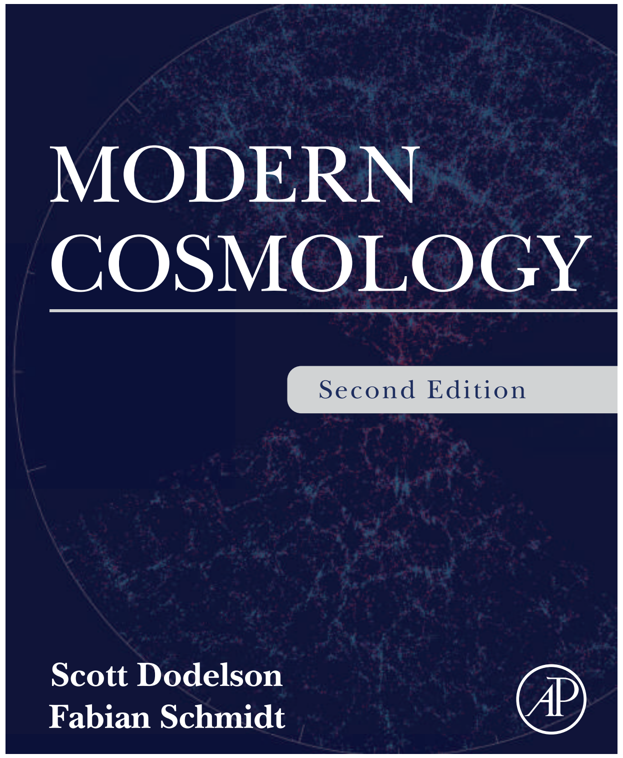
Structure Formation
Lecture 1

Fabian Schmidt
MPA

All figures taken from *Modern Cosmology, Second Edition*, unless otherwise noted



ICTP-SAIFR School on Cosmology, January 2021

Motivation

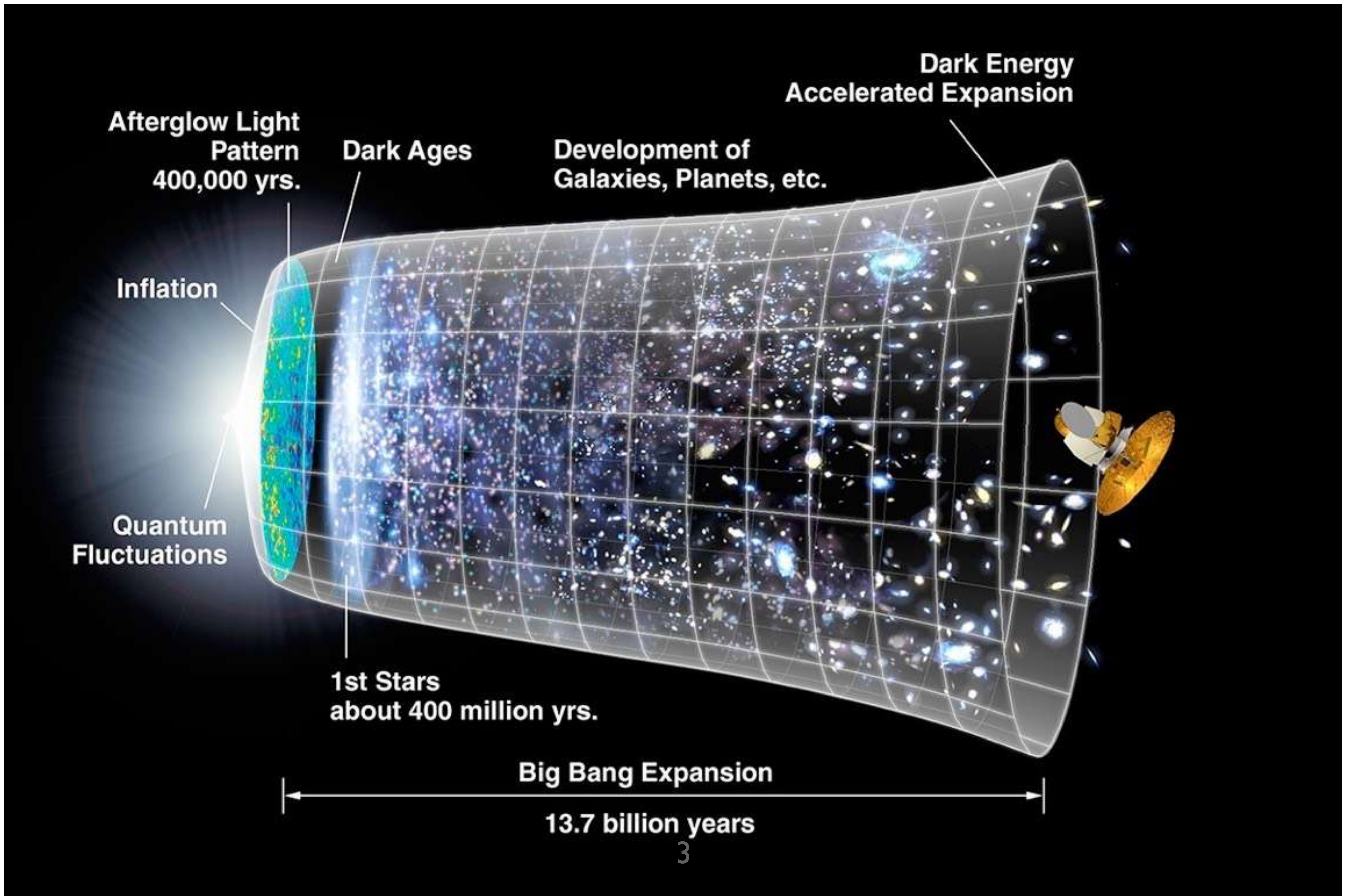
- The large-scale structure (LSS) is historically one of the key probes of cosmology

Peebles; Efstathiou+ ’90 predicted a positive cosmological constant Λ from LSS observations

- Now, we are really in a golden age of LSS with plenty of experiments under way: eBOSS, DES, DESI, PFS, SphereX, Euclid, WFIRST, ...

Motivation

- Using large-scale structure, we can learn about



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- Inflation: reconstruct the properties of the initial conditions, and look for gravitational waves

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- Dark Energy and Gravity: the growth of structure depends sensitively on the expansion history of the Universe, and the nature of gravity

