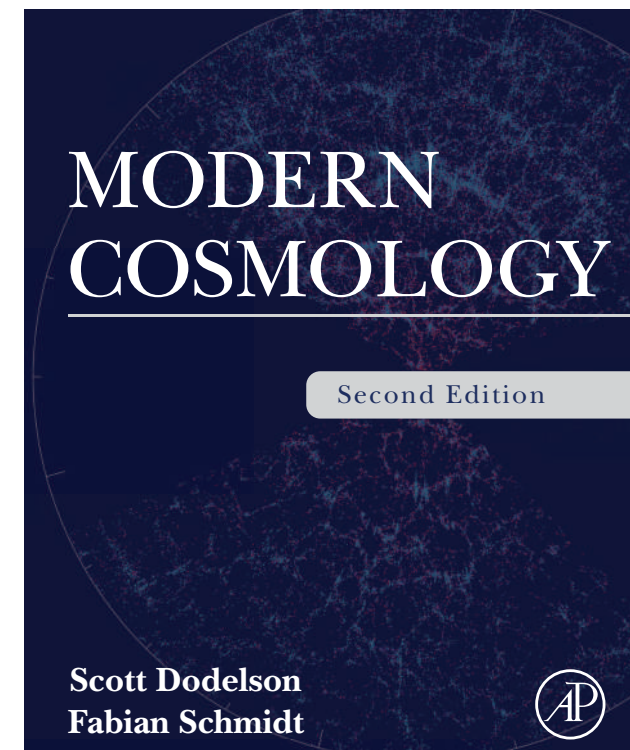


# Structure Formation

## Lecture I

Fabian Schmidt  
MPA

All figures taken from *Modern Cosmology, Second Edition*, unless otherwise noted



# Motivation

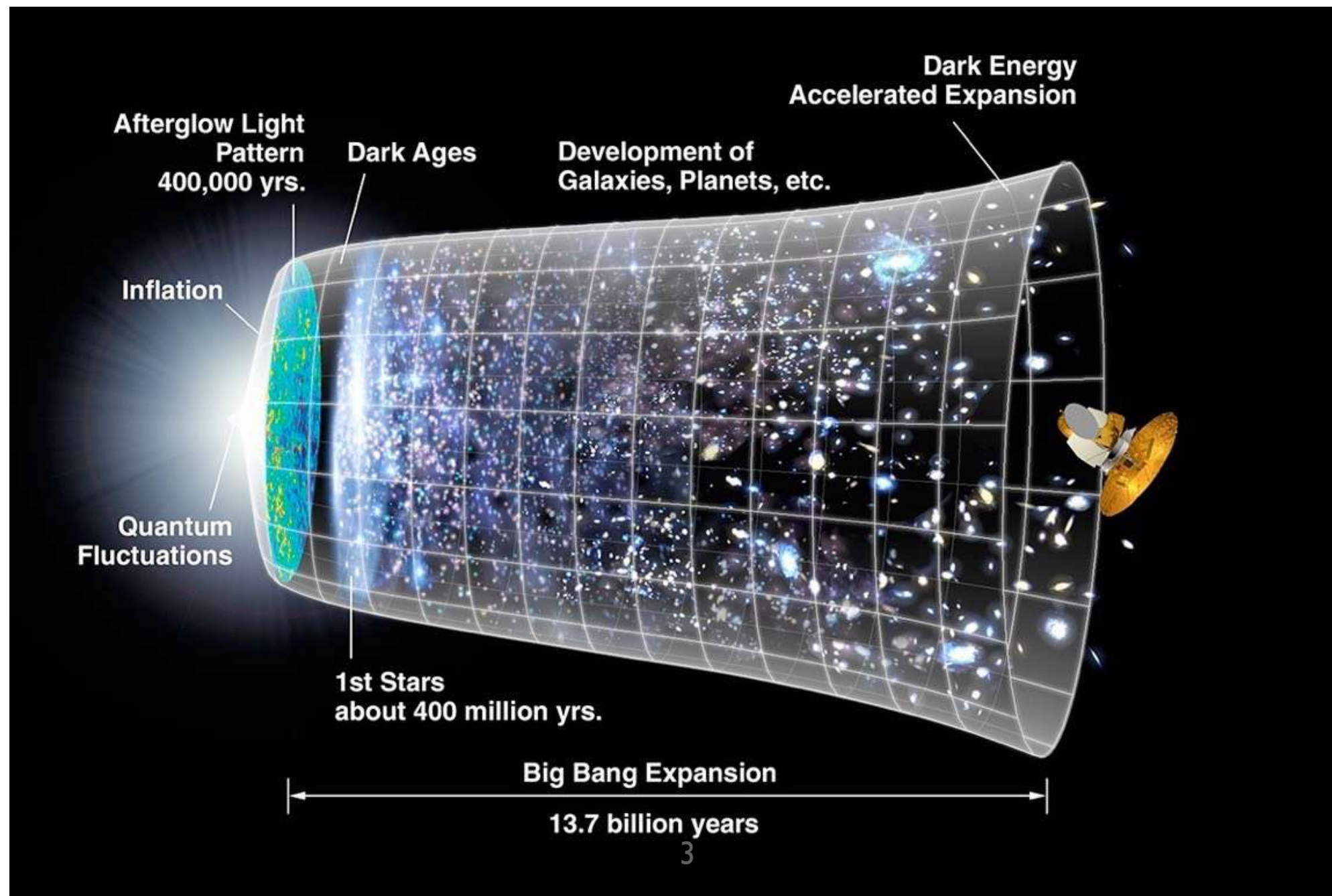
- The **large-scale structure (LSS)** is historically one of the key probes of cosmology

Peebles; Efstathiou+ '90 predicted a positive cosmological constant  $\Lambda$  from LSS observations

- Now, we are really in a golden age of LSS with plenty of experiments under way:  
**eBOSS, DES, DESI, PFS, SphereX, Euclid, WFIRST, ...**

# Motivation

- Using large-scale structure, we can learn about



# Motivation

- **Inflation:** reconstruct the properties of the initial conditions, and look for gravitational waves

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- **Inflation**: reconstruct the properties of the initial conditions, and look for gravitational waves
- **Dark Energy and Gravity**: the growth of structure depends sensitively on the **expansion history** of the Universe, and the nature of **gravity**

