Δ and concentration, and solve for ρ_s. You now have a reasonably accurate expression for the density profile of a halo of mass M_Δ and concentration c_Δ. Plot the profile for a halo of mass $M_{200} = 10^{12} M_\odot$ ($\Delta = 200$), and for concentrations $c_{200} \in \{4, 8, 16\}$. That is, make the plot for Sect. 12.4.1 that we were too lazy to create!

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12.9 Derive the spherical collapse threshold δ_{cr} and the virial overdensity Δ_{vir} by solving Eq. (12.67) without considering Λ. Follow these steps:

(a) Show that Eq. (12.67) can be rewritten as

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}\bar{\rho}_i[1+\delta_i]\left(\frac{r_i}{r}\right)^3 \tag{12.107}$$

where r_i, $\bar{\rho}_i$ are, respectively, the radius of the spherical region and the background matter density at the initial time, and δ_i is the initial overdensity.

(b) Show that, when the initial expansion rate is given by $\dot{r}_i = H_i r_i(1-\delta_i/3)$, the maximum radius r_{ta} (the turn-around radius) that the spherical region reaches is given by

$$r_{ta} = r_{ta} = \frac{3}{5}\left(\frac{1+\delta_i}{\delta_i}\right) r_i. \tag{12.108}$$

(c) Show that the parametric solution (cycloid) of Eq. (12.68) is a solution of Eq. (12.107). What is t_{ta} in terms of the initial conditions, r_i, δ_i, $\bar{\rho}_i$?

(d) Find the expression for the nonlinear overdensity $\delta(\theta)$. What is the nonlinear density contrast at the time of turn around? Plot $\delta(t)$ as a function of $\delta^{(1)}(t)$, the initial overdensity evolved forward using the linear growth factor. Derive the expansion of $\delta(\delta^{(1)})$ to third order in $\delta^{(1)}$.

(e) Assume that, by some magic (which we call violent relaxation), the object virializes. Find the virial radius in terms of the turn around radius. Using that, give the density contrast $\Delta_{vir} \equiv 1+\delta(t_{vir})$ expected after virialization. Assuming that collapse is completed at $\theta = 2\pi$ [that is, $t_{vir} = t(\theta = 2\pi)$], what is the value of $\delta^{(1)}(t)$ at collapse? This is the collapse threshold δ_{cr}.

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12.10 Derive the correlation function of thresholded regions Eq. (12.82) in the linear density field.

(a) Define the scaled density field $\nu(\boldsymbol{x}) \equiv \delta_R^{(1)}(\boldsymbol{x})/\sigma(R)$ (note that this is a *field*, and not to be confused with the parameter $\nu = \delta_{\rm cr}/\sigma(R)$, which we shall indicate with $\nu_{\rm cr}$ in this exercise). Show that $\nu(\boldsymbol{x})$ at an arbitrary fixed location follows the normal distribution

$$p(\nu) = \frac{1}{\sqrt{2\pi}} e^{-\nu^2/2}, \qquad (12.109)$$

and that the joint distribution of ν_1, ν_2, where $\nu_i \equiv \nu(\boldsymbol{x}_i)$, is a bivariate Gaussian

$$p(\nu(\boldsymbol{x}_1), \nu(\boldsymbol{x}_2)) = \frac{1}{2\pi\sqrt{1-\xi_{12}^2/\sigma^4(R)}} \exp\left[-\frac{1}{2}(\nu_1,\nu_2)^\top \mathrm{C}^{-1}(\nu_1,\nu_2)\right] \qquad (12.110)$$

$$\text{where} \quad \mathrm{C} = \begin{pmatrix} 1 & \xi_{12}/\sigma^2(R) \\ \xi_{12}/\sigma^2(R) & 1 \end{pmatrix} \qquad (12.111)$$

and $\xi_{12} = \xi_R^{(1)}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) = \left\langle \delta_R^{(1)}(\boldsymbol{x}_1)\delta_R^{(1)}(\boldsymbol{x}_2)\right\rangle$ is the correlation function of the linear, smoothed density field.

(b) Using this result, show that the one-point probability (or volume fraction) becomes

$$p(\delta_R^{(1)} > \delta_{\rm cr}) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\nu_{\rm cr}}{\sqrt{2}}\right). \qquad (12.112)$$

Here, $\nu_{\rm cr} \equiv \delta_{\rm cr}/\sigma(R)$ and the complementary error function is defined in Eq. (C.31). Obtain the corresponding expression for the joint probability

$$p\left(\delta_R^{(1)}(\boldsymbol{x}_1) > \delta_{\rm cr}, \delta_R^{(1)}(\boldsymbol{x}_2) > \delta_{\rm cr}\right), \qquad (12.113)$$

and for $\xi_{\rm thr}(r)$ from Eq. (12.82). Notice that one of the two integrals can be done analytically.

(c) Using the fact that the matter correlation function goes to zero at large r, expand your result in the small quantity $\xi(r)$. Show that the first two terms can be written as Eq. (12.83), and derive the expression for b_1 and b_2, as well as their limiting form for very rare halos, $\nu_{\rm cr} \gg 1$.

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12.11 Continue the expansion in Eq. (12.80) to second order in δ_ℓ. The second-order bias is defined by

$$\delta_{\mathrm{h},\ell} = b_1 \delta_\ell + \frac{1}{2} b_2 \delta_\ell^2 + \cdots . \tag{12.114}$$

What is the expression for b_2 in terms of $\sigma(M, z)$ and $f(\nu)$? Derive $b_2(\nu)$ for the Press–Schechter mass function Eq. (12.73).

12.12 Derive the second-order perturbation theory kernel for the galaxy density $\delta_{\mathrm{g}}^{(2)}$.

(a) Define the scaled tidal field through

$$K_{ij}(\boldsymbol{x}, \eta) = \frac{1}{4\pi G a^2(\eta)} \left[\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right] \Psi(\boldsymbol{x}, \eta). \tag{12.115}$$

Use this definition to relate K_{ij} is to the matter density in real and Fourier space.

(b) Begin with the real-space expression of the second-order galaxy density,

$$\delta_{\mathrm{g}}^{(2)}(\boldsymbol{x}, \eta) = b_1 \delta^{(2)} + \frac{1}{2} b_2 (\delta^{(1)})^2 + b_{K^2} K_{ij}^{(1)} K^{(1)ij}, \tag{12.116}$$

where on the right-hand side all fields are evaluated at $(\boldsymbol{x}, \eta)$, and the bias parameters b_1, b_2, b_{K^2} are defined at η. Why does the tidal field only appear at second order and in this particular combination? Now pull out the time dependence contained in the growth factors, and Fourier transform Eq. (12.116) to arrive at Eq. (12.87).

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