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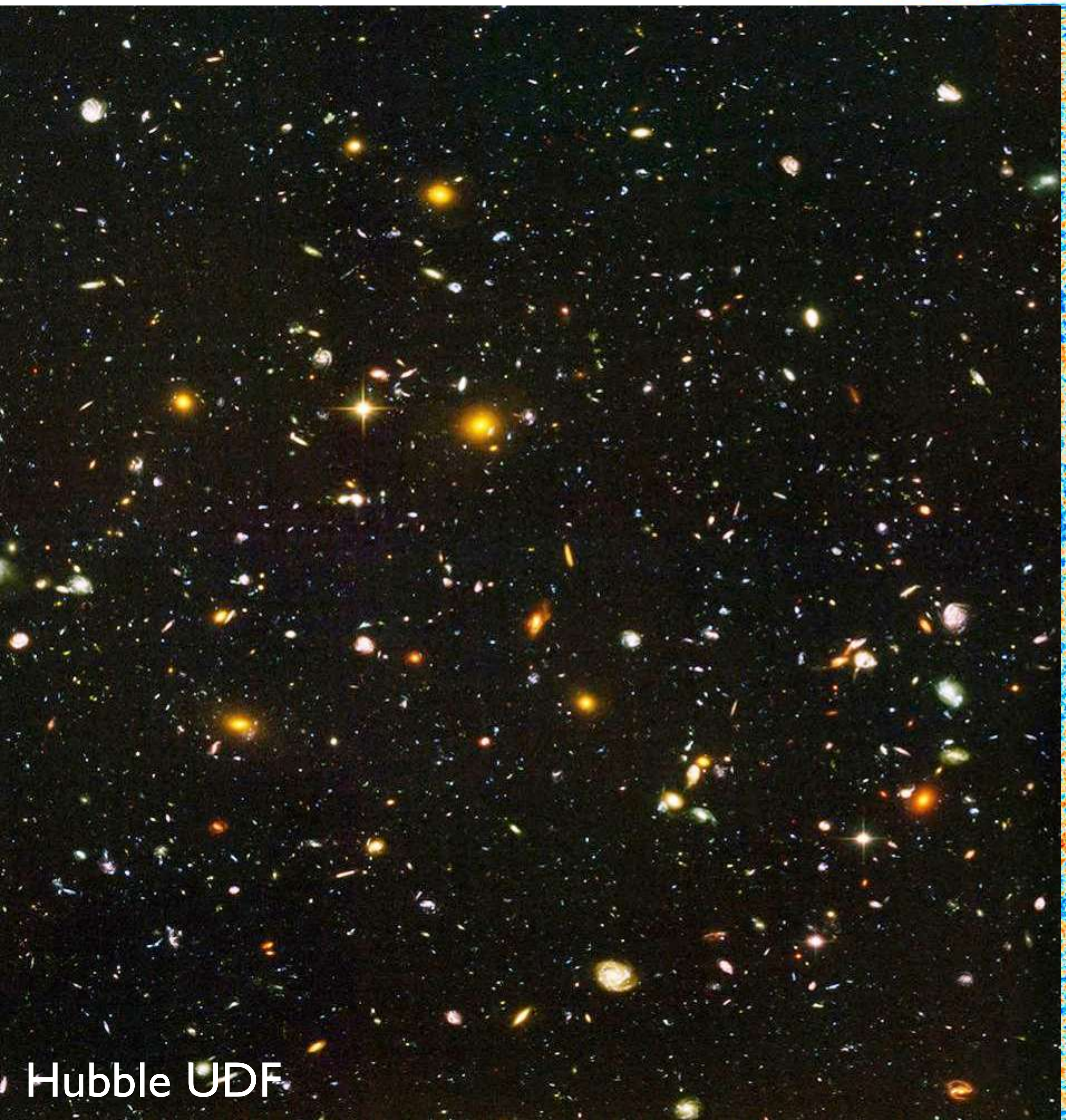
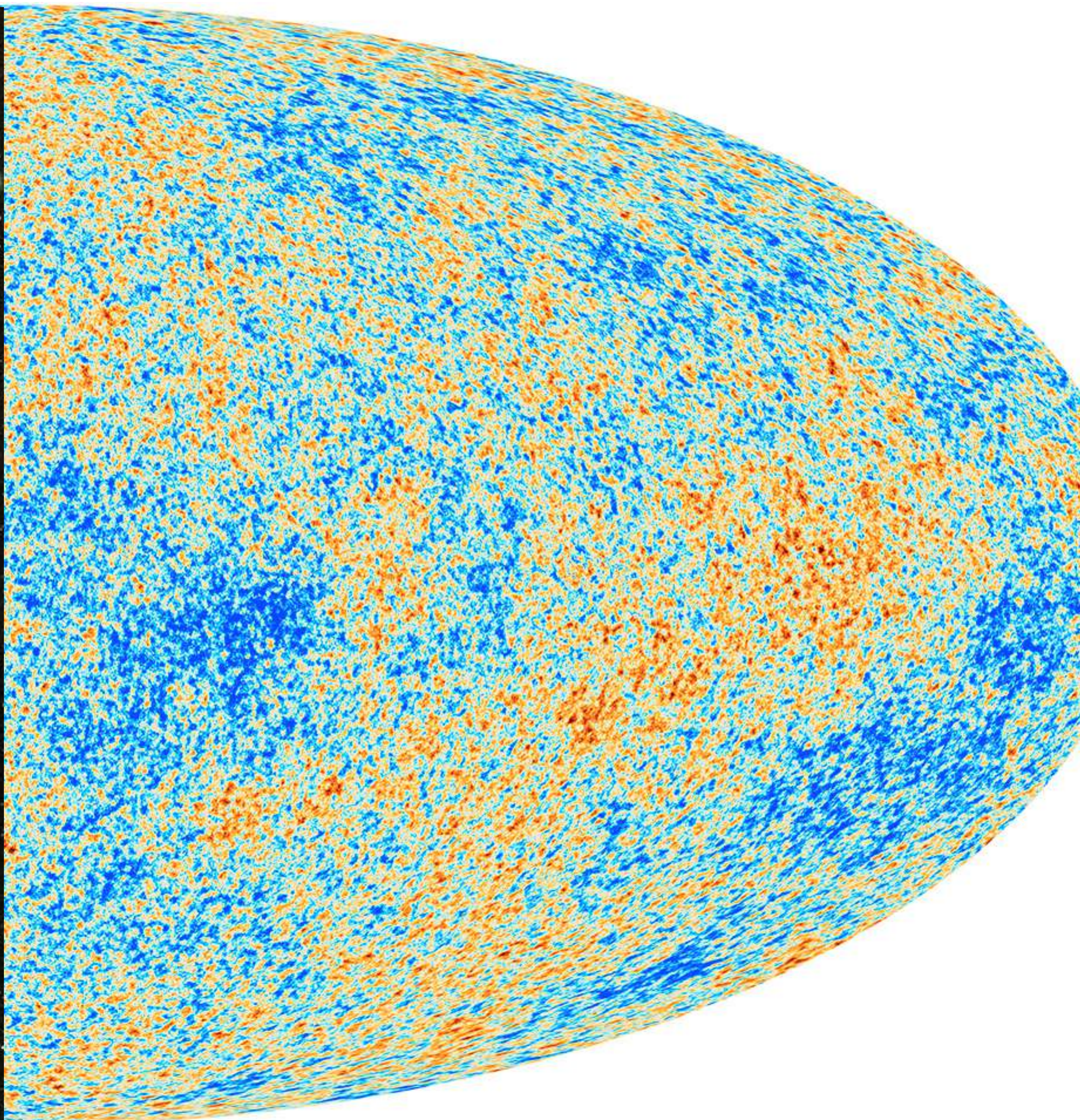
- Dark Energy and Gravity: the growth of structure depends sensitively on the expansion history of the Universe, and the nature of gravity

  Growth equation: $D'' + aH\,D' = 4\pi\,G\,\bar{\rho}\,D$
- Dark Matter: how "cold" is cold dark matter ? What is the sum of neutrino masses ?

Challenge: unlike the CMB, every data point is nonlinear!