Growth equation:  $D'' + aH D' = 4\pi G \bar{\rho} D$

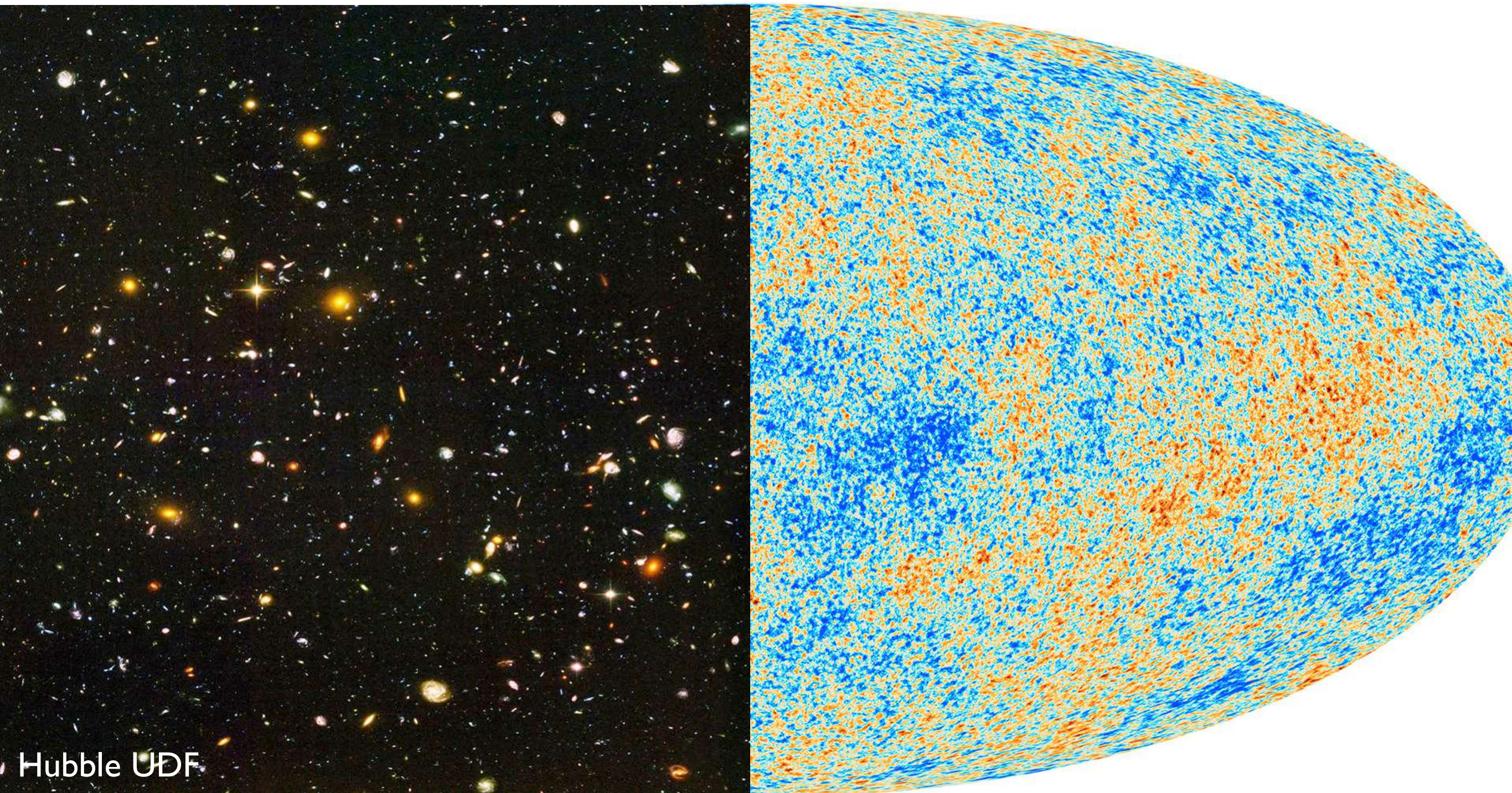
# Motivation

- **Inflation:** reconstruct the properties of the initial conditions, and look for gravitational waves
- **Dark Energy and Gravity:** the growth of structure depends sensitively on the **expansion history** of the Universe, and the nature of **gravity**

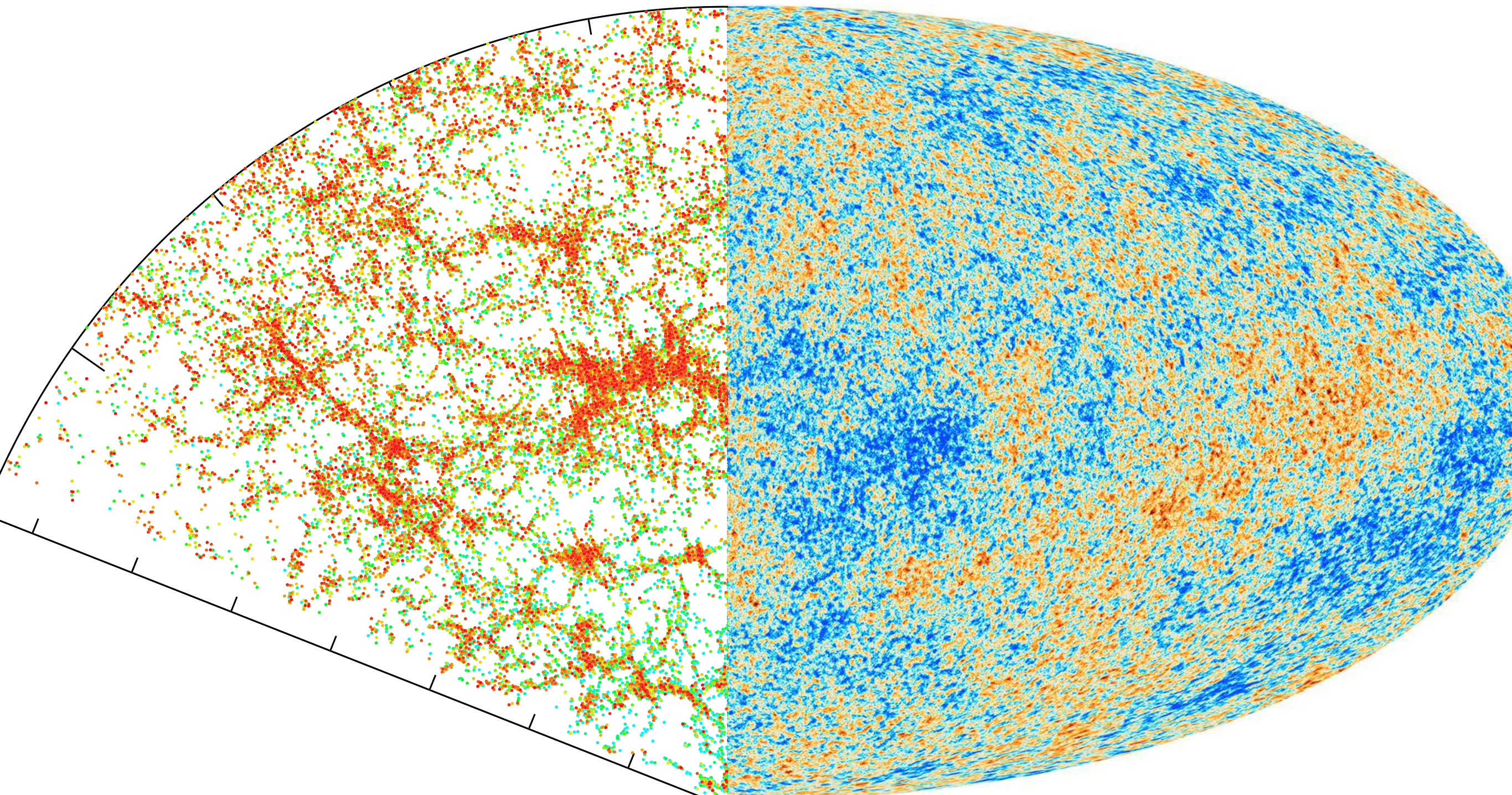
Growth equation:  $D'' + aH D' = 4\pi G \bar{\rho} D$

- **Dark Matter:** how “cold” is cold dark matter ?  
What is the sum of neutrino masses ?

Challenge: unlike the CMB,  
every data point is nonlinear!



