Structure Formation
Lecture 2

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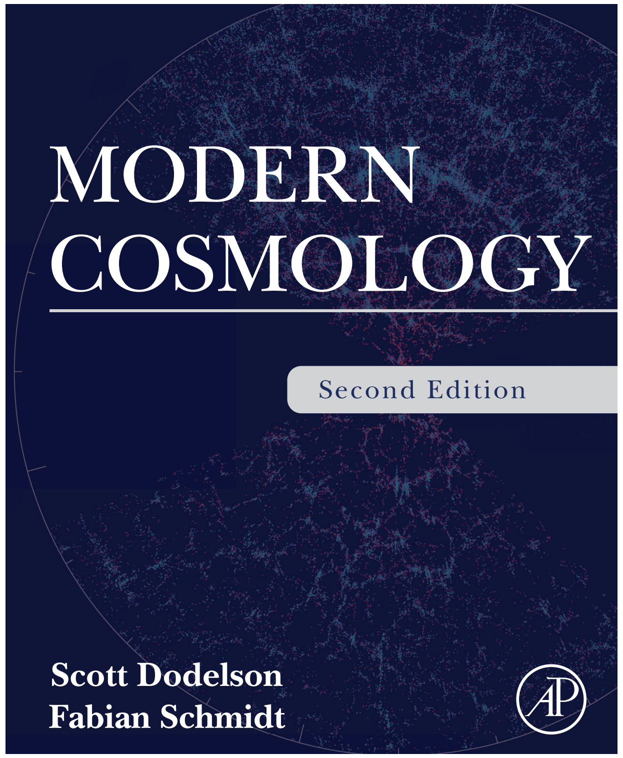
All figures taken from *Modern Cosmology, Second Edition*, unless otherwise noted



ICTP-SAIFR School on Cosmology, January 2021

Outline of lectures

1. The problem: collisionless Boltzmann equation and fluid approximation
   1. Linear evolution
2. Nonlinear evolution of matter
   1. Perturbation theory <- HERE
   2. Simulations
   3. Phenomenology of nonlinear matter distribution
3. Formation and distribution of galaxies
   1. Galaxy formation in a nutshell
   2. Spherical collapse model
   3. Physical clustering of halos and galaxies; bias
   4. Observed clustering of galaxies
4. Beyond ΛCDM

Notation

$$ds^2 = -(1 + 2\Psi(\boldsymbol{x}, t))dt^2 + a^2(t)(1 + 2\Phi(\boldsymbol{x}, t))d\boldsymbol{x}^2$$

- Comoving coordinates: $d\boldsymbol{r} = a(t)d\boldsymbol{x}$
- Conformal time: $d\eta = \frac{dt}{a(t)} = \frac{da}{a^2 H(a)} = \frac{d\ln a}{aH(a)}.$
  - Comoving distance: $d\chi = -d\eta = \frac{dz}{H(z)}$
- Particle velocity/momentum: $\boldsymbol{v} = \frac{\boldsymbol{p}}{m} = a\frac{d\boldsymbol{x}}{dt} = \boldsymbol{x}'$
- Fluid velocity; divergence: $\boldsymbol{u}; \quad \theta = \partial_i u^i$
- Gravitational potential: Ψ

Recap

- In Lecture 1, we derived the collisionless Boltzmann equation for DM and baryons
- Combined with Poisson equation for gravitational potential, these govern all of cosmological structure formation at late times
- We then took moments to obtain the fluid equations (continuity & Euler), and dropped the curl velocity
- Result:

$$\delta_{\rm m}{}' + \theta_{\rm m} = -\delta_{\rm m}\theta_{\rm m} - u^j_{\rm m}\frac{\partial}{\partial x^j}\delta_{\rm m},$$

$$\theta_{\rm m}{}' + aH\theta_{\rm m} + \nabla^2\Psi = -u^j_{\rm m}\frac{\partial}{\partial x^j}\theta_{\rm m} - (\partial_i u^j_{\rm m})(\partial_j u^i_{\rm m}).$$