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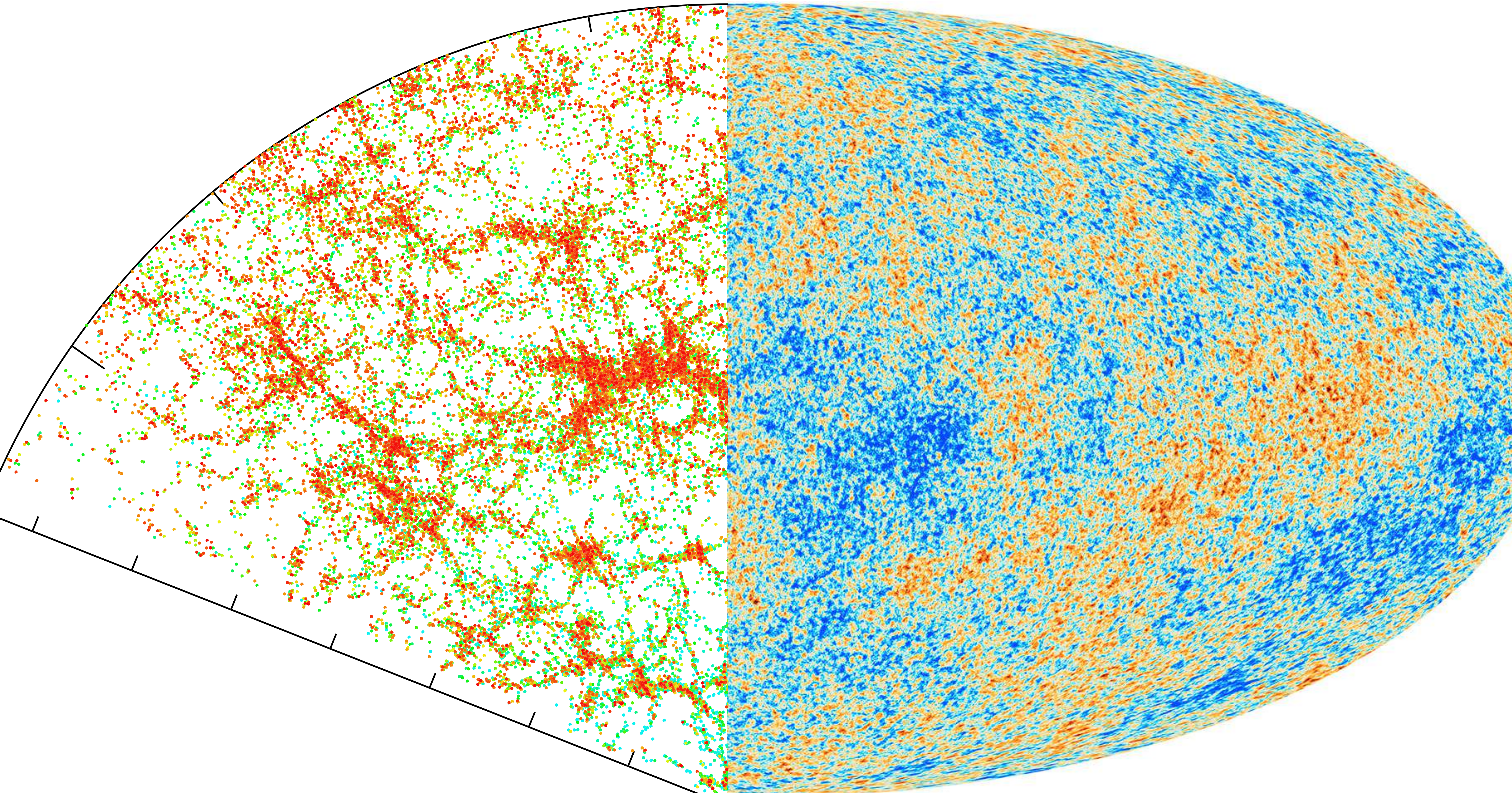
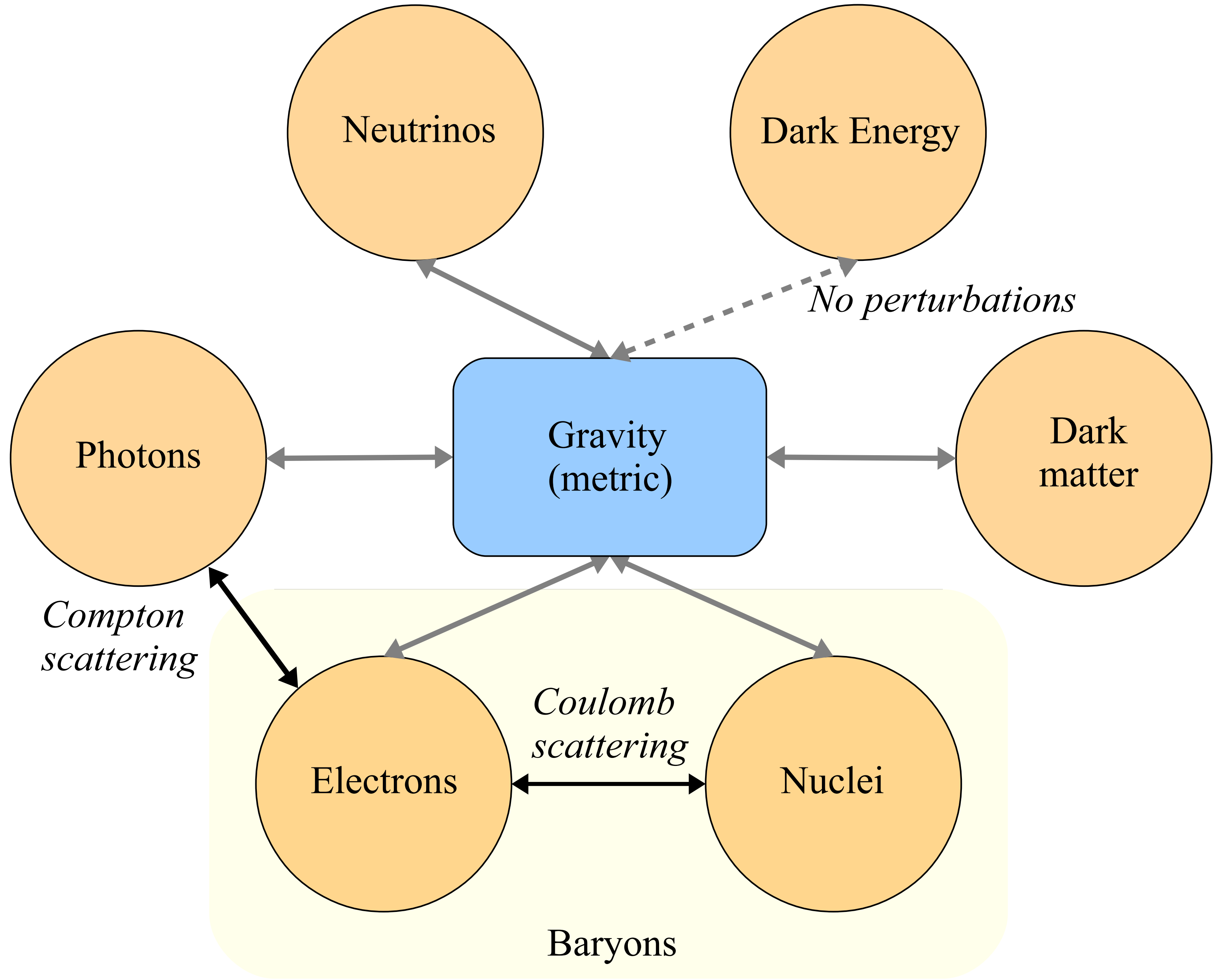


Preliminaries

- In bulk of lectures, we'll be assuming "vanilla" Euclidean (flat) ΛCDM cosmology
  - Gaussian, adiabatic, almost scale-invariant initial perturbations
  - Dark Energy equation of state $w=-1$, although results hold for general smooth DE as well
  - Mostly neglect effect of massive neutrinos
- We will (hopefully) discuss the effect of going beyond these assumptions in the 5th lecture

Preliminaries

Gravity <-> Matter
(Einstein eq.) Boltzmann equations



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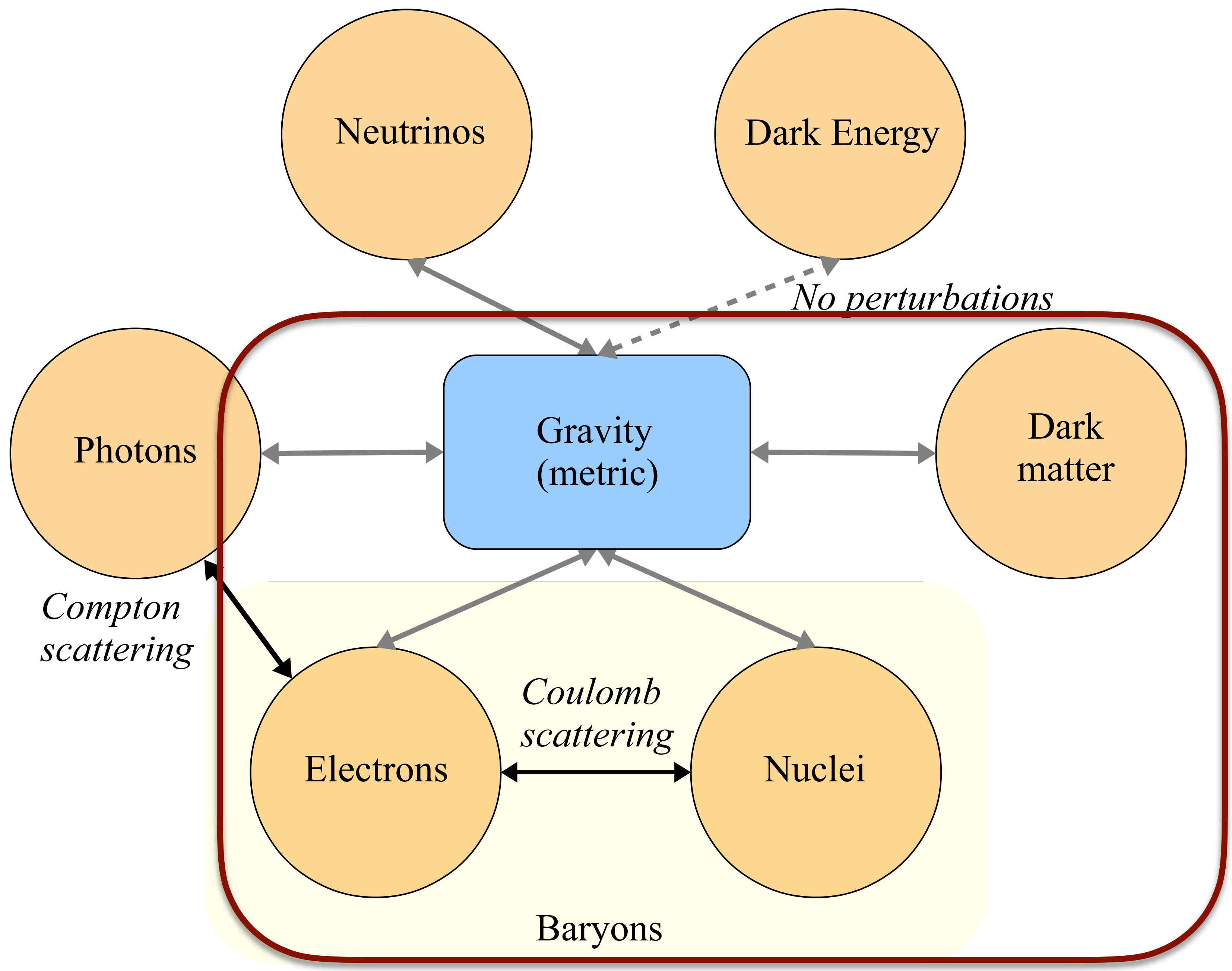