**Challenge: unlike the CMB,  
every data point is nonlinear!**



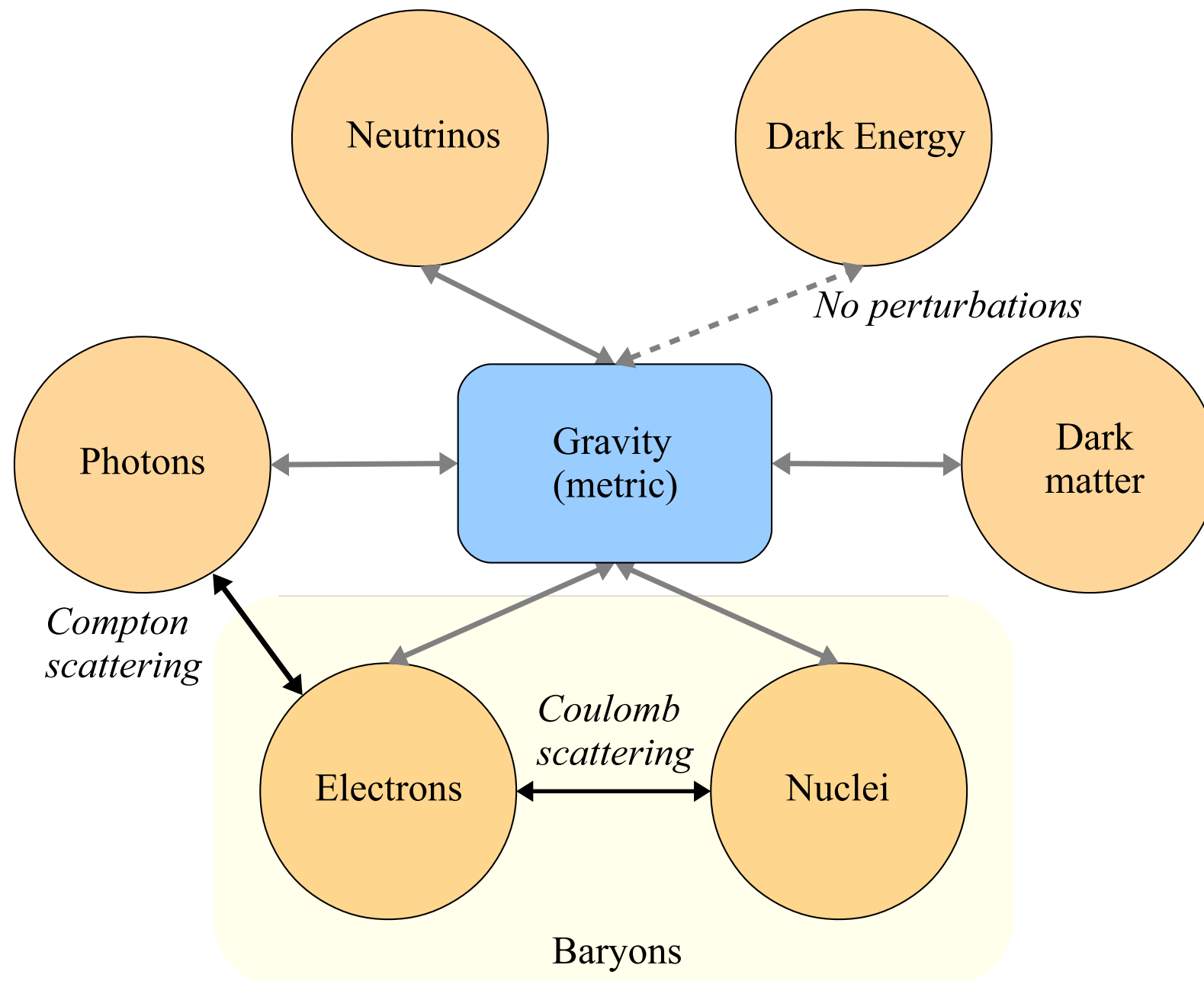
# Preliminaries

- In bulk of lectures, we'll be assuming “vanilla” Euclidean (flat)  $\Lambda$ CDM cosmology
  - Gaussian, adiabatic, almost scale-invariant initial perturbations
  - Dark Energy equation of state  $w=-1$ , although results hold for general smooth DE as well
  - Mostly neglect effect of massive neutrinos
- We will (hopefully) discuss the effect of going beyond these assumptions in the 5th lecture

# Preliminaries

Gravity  
(Einstein eq.)

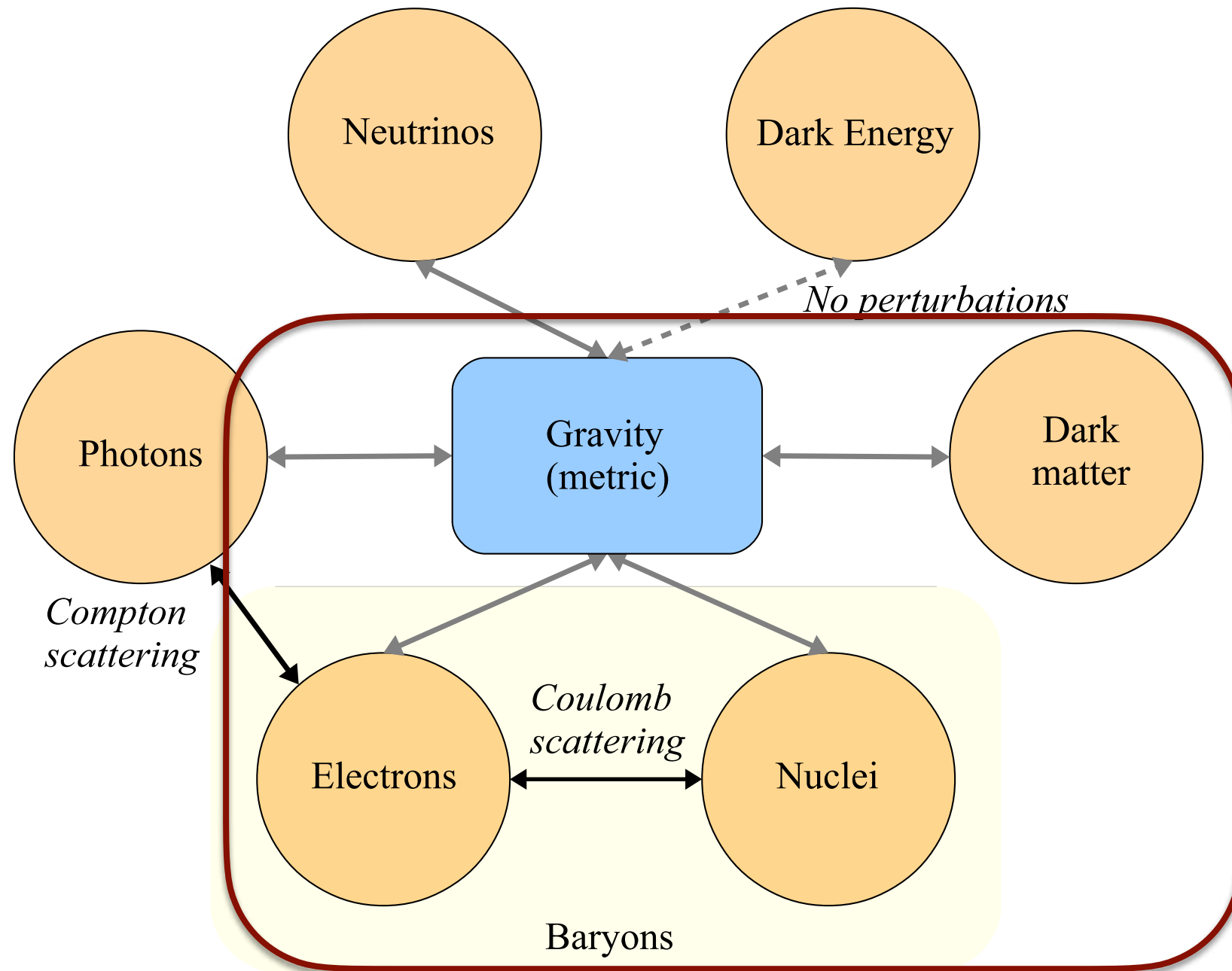
$\leftrightarrow$  Matter  
Boltzmann equations



# Preliminaries

Gravity  
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<-> Matter  
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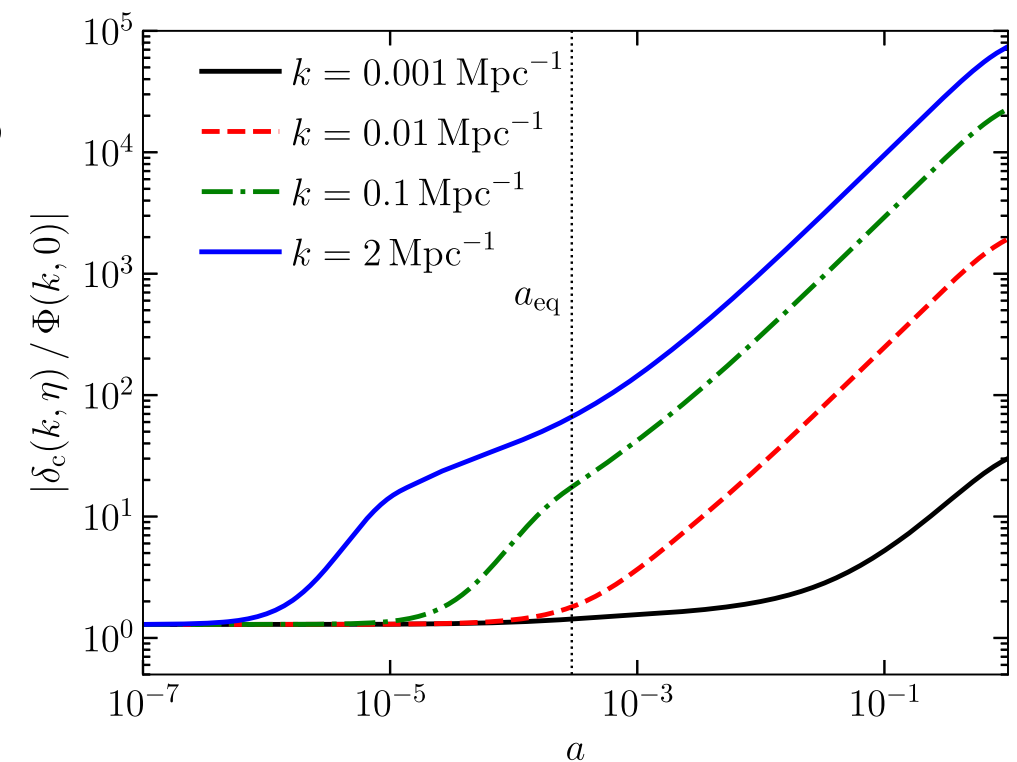
# Preliminaries