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta_{\rm m}.$$

Recap

- We then derived the *linear approximation*, when all of δ,θ,Ψ are small:

$$\delta^{(1)}(\boldsymbol{x},\eta) = D(\eta)\delta_0(\boldsymbol{x})$$

$$D'' + aHD' = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2 D(\eta)$$

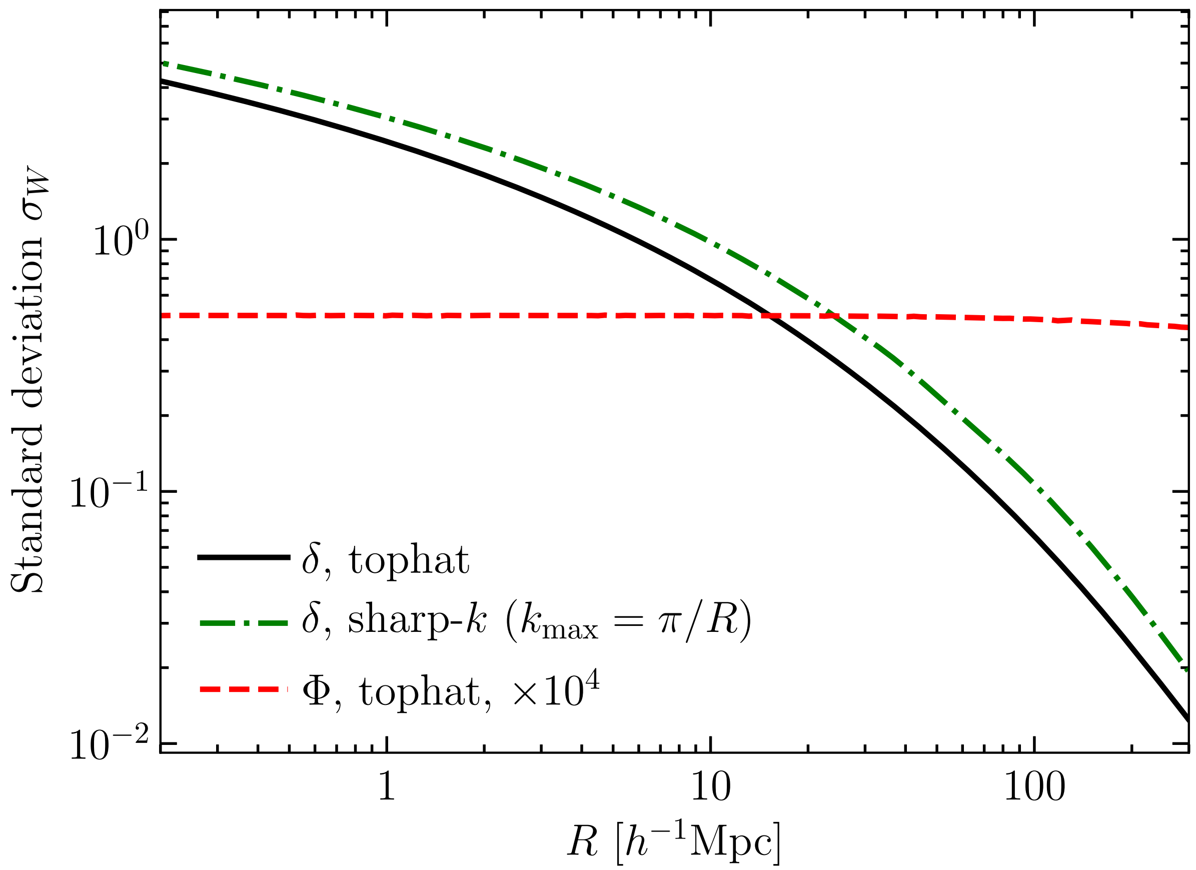
$$\Omega_{\rm m}(\eta) = \frac{\rho_{\rm m}(\eta)}{\rho_{\rm cr}(\eta)}$$

Time-dependent density parameter;
=0.3 today, =1 in the past

The density at all points in (real or Fourier) space evolves independently!