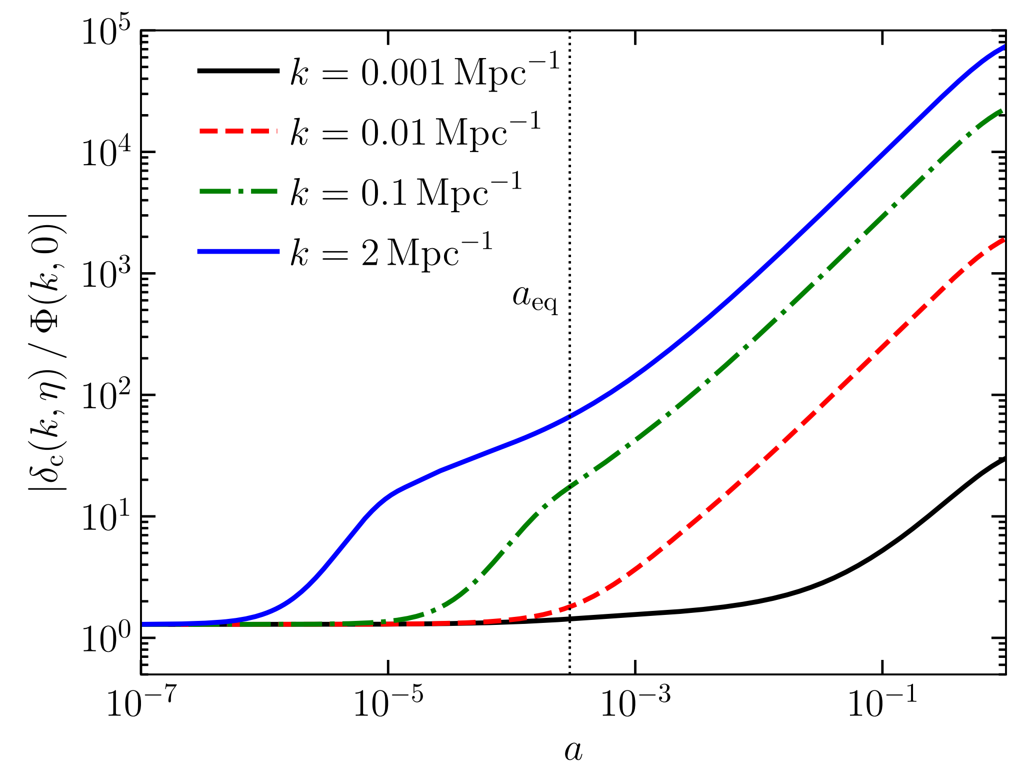
Preliminaries

- Baryons and CDM are "cold": the constituent particles are non-relativistic
- Most of structure formation happens well within the Hubble horizon: sub horizon approximation
- These two facts simplify equations substantially!
- Can often use our intuition for Newtonian gravity

Preliminaries

- Will not study early universe evolution here
- Early evolution starts when perturbation "enters the horizon"
- Evolution depends on whether this happens in radiation domination (slower growth) or matter domination (faster growth)
- Small-scale modes enter horizon earlier

Evolution of modes of different wavelengths at early times ($k=2\pi/\lambda$)



Cold dark matter component only

Notation

$$ds^2 = -(1+2\Psi(\boldsymbol{x},t))dt^2 + a^2(t)(1+2\Phi(\boldsymbol{x},t))d\boldsymbol{x}^2$$

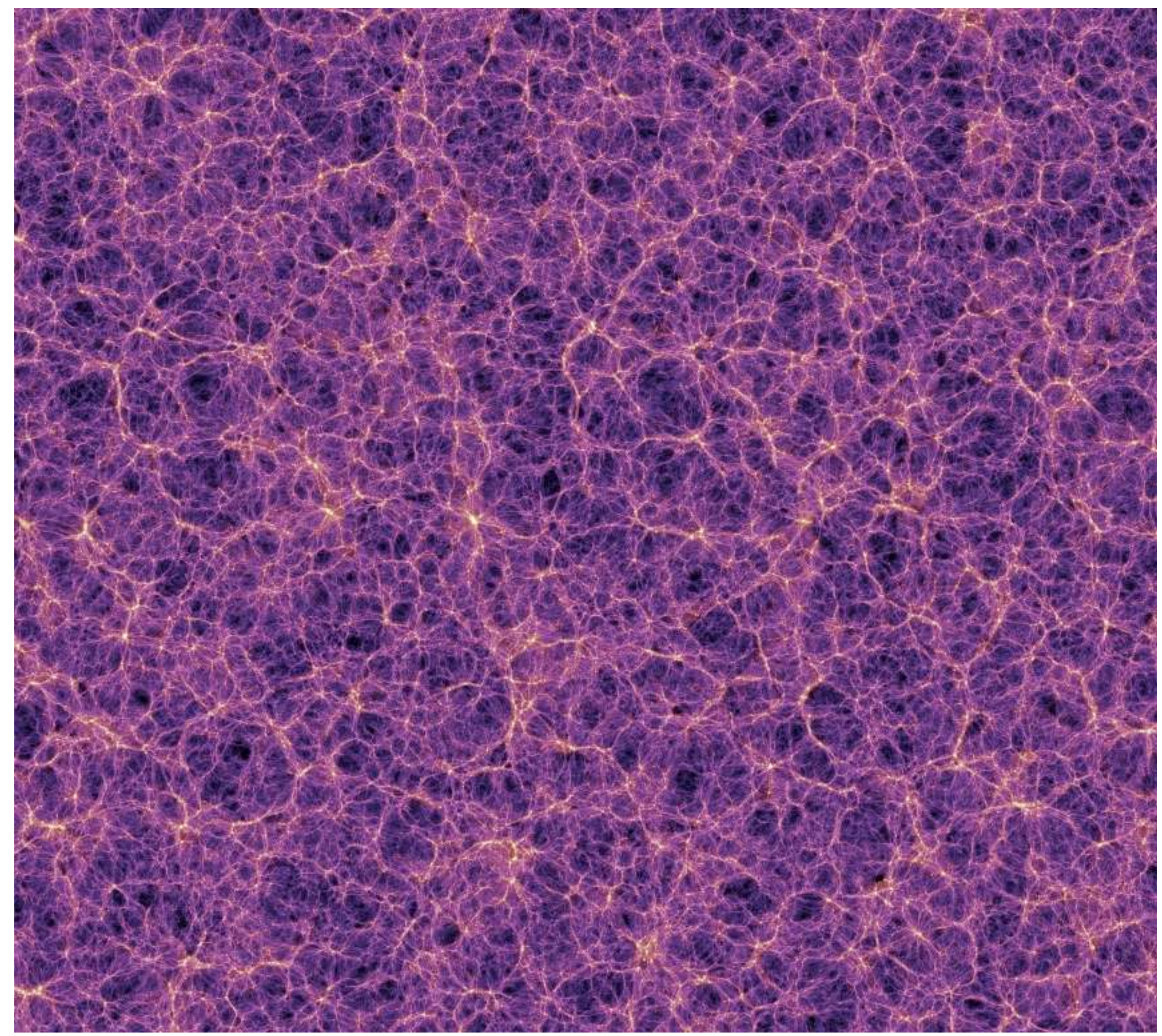
- Comoving coordinates: $d\boldsymbol{r} = a(t)d\boldsymbol{x}$
- Conformal time: $d\eta = \dfrac{dt}{a(t)} = \dfrac{da}{a^2H(a)} = \dfrac{d\ln a}{aH(a)}.$

  Primes denote derivative w.r.t conformal time

  - Comoving distance: $d\chi = -d\eta = \dfrac{dz}{H(z)}$
- Particle velocity/momentum: $\boldsymbol{v} = \dfrac{\boldsymbol{p}}{m} = a\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{x}'$
- Fluid velocity; divergence: $\boldsymbol{u}; \quad \theta = \partial_i u^i$
- Gravitational potential: Ψ

Cold Dark Matter cosmology in a nutshell

- Large-scale fluctuations are small (still linear today)
- Structure forms *hierarchically from small to large* scales
- *Perturbative expansion* in fluctuations on large scales
- Simulations of large volumes can assume background cosmology