- Baryons and CDM are “cold”: the constituent particles are non-relativistic
- Most of structure formation happens well within the Hubble horizon: sub horizon approximation
- These two facts simplify equations substantially!
- Can often use our intuition for Newtonian gravity

# Preliminaries

- Will not study early universe evolution here
- Early evolution starts when perturbation “enters the horizon”
- Evolution depends on whether this happens in radiation domination (slower growth) or matter domination (faster growth)
- Small-scale modes enter horizon earlier

Evolution of modes of different wavelengths at early times ( $k=2\pi/\lambda$ )



Cold dark matter component only

# Notation

$$ds^2 = -(1 + 2\Psi(\boldsymbol{x}, t))dt^2 + a^2(t)(1 + 2\Phi(\boldsymbol{x}, t))d\boldsymbol{x}^2$$

- Comoving coordinates:

$$d\boldsymbol{r} = a(t)d\boldsymbol{x}$$

- Conformal time:

Primes denote derivative w.r.t conformal time

$$d\eta = \frac{dt}{a(t)} = \frac{da}{a^2 H(a)} = \frac{d \ln a}{a H(a)}.$$

- Comoving distance:

$$d\chi = -d\eta = \frac{dz}{H(z)}.$$

- Particle velocity/momentum:  $\boldsymbol{v} = \frac{\boldsymbol{p}}{m} = a \frac{d\boldsymbol{x}}{dt} = \boldsymbol{x}'$

- Fluid velocity; divergence:

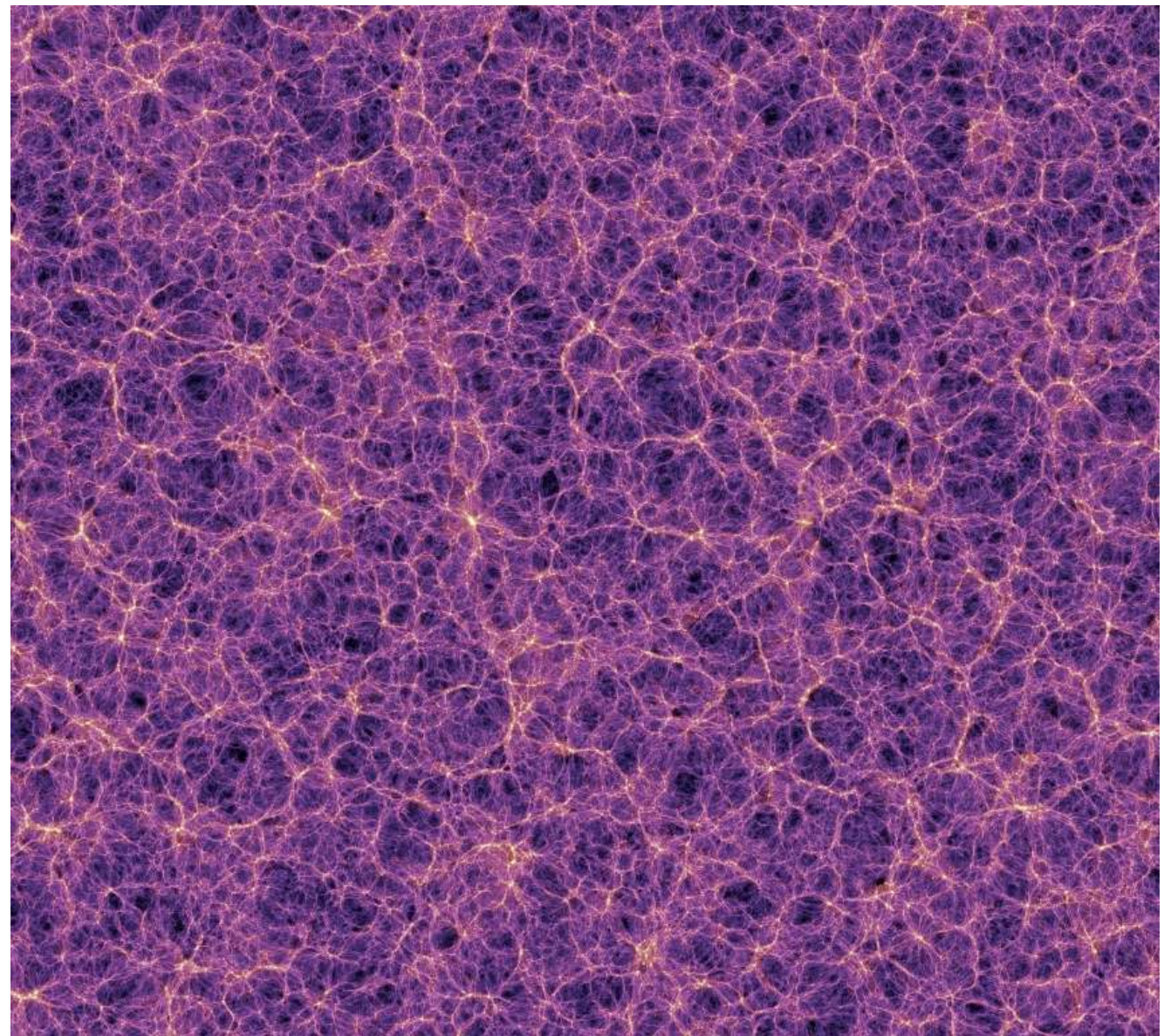
$$\boldsymbol{u}; \quad \theta = \partial_i u^i$$

- Gravitational potential:

$$\Psi$$

# Cold Dark Matter cosmology in a nutshell

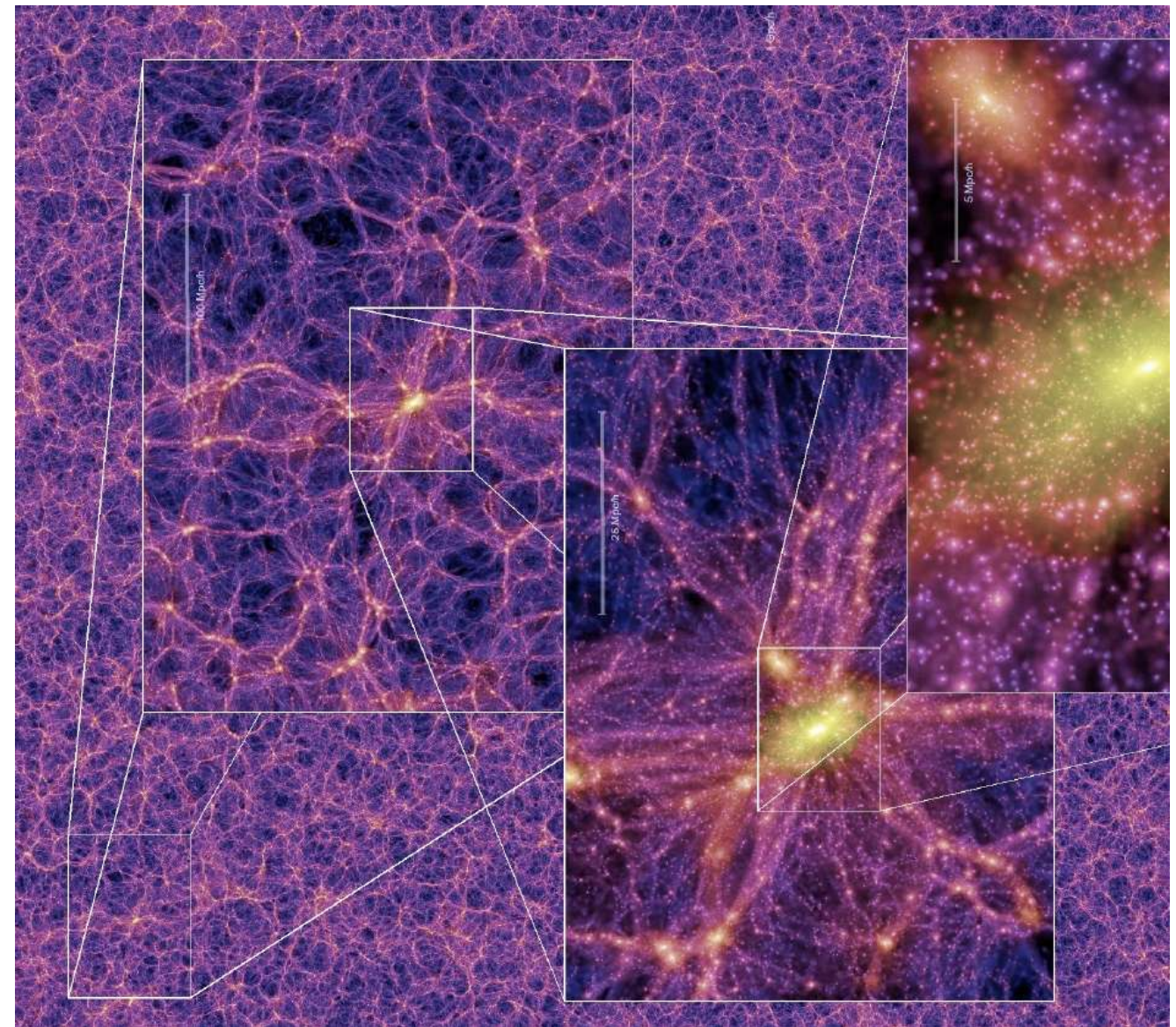
- Large-scale fluctuations are small (still linear today)
- Structure forms *hierarchically from small to large scales*
- *Perturbative expansion* in fluctuations on large scales
- Simulations of large volumes can assume background cosmology



Millennium simulation / MPA

# Cold Dark Matter cosmology in a nutshell

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Millennium simulation / MPA

# How do we compare theory with data?

- Assume we observe the matter density field  $\rho(\boldsymbol{x}) = \bar{\rho}[1 + \delta(\boldsymbol{x})]$   $\delta$ : fractional matter density perturbation
- Given **cosmological parameters**  $\theta$ , theory predicts
  1. Statistics of initial conditions (Gaussian)
  2. How a given  $\delta_{\text{in}}(\boldsymbol{x})$  evolves into the final density field  $\delta$
- In cosmology, we are always dealing with statistical fields!

# Characterizing Statistical Field

- Consider  $\delta(\mathbf{x})$ , and its Fourier-space version  $\delta(\mathbf{k})$
- Simplest statistical field: the field values at each point are independent Gaussian random variables (with vanishing mean)
- In cosmology, we often encounter these simplest fields - where we have independent Fourier modes
- Statistics of field is completely described in terms of the variance of the Fourier modes, as a function of  $k$ : the power spectrum

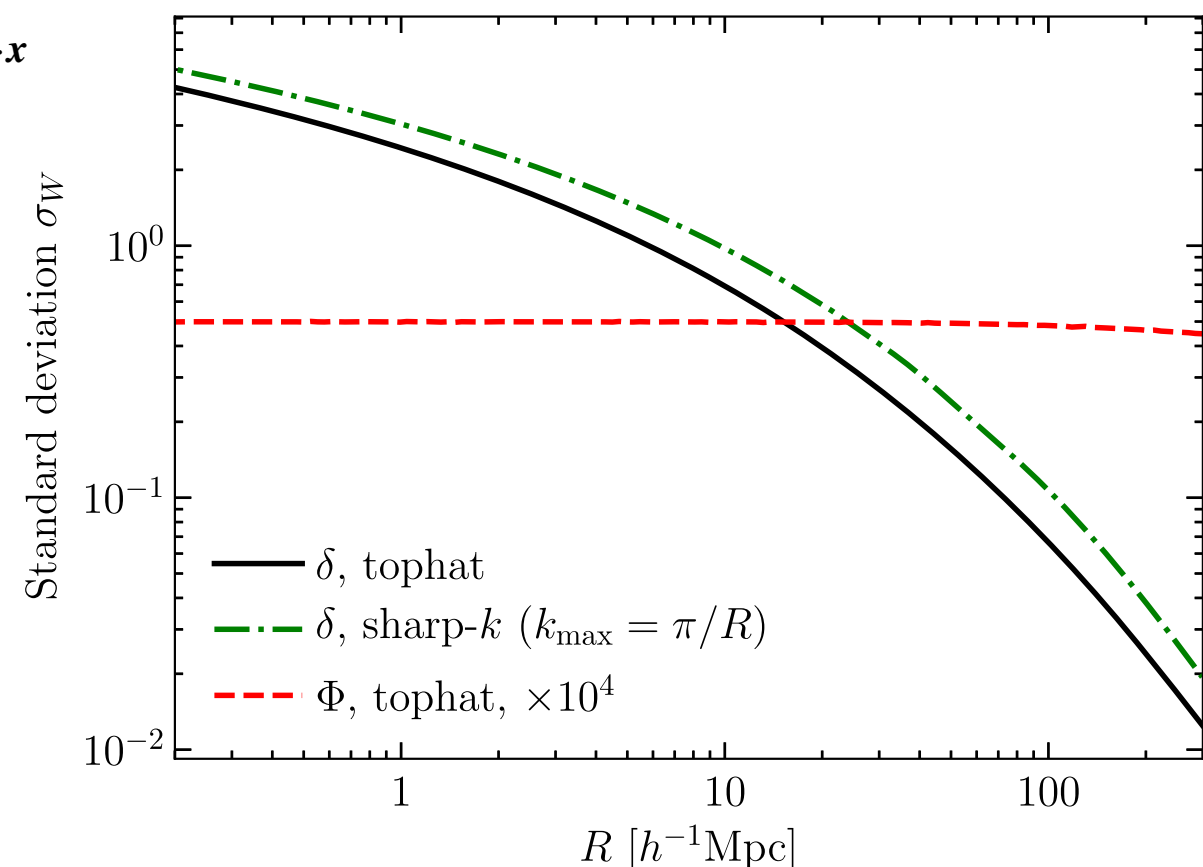
$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k)$$

# Characterizing Statistical Field

- So let's characterize large-scale matter density field
- Consider variance of matter density field filtered on different scales:

$$\begin{aligned}
 \sigma_W^2 &\equiv \langle (\delta_W)^2(\mathbf{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \delta_W(\mathbf{k}) \delta_W^*(\mathbf{k}') \rangle e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
 &= \int \frac{d^3k}{(2\pi)^3} P_L(k) |W(k)|^2 \\
 &= \frac{1}{2\pi^2} \int d\ln k k^3 P_L(k) |W(k)|^2.
 \end{aligned}$$