Going beyond linear theory

- We looked at the variance of matter density field filtered on different scales:
  - Shape is consequence of initial conditions from inflation



- Clearly, to describe universe on scales smaller than hundreds of Mpc, we need to go beyond linear theory!

Going beyond linear theory

- Let's go back to full fluid equations
- They contain nonlinear terms, specifically quadratic terms, moved here to the r.h.s.:

$$\delta_{\rm m}{}' + \theta_{\rm m} = -\delta_{\rm m}\theta_{\rm m} - u_{\rm m}^j \frac{\partial}{\partial x^j}\delta_{\rm m},$$

$$\theta_{\rm m}{}' + aH\theta_{\rm m} + \nabla^2\Psi = -u_{\rm m}^j \frac{\partial}{\partial x^j}\theta_{\rm m} - (\partial_i u_{\rm m}^j)(\partial_j u_{\rm m}^i).$$

$$\nabla^2\Psi = \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta_{\rm m}.$$ is just linear!

Going beyond linear theory

- That structure suggests iterative approach: plug in linear solution to nonlinear source terms, and solve for second order:

$$\delta^{(2)\prime} + \theta^{(2)} = -\delta^{(1)}\theta^{(1)} - (u^{(1)})^j \frac{\partial}{\partial x^j}\delta^{(1)},$$

$$\theta^{(2)\prime} + aH\theta^{(2)} + \frac{3}{2}\Omega_{\rm m}(\eta)(aH)^2\delta^{(2)} = -(u^{(1)})^j \frac{\partial}{\partial x^j}\theta^{(1)} - [\partial_i (u^{(1)})^j][\partial_j (u^{(1)})^i],$$

where we have used the Poisson equation for $\nabla^2\Psi^{(2)}$

Perturbation theory

- Idea: expand all fields according to:

$$\delta_{\mathrm{m}}(\boldsymbol{x},\eta) = \delta^{(1)}(\boldsymbol{x},\eta) + \delta^{(2)}(\boldsymbol{x},\eta) + \cdots + \delta^{(n)}(\boldsymbol{x},\eta)$$
$$\theta_{\mathrm{m}}(\boldsymbol{x},\eta) = \theta^{(1)}(\boldsymbol{x},\eta) + \theta^{(2)}(\boldsymbol{x},\eta) + \cdots + \theta^{(n)}(\boldsymbol{x},\eta)$$

- Each order collects all terms that have the same number of *linear fields* $\delta^{(1)}$, $\theta^{(1)}$
- This approach is expected to work as long as each successive term in the series is smaller than the previous one
- Of course, in practice we always stop at some *n*

Second order

- So let's proceed with solving at second order:

$$\delta^{(2)\prime} + \theta^{(2)} = -\delta^{(1)}\theta^{(1)} - (u^{(1)})^j \frac{\partial}{\partial x^j}\delta^{(1)},$$

$$\theta^{(2)\prime} + aH\theta^{(2)} + \frac{3}{2}\Omega_{\mathrm{m}}(\eta)(aH)^2\delta^{(2)} = -(u^{(1)})^j \frac{\partial}{\partial x^j}\theta^{(1)} - [\partial_i (u^{(1)})^j][\partial_j (u^{(1)})^i],$$

- R.h.s. involves derivatives and velocity u: more easily solved in *Fourier space*
- The linear velocity is given by

$$(u^{(1)})^i(\boldsymbol{k}, \eta) = \frac{ik^i}{k^2} aHf\delta^{(1)}(\boldsymbol{k}, \eta) \qquad f \equiv d\ln D/d\ln a$$

Second order

- Fourier transform, and pull out time dependence of source term (important that we can do that!)

$$\delta^{(2)\prime}(\boldsymbol{k},\eta) + \theta^{(2)}(\boldsymbol{k},\eta) = aHfD^2(\eta)S_\delta(\boldsymbol{k})$$

$$\theta^{(2)\prime}(\boldsymbol{k},\eta) + \frac{3}{2}\Omega_m(\eta)(aH)^2\delta^{(2)}(\boldsymbol{k},\eta) = (aHf)^2D^2(\eta)S_\theta(\boldsymbol{k})$$