Millennium simulation / MPA

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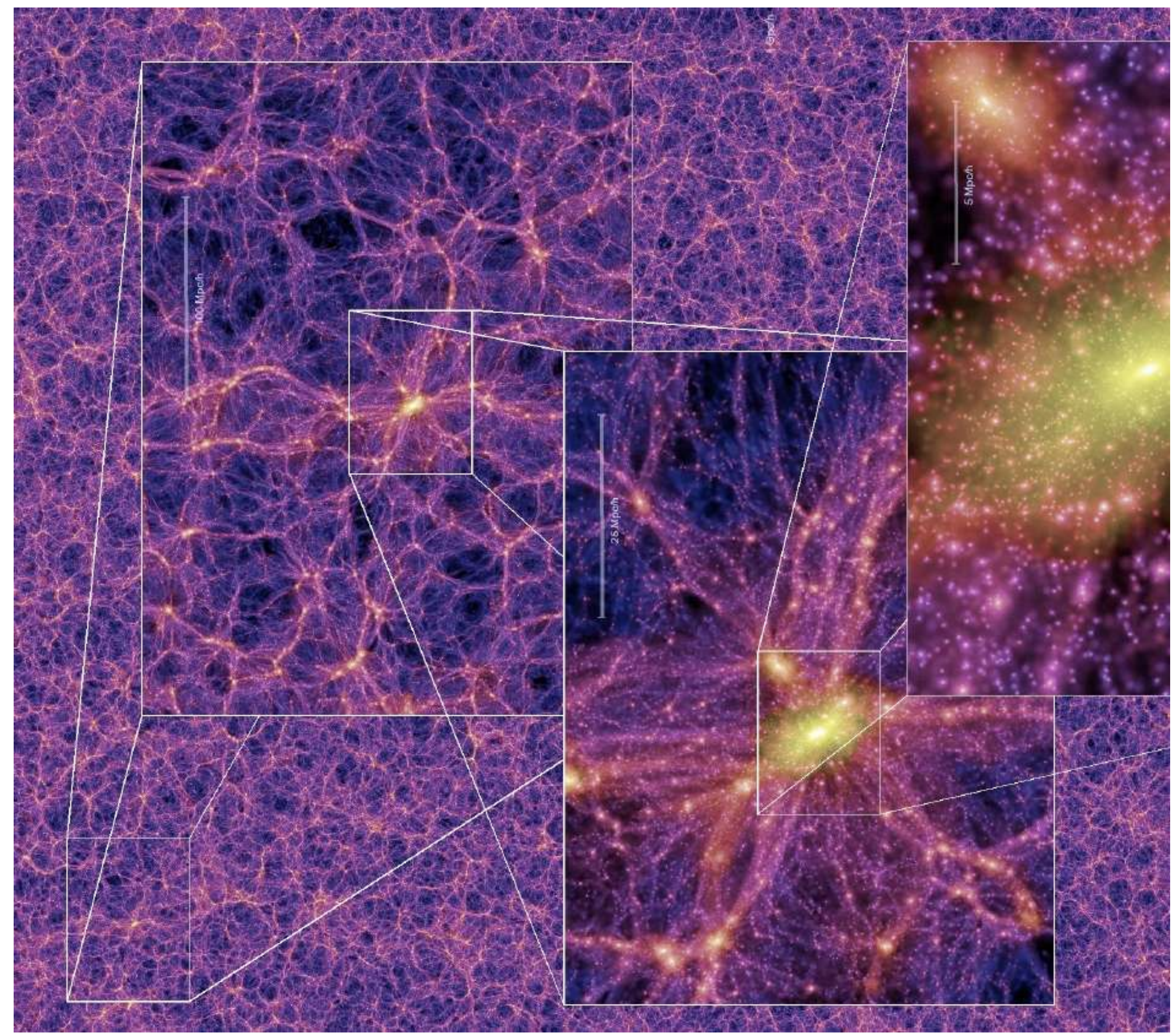


Millennium simulation / MPA

How do we compare theory with data?

- Assume we observe the matter density field $\rho(\boldsymbol{x}) = \bar{\rho}[1 + \delta(\boldsymbol{x})]$

δ: fractional matter density perturbation

- Given cosmological parameters θ, theory predicts

1. Statistics of initial conditions (Gaussian)
2. How a given $\delta_{\rm in}(\boldsymbol{x})$ evolves into the final density field δ

- In cosmology, we are always dealing with statistical fields!

Characterizing Statistical Field

- Consider δ(x), and its Fourier-space version δ(k)
- Simplest statistical field: the field values at each point are independent Gaussian random variables (with vanishing mean)
- In cosmology, we often encounter these simplest fields - where we have independent *Fourier* modes
- Statistics of field is completely described in terms of the variance of the Fourier modes, as a function of k: the *power spectrum*

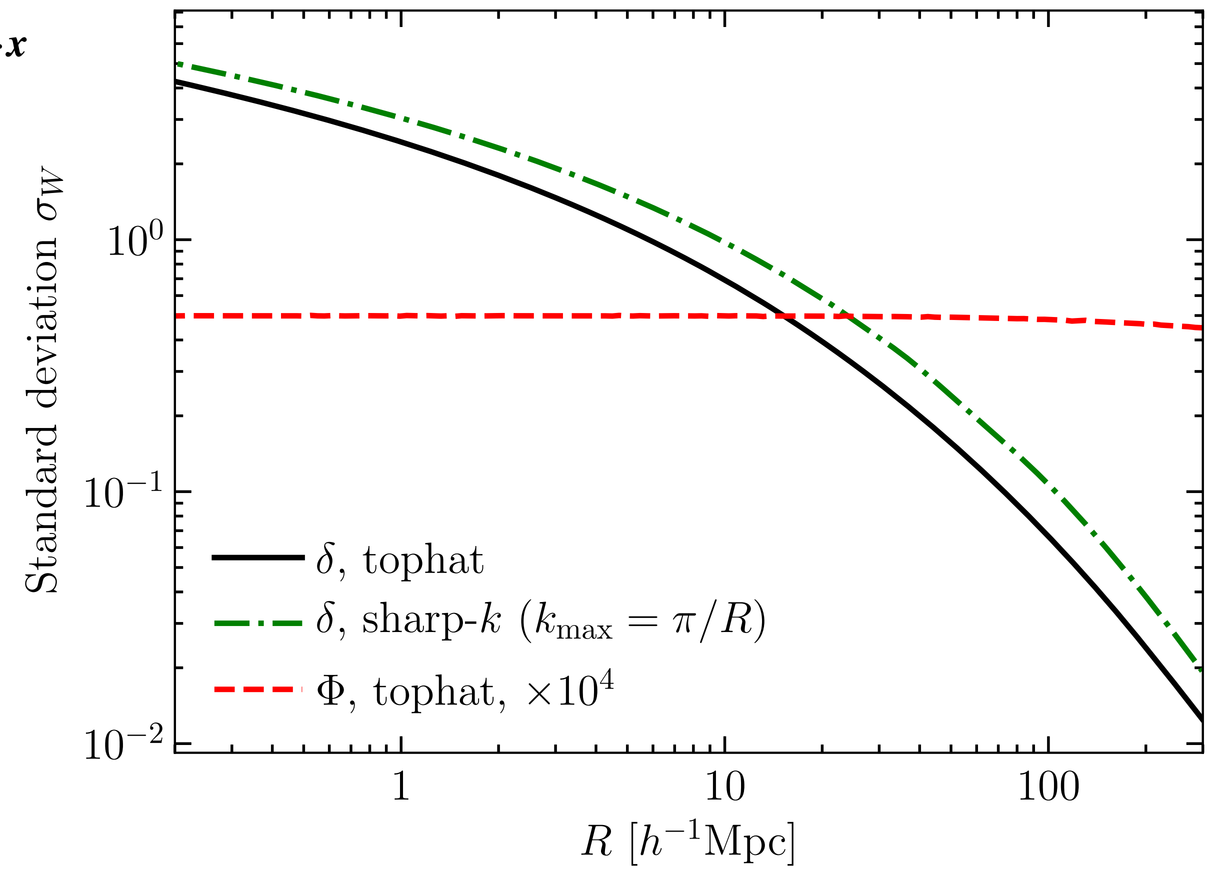
$$\langle \delta(\boldsymbol{k})\delta^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta_D(\boldsymbol{k}-\boldsymbol{k}')\,P(k)$$

Characterizing Statistical Field

- So let's characterize large-scale matter density field

- Consider variance of matter density field filtered on different scales:

$$\sigma_W^2 \equiv \left\langle (\delta_W)^2(\boldsymbol{x}) \right\rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left\langle \delta_W(\boldsymbol{k}) \delta_W^*(\boldsymbol{k}') \right\rangle e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{x}}$$
$$= \int \frac{d^3k}{(2\pi)^3} P_{\rm L}(k) |W(k)|^2$$
$$= \frac{1}{2\pi^2} \int d\ln k\, k^3 P_{\rm L}(k) |W(k)|^2.$$