- Variance is small for large smoothing scales

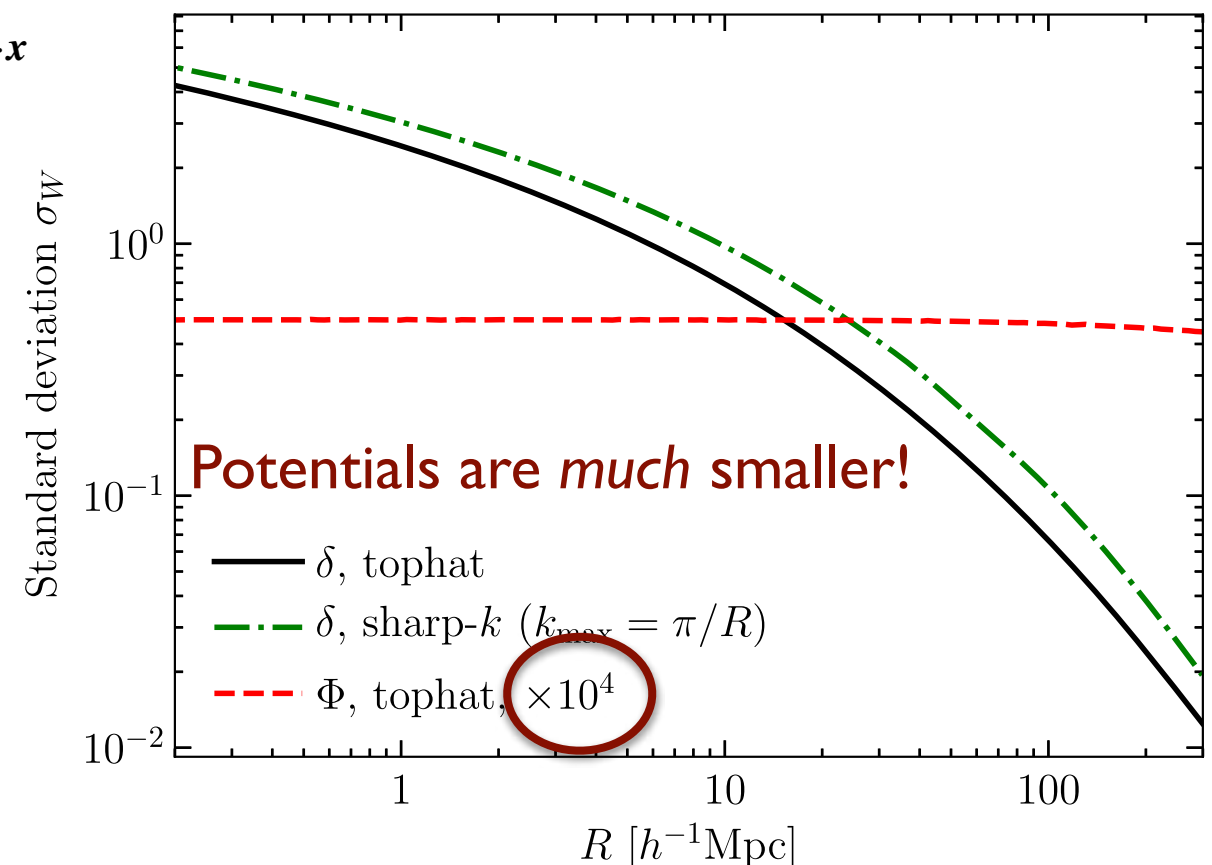


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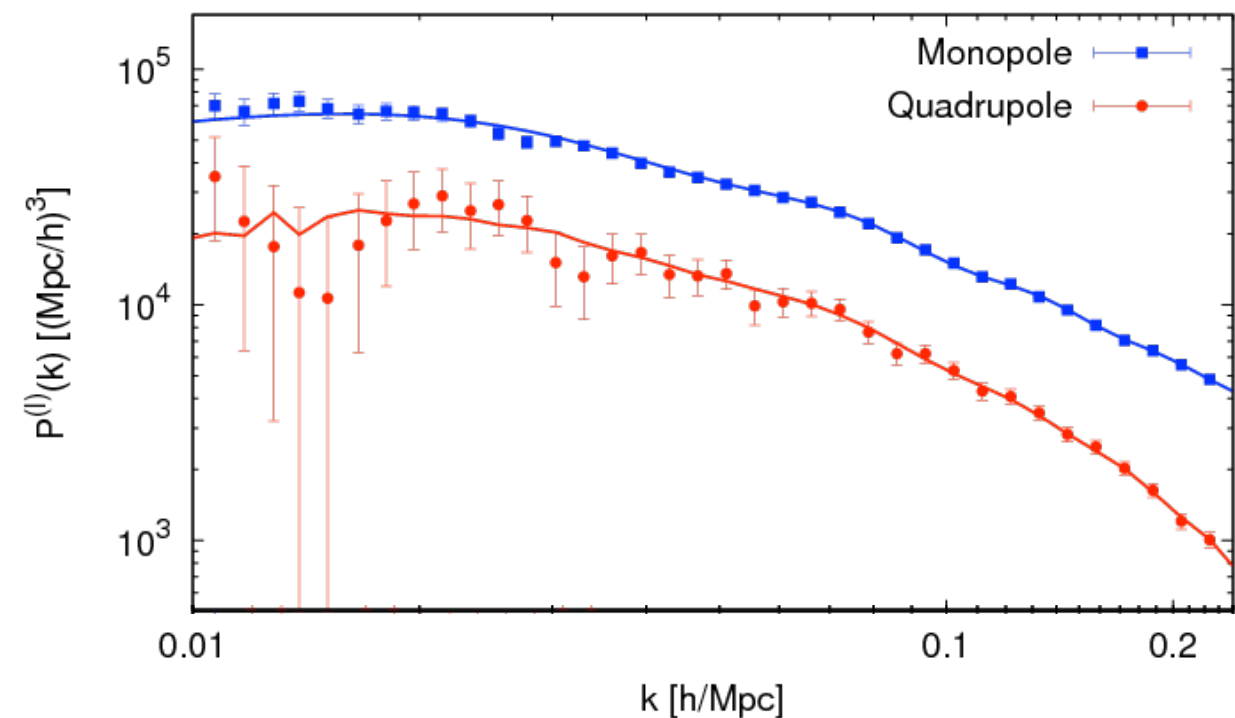
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# How do we compare theory with data?

- Goal: compute power spectrum of matter and galaxies
- And also other statistics of LSS

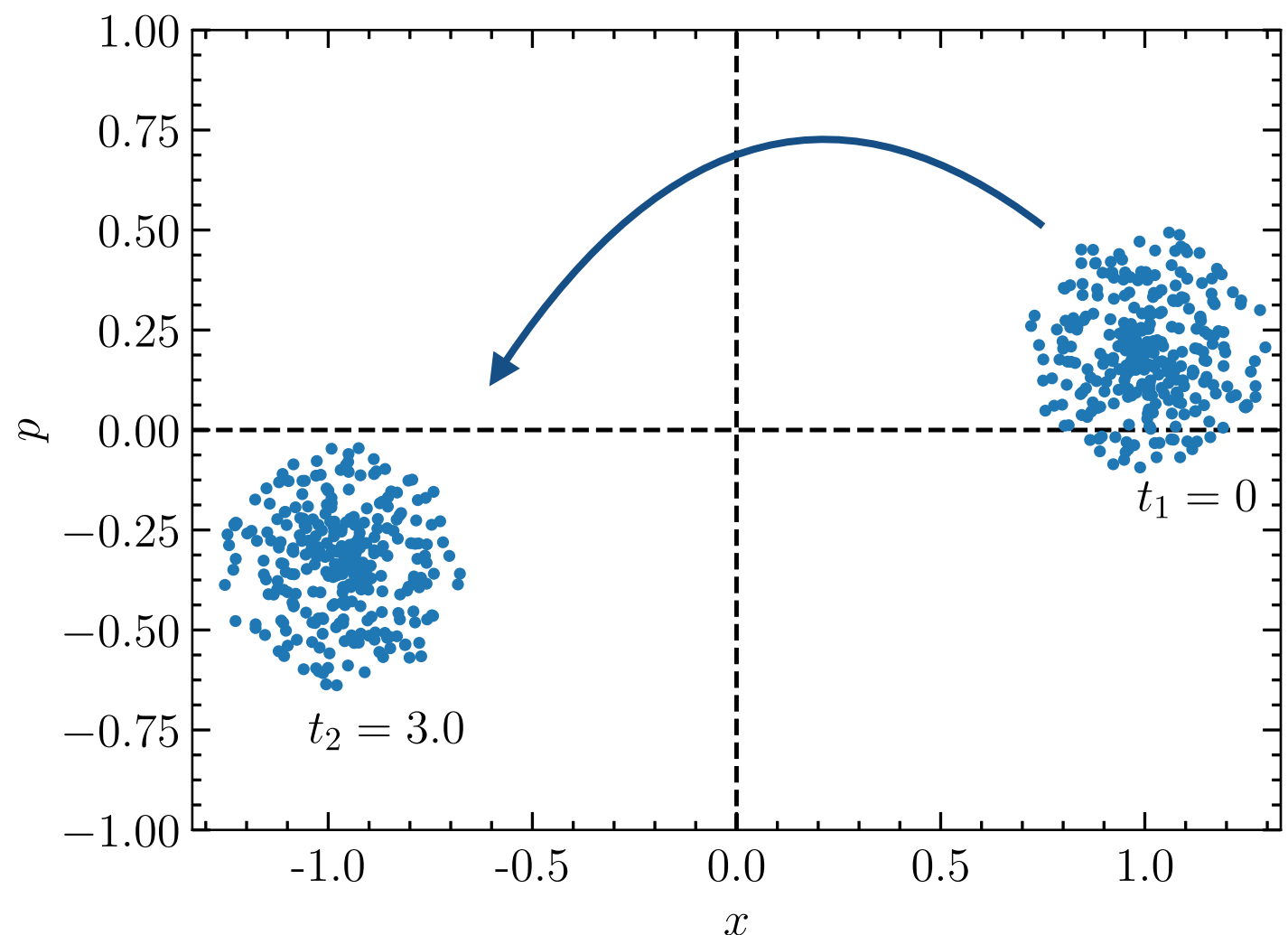
Galaxy Power spectrum measured in BOSS  
CMASS ( $z_{\text{eff}}=0.57$ )