Second order

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$$S_\delta(\boldsymbol{k})=\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\times\left[1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}\right]\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2),$$

$$S_\theta(\boldsymbol{k})=-\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\times\left[\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}+\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{k_1^2k_2^2}\right]\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)$$

where we used

$$\delta^{(1)}(\boldsymbol{k},\eta)=D(\eta)\delta_0(\boldsymbol{k})$$

Second order

- So we can separate the time- and **k**-dependent parts even at second order!

$$\delta^{(2)\prime}(\boldsymbol{k},\eta)+\theta^{(2)}(\boldsymbol{k},\eta)=aHfD^2(\eta)S_\delta(\boldsymbol{k})$$

$$\theta^{(2)\prime}(\boldsymbol{k},\eta)+\frac{3}{2}\Omega_m(\eta)(aH)^2\delta^{(2)}(\boldsymbol{k},\eta)=(aHf)^2D^2(\eta)S_\theta(\boldsymbol{k})$$

where we used

$$\delta^{(1)}(\boldsymbol{k},\eta)=D(\eta)\delta_0(\boldsymbol{k})$$

$$S_\delta(\boldsymbol{k})=\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\times\left[1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}\right]\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2),$$

$$S_\theta(\boldsymbol{k})=-\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\times\left[\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1^2}+\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{k_1^2k_2^2}\right]\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)$$

Second order

- Solving coupled set of sourced first-order ODE using standard techniques* yields:

$$\delta^{(2)}(\boldsymbol{k},\eta) = D_+^2(\eta)\int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \times F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2),$$

$$F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{k_1^2k_2^2} + \frac{1}{2}\boldsymbol{k}_1\cdot\boldsymbol{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)$$

time-independent perturbation theory kernel

- Velocity divergence θ obeys similar equation

* Assume matter domination when integrating equations; accurate to better than 1%.

Second order

- Solving coupled set of sourced first-order ODE using standard techniques* yields:

grows twice as fast as linear density

$$\delta^{(2)}(\boldsymbol{k},\eta) = D_+^2(\eta)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)$$
$$\times F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2),$$

$$F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{5}{7}+\frac{2}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{k_1^2k_2^2}+\frac{1}{2}\boldsymbol{k}_1\cdot\boldsymbol{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)$$

time-independent perturbation theory kernel

- Velocity divergence θ obeys similar equation

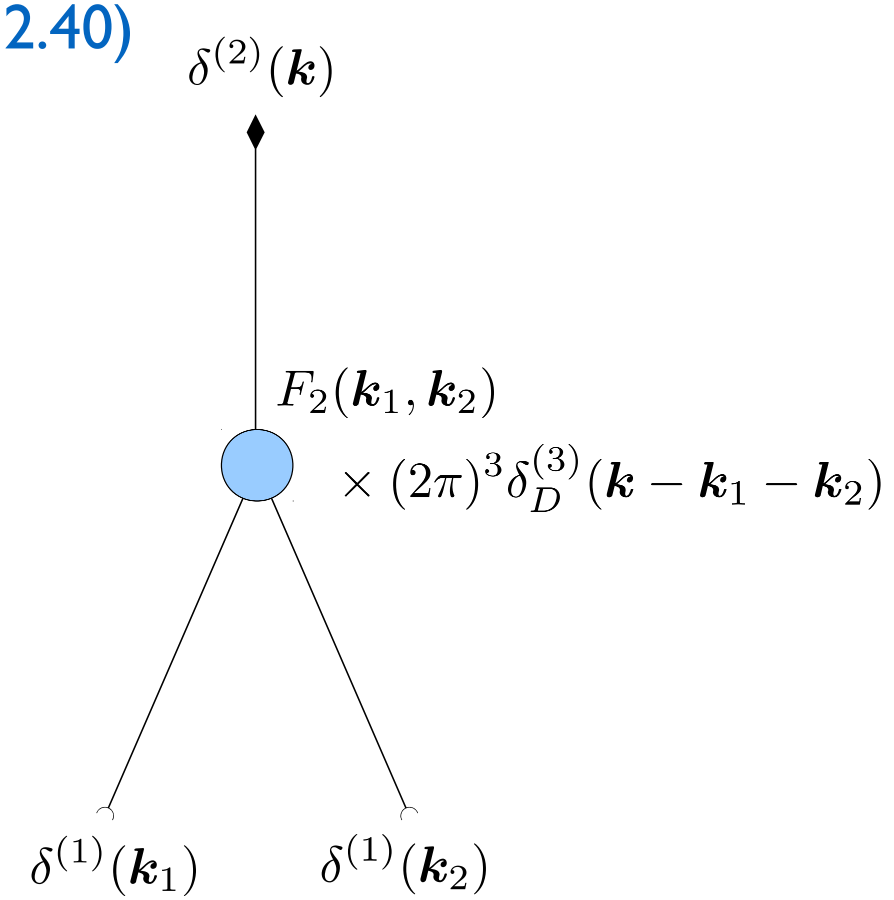
* Assume matter domination when integrating equations; accurate to better than 1%.