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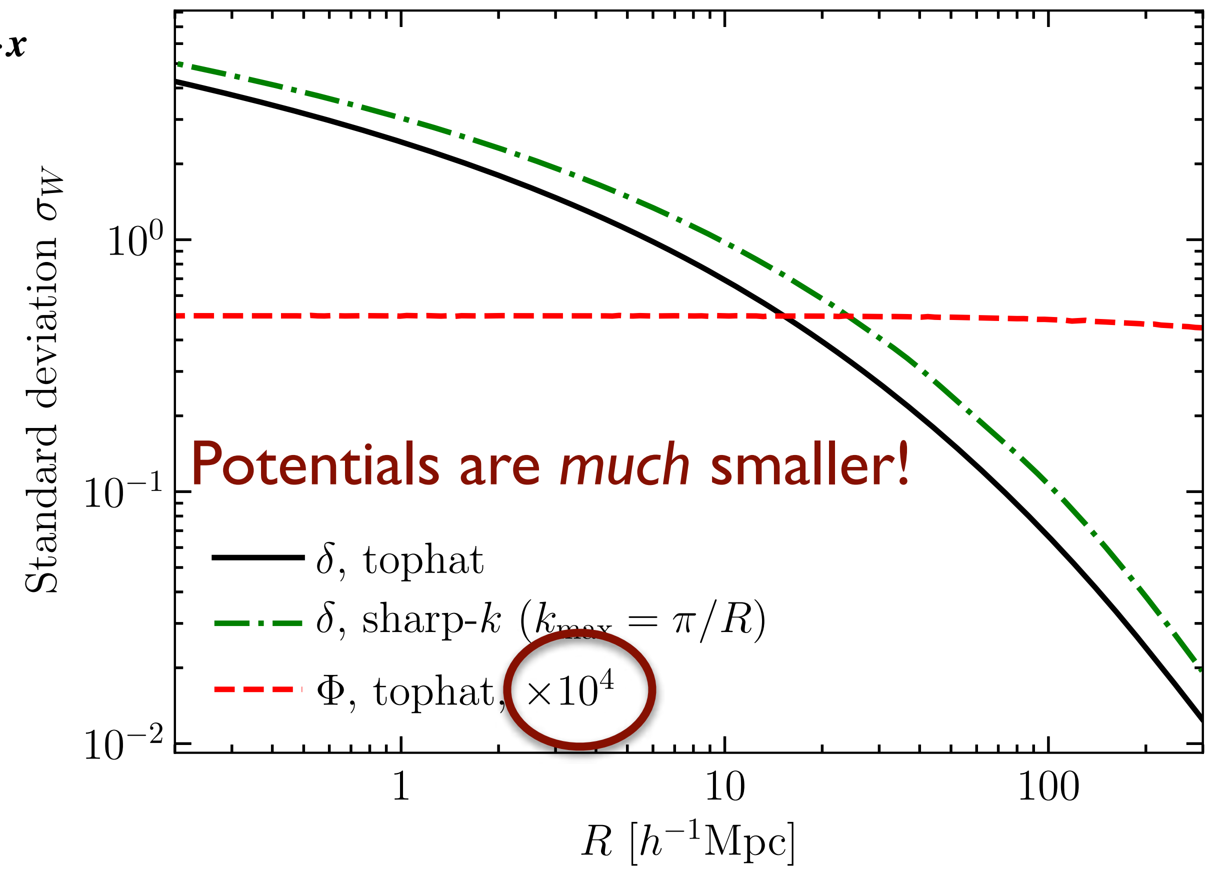
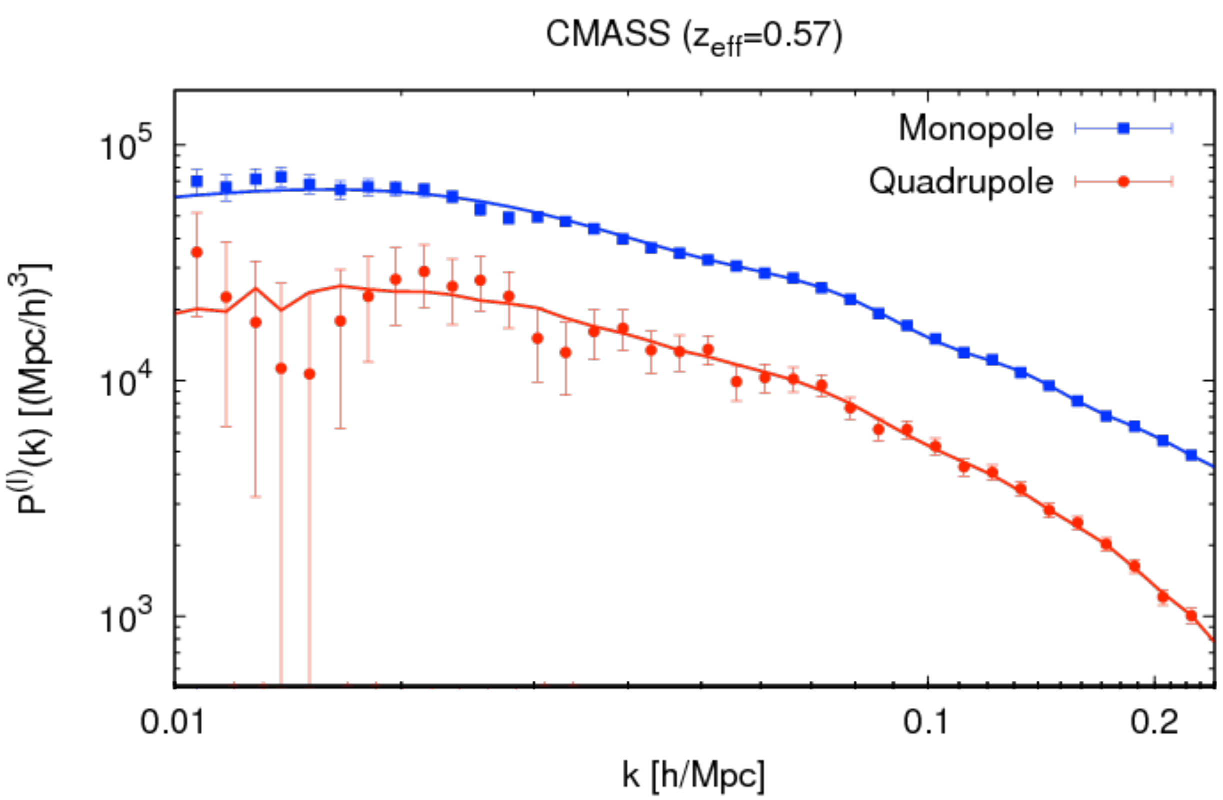


- Variance is small for large smoothing scales

How do we compare theory with data?

- Goal: compute power spectrum of matter and galaxies
- And also other statistics of LSS