$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k)$$

# The Boltzmann equation for cold, collisionless matter

- Fundamental quantity: distribution function  $f_m(\mathbf{x}, \mathbf{p}, t)$
- Boltzmann equation describes its evolution
- Dark matter: no interactions! Baryons: neglect interactions...
- Then, can lump dark matter and baryons together



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Geodesic equations: just Newtonian plus factors of  $a$

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{p^i}{am} \\ \frac{dp^i}{dt} &= -Hp^i - \frac{m}{a} \partial_i \Psi \end{aligned}$$

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Supplemented with the Poisson equation for the gravitational potential:

$$\nabla^2 \Psi = \frac{3}{2} \Omega_m(\eta) (aH)^2 \delta_m.$$

00-component of Einstein eq. in the subhorizon limit

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$$\frac{df_m}{dt} = \frac{\partial f_m}{\partial t} + \frac{\partial f_m}{\partial x^j} \frac{p^j}{ma} - \frac{\partial f_m}{\partial p^j} \left[ H p^j + \frac{m}{a} \frac{\partial \Psi}{\partial x^j} \right] = 0.$$

$$\nabla^2 \Psi = \frac{3}{2} \Omega_m(\eta) (aH)^2 \delta_m.$$

These equations will govern almost everything in these lectures!

# The Boltzmann equation for cold, collisionless matter

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Initial conditions: cold

$$f_m(\mathbf{x}, \mathbf{p}, t) = \frac{\rho_m(\mathbf{x}, t)}{m} (2\pi)^3 \delta_D^{(3)}(\mathbf{p} - m\mathbf{u}_m(\mathbf{x}, t)) \quad \text{Eq (12.9)}$$

$\Leftrightarrow$  no velocity dispersion

# Taking moments of the Boltzmann equation

- Boltzmann equation: 6+1 dim; plus we need to integrate  $f_m$  to obtain  $\delta$  for  $\Psi$
- Extremely difficult to solve. Let's try different approach: taking moments
- That means we integrate the equation (multiplied by  $p$ ,  $p^2$ ) over  $d^3p$

# Taking moments of the Boltzmann equation

- Define:
- Zeroth moment yields density:
- First moment yields bulk velocity:

$$\langle A \rangle_{f_m}(\mathbf{x}, t) \equiv \int \frac{d^3 p}{(2\pi)^3} A(\mathbf{x}, \mathbf{p}, t) f_m(\mathbf{x}, \mathbf{p}, t)$$

$$\langle 1 \rangle_{f_m}(\mathbf{x}, t) = n(\mathbf{x}, t) = \frac{\rho_m(\mathbf{x}, t)}{m}$$

$$u_m^i(\mathbf{x}, t) \equiv \frac{\langle p^i \rangle_{f_m}}{\langle m \rangle_{f_m}}$$

Homework: take the moments of the Boltzmann equation to derive the fluid equations. Use:

$$\frac{1}{m} \langle p^i p^j \rangle_{f_m} = \rho_m u_m^i u_m^j + \sigma_m^{ij}. \quad \text{Eq (12.17)}$$

# Result: the fluid equations (Euler-Poisson system)

$$\begin{aligned}\delta_m' + \frac{\partial}{\partial x^j} \left[ (1 + \delta_m) u_m^j \right] &= 0, \\ u_m^{i'} + u_m^j \frac{\partial}{\partial x^j} u_m^i + a H u_m^i + \frac{\partial \Psi}{\partial x^i} &= 0, \\ \nabla^2 \Psi &= \frac{3}{2} \Omega_m(\eta) (a H)^2 \delta_m.\end{aligned}\tag{Eq (12.23)}$$

Primes denote derivative w.r.t conformal time

- *Much* nicer: 3+1 dim; no integrals involved
- How did this magic happen? Neglected higher moments, in particular a contribution to Euler equation from velocity dispersion (anisotropic stress)  $\rho_m^{-1} \partial_j \sigma_m^{ij}$
- Fine on large scales, as we will see.

# Result: the fluid equations (Euler-Poisson system)

- Now, take divergence of Euler equation, and separate linear and nonlinear terms
- Curl component decays if not sourced (Homework)

$$\delta_m' + \theta_m = -\delta_m \theta_m - u_m^j \frac{\partial}{\partial x^j} \delta_m,$$

$$\theta_m' + aH\theta_m + \nabla^2 \Psi = -u_m^j \frac{\partial}{\partial x^j} \theta_m - (\partial_i u_m^j)(\partial_j u_m^i).$$

$$\nabla^2 \Psi = \frac{3}{2} \Omega_m(\eta) (aH)^2 \delta_m.$$

# Linearizing the fluid equations

- If all of  $\delta, \theta, \Psi$  are small, we can neglect the nonlinear terms on the right-hand side:

$$\begin{aligned}
 \delta_m' + \theta_m &= -\delta_m \theta_m - u_m^j \frac{\partial}{\partial x^j} \delta_m, \\
 \theta_m' + aH\theta_m + \nabla^2 \Psi &= -u_m^j \frac{\partial}{\partial x^j} \theta_m - (\partial_i u_m^j)(\partial_j u_m^i), \\
 \nabla^2 \Psi &= \frac{3}{2} \Omega_m(\eta) (aH)^2 \delta_m.
 \end{aligned}$$

0

# Linearizing the fluid equations

- Then, we can combine all three equations into a single, second-order ODE for the density  $\delta$ :

$$\delta''(\boldsymbol{x}, \eta) + aH\delta'(\boldsymbol{x}, \eta) = \frac{3}{2}\Omega_{\text{m}}(\eta)(aH)^2\delta(\boldsymbol{x}, \eta)$$

$$\Omega_{\text{m}}(\eta) = \frac{\rho_{\text{m}}(\eta)}{\rho_{\text{cr}}(\eta)}$$

Primes denote derivative w.r.t conformal time

The density at all points in (real or Fourier) space evolves independently!