Diagrammatic representation

- F_2 corresponds to interaction vertex (with 3-momentum conservation) coupling two incoming δ_0

$$\delta^{(2)}(\boldsymbol{k},\eta) = D_+^2(\eta)\int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \times F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2),$$

Eq. (12.40)

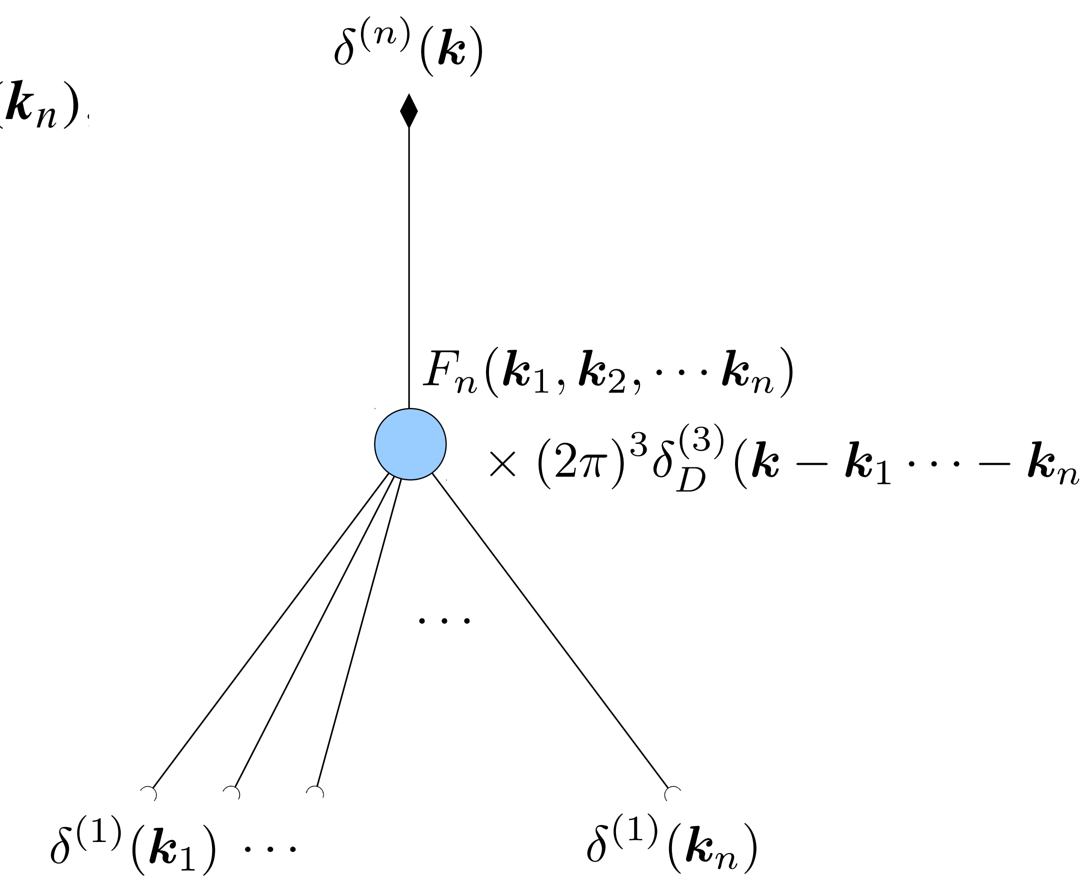


Diagrammatic representation

- Similarly, we can go to higher orders:

$$\delta^{(n)}(\boldsymbol{k},\eta) = D_+^n(\eta)\left[\prod_{i=1}^{n}\int\frac{d^3k_i}{(2\pi)^3}\right](2\pi)^3\delta_{\rm D}^{(3)}\left(\boldsymbol{k}-\sum_{i=1}^{n}\boldsymbol{k}_i\right)$$
$$\times F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_n).$$

PT kernels F_n obey recursion relation.



Matter power spectrum

- Since we don't know the initial conditions at the field level, let's compute statistics
- Power spectrum:

$$\left\langle \delta_{\rm m}(\boldsymbol{k},\eta)\delta_{\rm m}(\boldsymbol{k}',\eta)\right\rangle = D_+^2(\eta)\left\langle \delta_0(\boldsymbol{k})\delta_0(\boldsymbol{k}')\right\rangle + \left\langle \delta^{(2)}(\boldsymbol{k},\eta)\delta^{(2)}(\boldsymbol{k}',\eta)\right\rangle + 2\left\langle \delta^{(1)}(\boldsymbol{k},\eta)\delta^{(3)}(\boldsymbol{k}',\eta)\right\rangle + \cdots .$$

Eq. (12.42)

- Why these terms and not others?
  - Count terms that have equal numbers of δ_0
  - Terms with odd number of δ_0 vanish

Matter power spectrum

- Terms with odd number of δ_0 vanish because δ_0 is ***Gaussian***

(this can be generalized to include small amount of ***primordial non-Gaussianity***)