Galaxy Power spectrum measured in BOSS

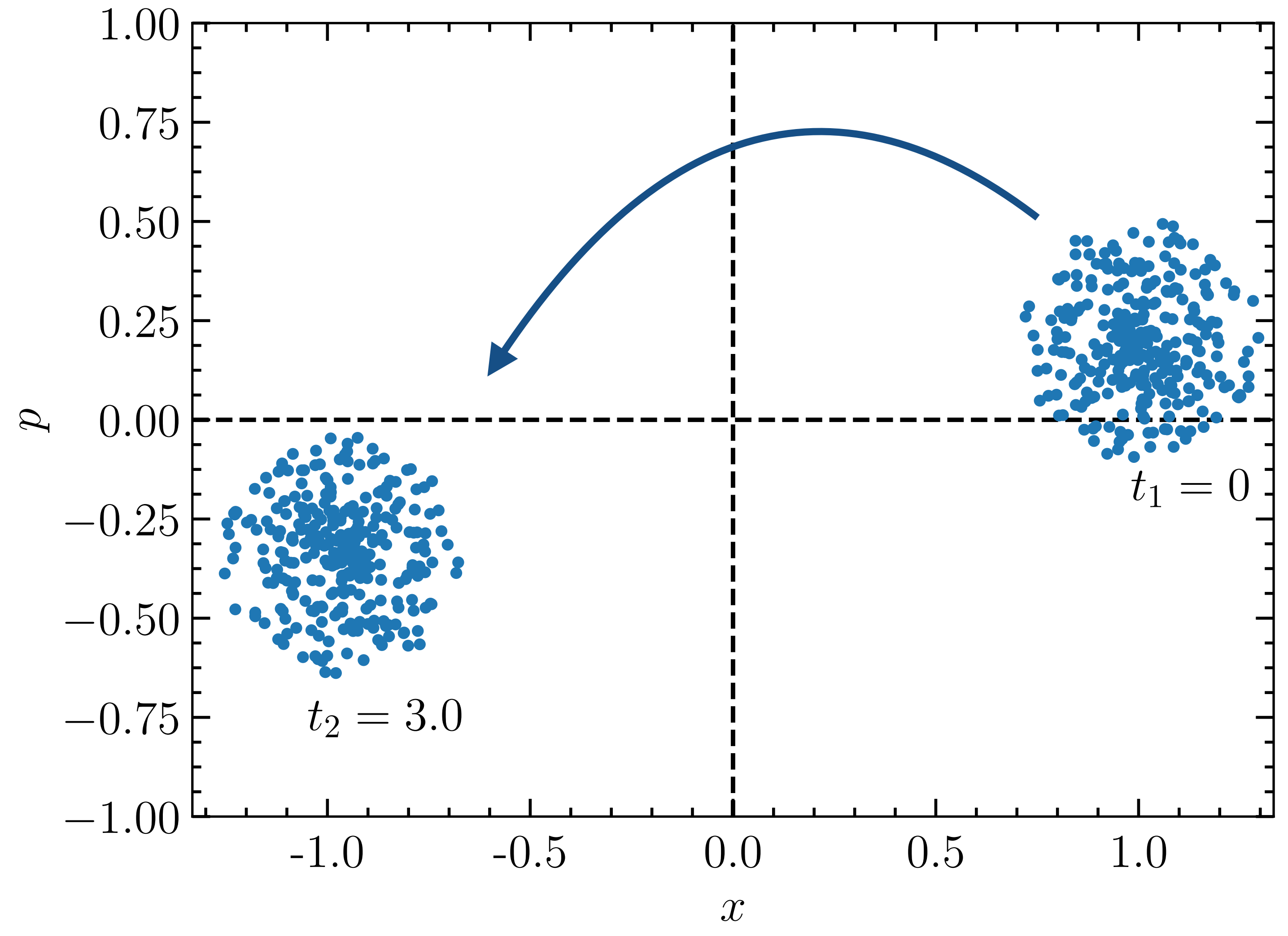


$$\langle \delta(\boldsymbol{k}) \delta^*(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_D(\boldsymbol{k} - \boldsymbol{k}')\, P(k)$$

Gil-Marin et al, 2016

The Boltzmann equation for cold, collision less matter

- Fundamental quantity: distribution function $f_{\mathrm{m}}(\boldsymbol{x}, \boldsymbol{p}, t)$
- Boltzmann equation describes its evolution
- Dark matter: no interactions! Baryons: neglect interactions…
- Then, can lump dark matter and baryons together



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$$\frac{d f_{\mathrm{m}}}{dt} = \frac{\partial f_{\mathrm{m}}}{\partial t} + \frac{\partial f_{\mathrm{m}}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{\mathrm{m}}}{\partial p^i}\frac{dp^i}{dt} = 0,$$

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Geodesic equations: just Newtonian plus factors of a

$$\frac{dx^i}{dt} = \frac{p^i}{am}$$
$$\frac{dp^i}{dt} = -Hp^i - \frac{m}{a}\partial_i\Psi$$

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Supplemented with the Poisson equation for the gravitational potential:

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta_{\rm m}.$$

00-component of Einstein eq. in the subhorizon limit

The Boltzmann equation for cold, collision less matter

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$$\frac{df_{\mathrm{m}}}{dt} = \frac{\partial f_{\mathrm{m}}}{\partial t} + \frac{\partial f_{\mathrm{m}}}{\partial x^j}\frac{p^j}{ma} - \frac{\partial f_{\mathrm{m}}}{\partial p^j}\left[Hp^j + \frac{m}{a}\frac{\partial \Psi}{\partial x^j}\right] = 0.$$

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\mathrm{m}}(\eta)(aH)^2\delta_{\mathrm{m}}.$$

These equations will govern almost everything in these lectures!

The Boltzmann equation for cold, collision less matter

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$$\frac{df_{\mathrm{m}}}{dt} = \frac{\partial f_{\mathrm{m}}}{\partial t} + \frac{\partial f_{\mathrm{m}}}{\partial x^j}\frac{p^j}{ma} - \frac{\partial f_{\mathrm{m}}}{\partial p^j}\left[Hp^j + \frac{m}{a}\frac{\partial \Psi}{\partial x^j}\right] = 0.$$

Initial conditions: *cold*

$$f_{\mathrm{m}}(\boldsymbol{x}, \boldsymbol{p}, t) = \frac{\rho_{\mathrm{m}}(\boldsymbol{x}, t)}{m}(2\pi)^3 \delta_{\mathrm{D}}^{(3)}\left(\boldsymbol{p} - m\boldsymbol{u}_{\mathrm{m}}(\boldsymbol{x}, t)\right)$$ Eq (12.9)

<=> no velocity dispersion

Taking moments of the Boltzmann equation

- Boltzmann equation: 6+1dim; plus we need to integrate f_m to obtain δ for Ψ
- Extremely difficult to solve. Let's try different approach: taking moments
- That means we integrate the equation (multiplied by p, p^2) over d^3p

Taking moments of the Boltzmann equation

- Define:

$$\langle A \rangle_{f_\mathrm{m}}(\boldsymbol{x}, t) \equiv \int \frac{d^3 p}{(2\pi)^3} A(\boldsymbol{x}, \boldsymbol{p}, t) f_\mathrm{m}(\boldsymbol{x}, \boldsymbol{p}, t)$$

- Zeroth moment yields density:

$$\langle 1 \rangle_{f_\mathrm{m}}(\boldsymbol{x}, t) = n(\boldsymbol{x}, t) = \frac{\rho_\mathrm{m}(\boldsymbol{x}, t)}{m}$$

- First moment yields *bulk* velocity:

$$u^i_\mathrm{m}(\boldsymbol{x}, t) \equiv \frac{\left\langle p^i \right\rangle_{f_\mathrm{m}}}{\langle m \rangle_{f_\mathrm{m}}}$$

Homework: take the moments of the Boltzmann equation to derive the fluid equations. Use:

$$\frac{1}{m}\left\langle p^i p^j \right\rangle_{f_\mathrm{m}} = \rho_\mathrm{m} u^i_\mathrm{m} u^j_\mathrm{m} + \sigma^{ij}_\mathrm{m}.$$

Eq (12.17)

Result: the fluid equations (Euler-Poisson system)

$$\delta_{\rm m}{}' + \frac{\partial}{\partial x^j}\left[(1+\delta_{\rm m})u^j_{\rm m}\right] = 0,$$

$$u^i_{\rm m}{}' + u^j_{\rm m}\frac{\partial}{\partial x^j}u^i_{\rm m} + aHu^i_{\rm m} + \frac{\partial \Psi}{\partial x^i} = 0,$$

Eq (12.23)

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta_{\rm m}.$$

Primes denote derivative w.r.t conformal time

- *Much* nicer: 3+1dim; no integrals involved
- How did this magic happen? Neglected higher moments, in particular a contribution to Euler equation from velocity dispersion (anisotropic stress) $\rho_{\rm m}^{-1}\partial_j\sigma^{ij}_{\rm m}$
- Fine on large scales, as we will see.

Result: the fluid equations (Euler-Poisson system)

- Now, take divergence of Euler equation, and separate linear and nonlinear terms
  - Curl component decays if not sourced (Homework)

$$\delta_{\mathrm{m}}{}' + \theta_{\mathrm{m}} = -\delta_{\mathrm{m}}\theta_{\mathrm{m}} - u^j_{\mathrm{m}} \frac{\partial}{\partial x^j}\delta_{\mathrm{m}},$$

$$\theta_{\mathrm{m}}{}' + aH\theta_{\mathrm{m}} + \nabla^2\Psi = -u^j_{\mathrm{m}}\frac{\partial}{\partial x^j}\theta_{\mathrm{m}} - (\partial_i u^j_{\mathrm{m}})(\partial_j u^i_{\mathrm{m}}).$$

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\mathrm{m}}(\eta)(aH)^2\delta_{\mathrm{m}}.$$

Linearizing the fluid equations

- If all of δ,θ, Ψ are small, we can neglect the nonlinear terms on the right-hand side:

0

$$\delta_{\rm m}{}' + \theta_{\rm m} = -\delta_{\rm m}\theta_{\rm m} - \cancel{u^j_{\rm m}\frac{\partial}{\partial x^j}}\delta_{\rm m},$$

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$$\nabla^2\Psi = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta_{\rm m}.$$

Linearizing the fluid equations

- Then, we can combine all three equations into a single, second-order ODE for the density δ:

$$\delta''(\boldsymbol{x},\eta) + aH\delta'(\boldsymbol{x},\eta) = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta(\boldsymbol{x},\eta)$$

$$\Omega_{\rm m}(\eta) = \frac{\rho_{\rm m}(\eta)}{\rho_{\rm cr}(\eta)}$$

Primes denote derivative w.r.t conformal time

The density at all points in (real or Fourier) space evolves independently!