# Linearizing the fluid equations

- Then, we can combine all three equations into a single, second-order ODE for the density  $\delta$ :

$$\delta^{(1)}(\boldsymbol{x}, \eta) = D(\eta) \delta_0(\boldsymbol{x})$$

$$D'' + aH D' = \frac{3}{2} \Omega_{\text{m}}(\eta) (aH)^2 D(\eta)$$

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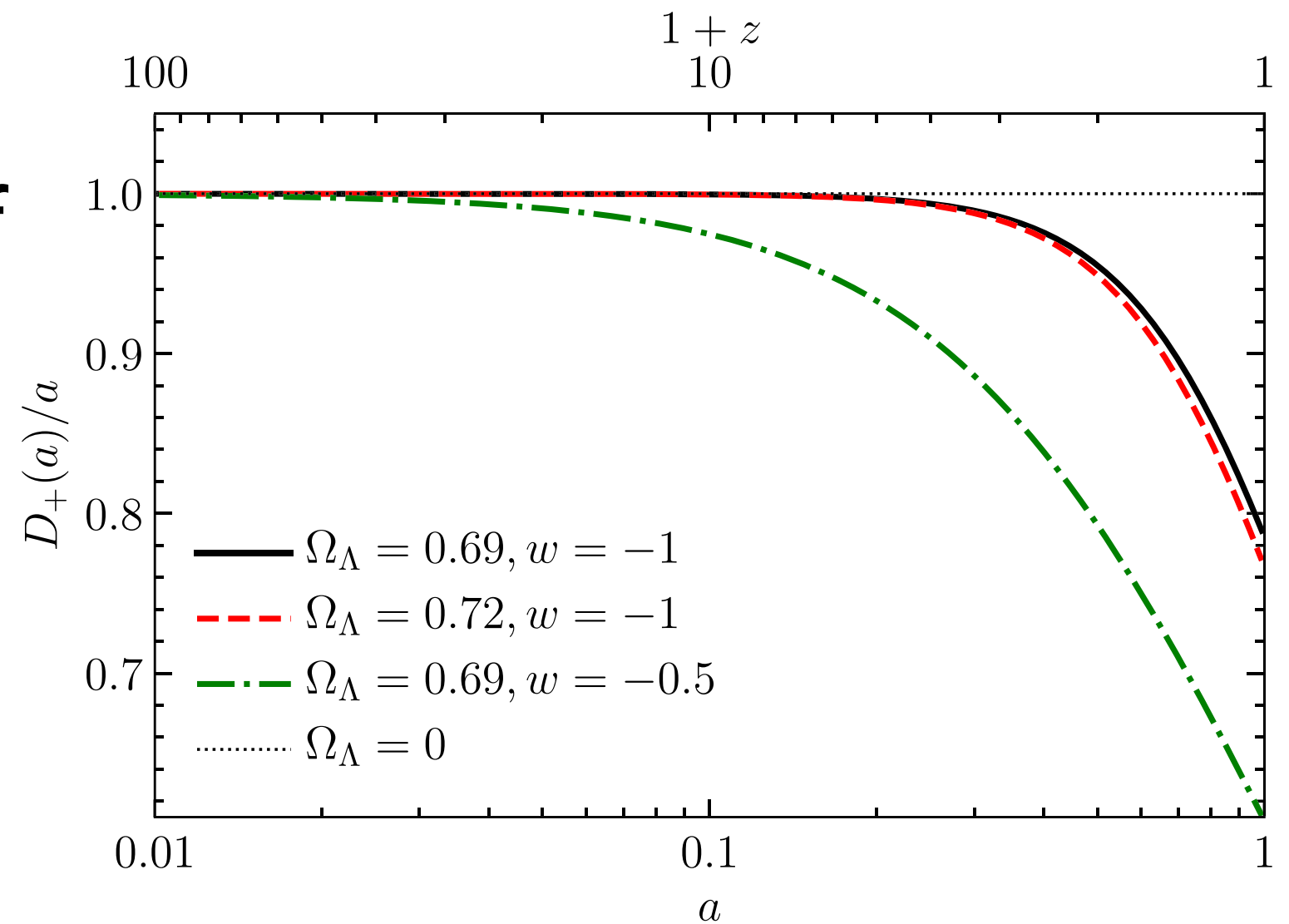
$$D'' + aH D' = \frac{3}{2} \Omega_m(\eta) (aH)^2 D(\eta)$$

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Linear velocity divergence:  $\theta^{(1)}(\mathbf{x}, \eta) = -\delta^{(1)'}(\mathbf{x}, \eta) = -aH f(\eta) \delta^{(1)}(\mathbf{x}, \eta)$ ,  $f \equiv d \ln D / d \ln a$

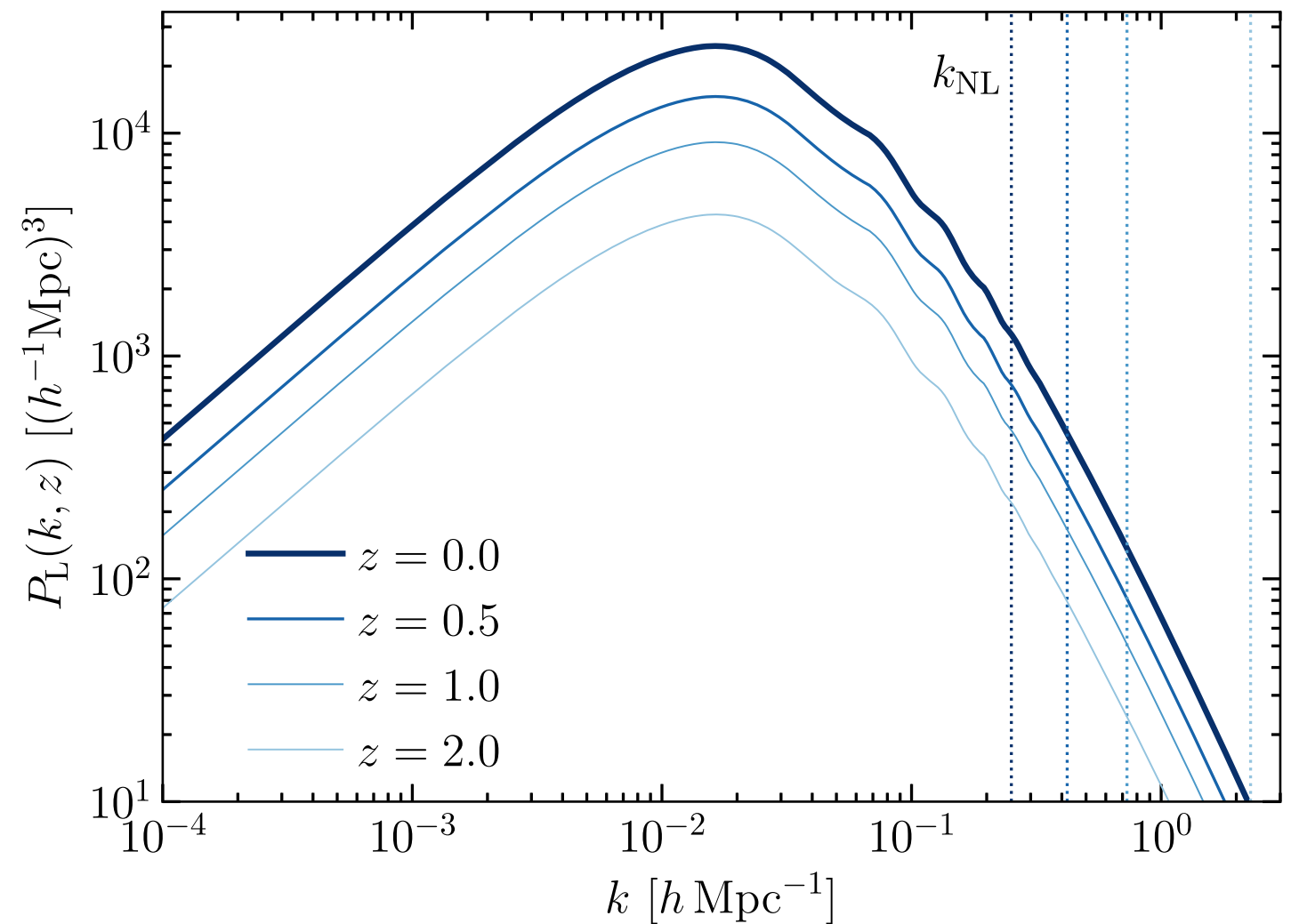
# Linear growth

- Growth is probe of dark energy



# Linear growth

- Together with initial conditions (transfer function), we can compute matter power spectrum



$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k)$$