Linearizing the fluid equations

- Then, we can combine all three equations into a single, second-order ODE for the density δ:

$$\delta^{(1)}(\boldsymbol{x},\eta) = D(\eta)\delta_0(\boldsymbol{x})$$

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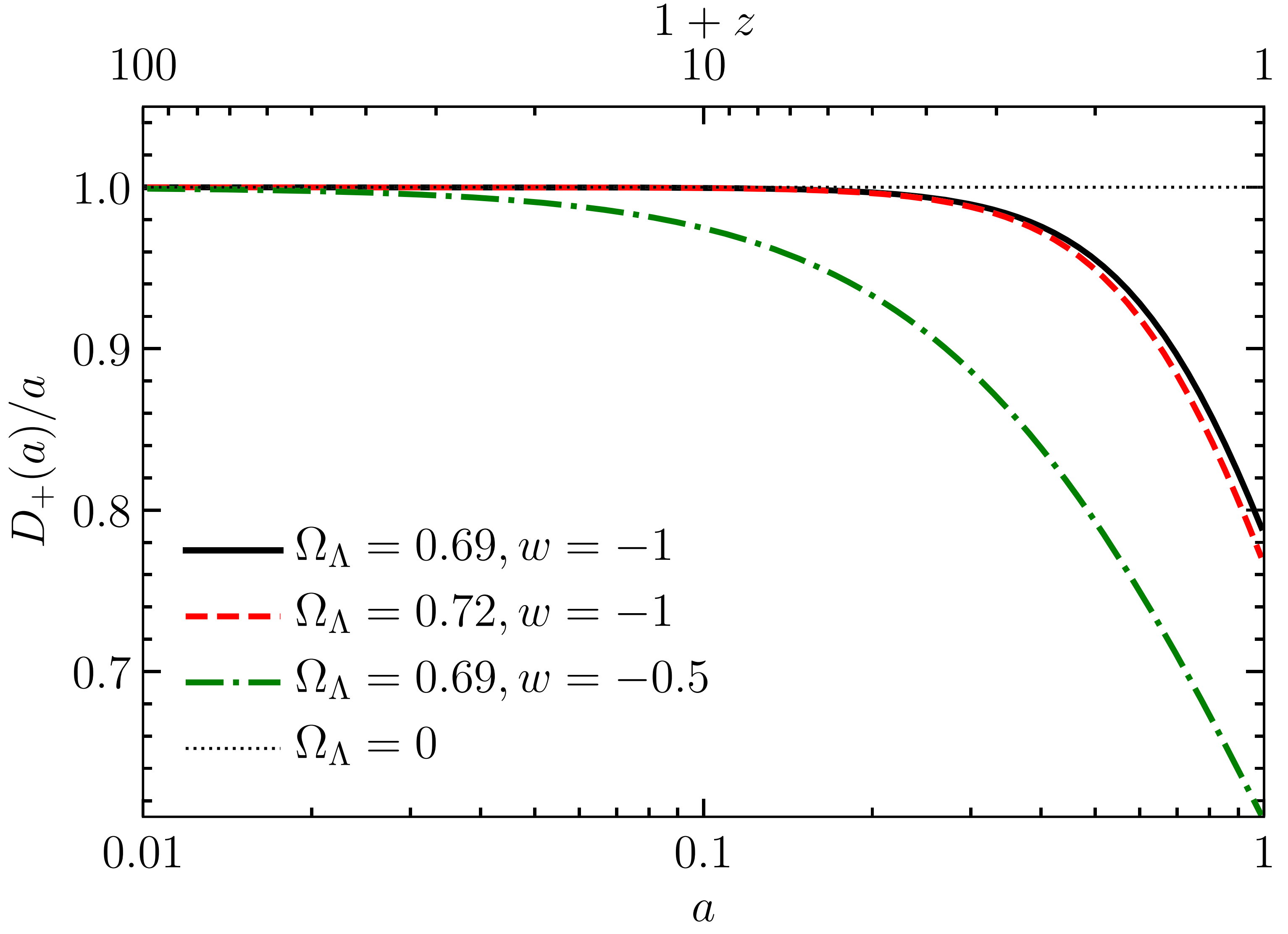
Linear velocity divergence: $\theta^{(1)}(\boldsymbol{x},\eta) = -\delta^{(1)\prime}(\boldsymbol{x},\eta) = -aHf(\eta)\delta^{(1)}(\boldsymbol{x},\eta),\quad f \equiv d\ln D/d\ln a$

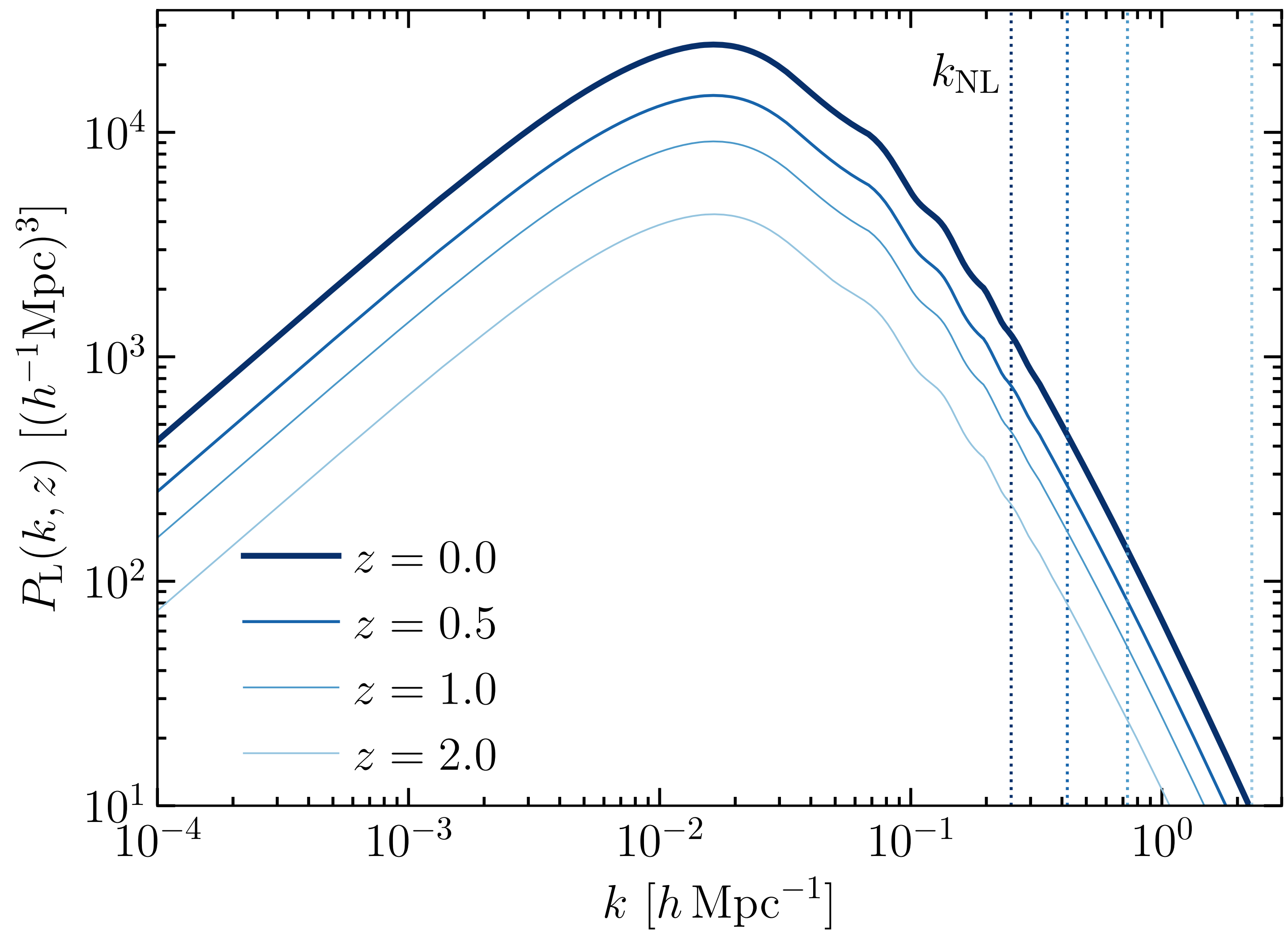
Linear growth

- Growth is probe of dark energy



Linear growth

- Together with initial conditions (transfer function), we can compute matter power spectrum



$$\langle \delta(\boldsymbol{k})\delta^*(\boldsymbol{k}')\rangle = (2\pi)^3\delta_D(\boldsymbol{k}-\boldsymbol{k}')\,P(k)$$