- For terms with even number, we use ***Wicks' theorem:***

$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\rangle = 0$$

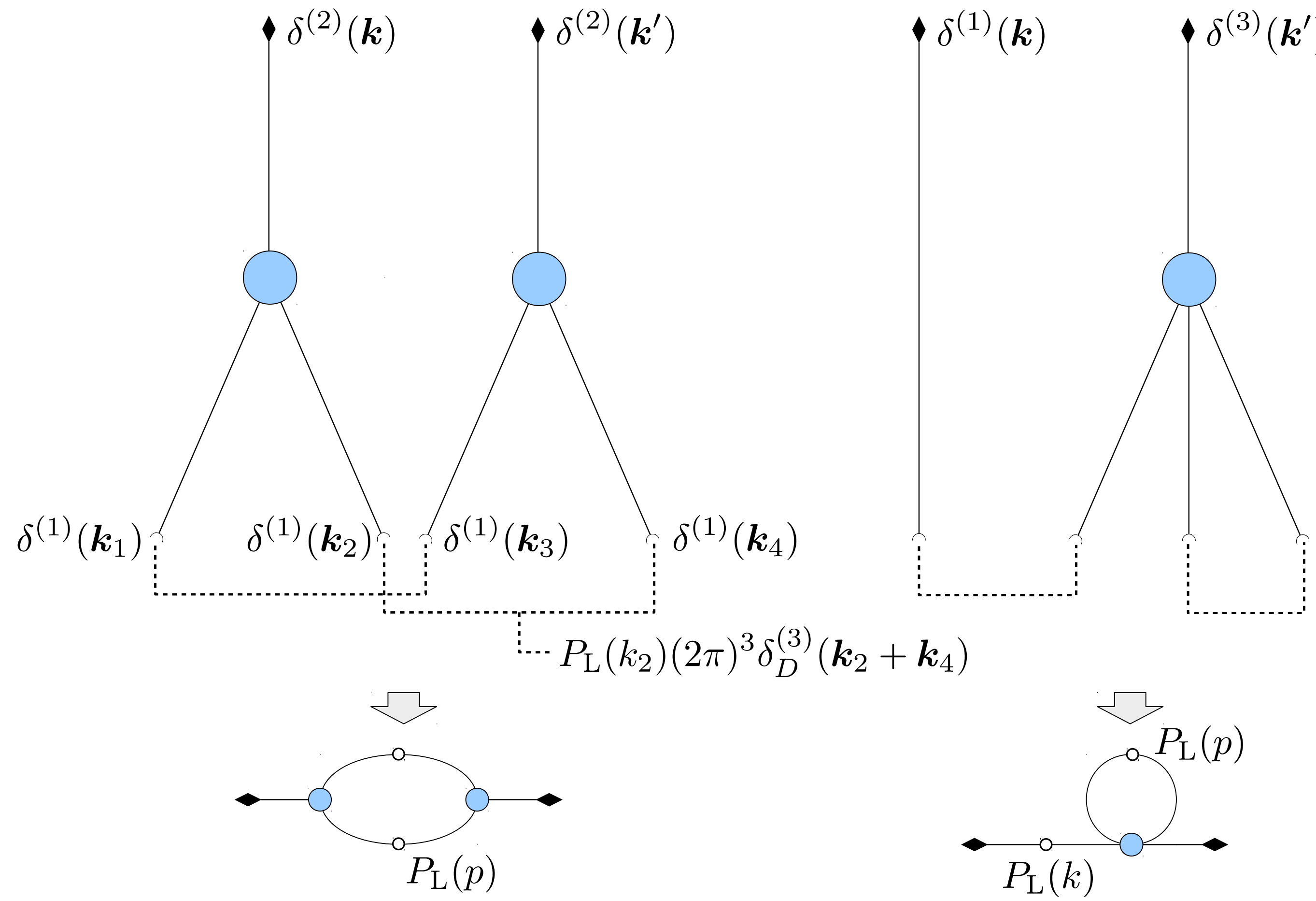
$$\begin{aligned}\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\delta_0(\boldsymbol{k}_4)\rangle = &(2\pi)^6\delta_{\rm D}^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\delta_{\rm D}^{(3)}(\boldsymbol{k}_3+\boldsymbol{k}_4)P(\boldsymbol{k}_1)P(\boldsymbol{k}_3)\\ &+(2\pi)^6\delta_{\rm D}^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_3)\delta_{\rm D}^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_4)P(\boldsymbol{k}_1)P(\boldsymbol{k}_2)\\ &+(2\pi)^6\delta_{\rm D}^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_4)\delta_{\rm D}^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_3)P(\boldsymbol{k}_1)P(\boldsymbol{k}_2).\end{aligned}$$

leads directly to

$$\begin{aligned}\left\langle\delta_{\rm m}(\boldsymbol{k},\eta)\delta_{\rm m}(\boldsymbol{k}',\eta)\right\rangle = &D_+^2(\eta)\left\langle\delta_0(\boldsymbol{k})\delta_0(\boldsymbol{k}')\right\rangle\\ &+\left\langle\delta^{(2)}(\boldsymbol{k},\eta)\delta^{(2)}(\boldsymbol{k}',\eta)\right\rangle+2\left\langle\delta^{(1)}(\boldsymbol{k},\eta)\delta^{(3)}(\boldsymbol{k}',\eta)\right\rangle+\cdots.\end{aligned}$$

Matter power spectrum

- Nicely represented using diagrams:



$$\left\langle \delta_{\rm m}(\boldsymbol{k},\eta)\delta_{\rm m}(\boldsymbol{k}',\eta)\right\rangle = D_+^2(\eta)\left\langle \delta_0(\boldsymbol{k})\delta_0(\boldsymbol{k}')\right\rangle + \left\langle \delta^{(2)}(\boldsymbol{k},\eta)\delta^{(2)}(\boldsymbol{k}',\eta)\right\rangle + 2\left\langle \delta^{(1)}(\boldsymbol{k},\eta)\delta^{(3)}(\boldsymbol{k}',\eta)\right\rangle + \cdots .$$

Eq. (12.42)

Matter power spectrum

- Use Feynman rules, or just plug in kernels to obtain:

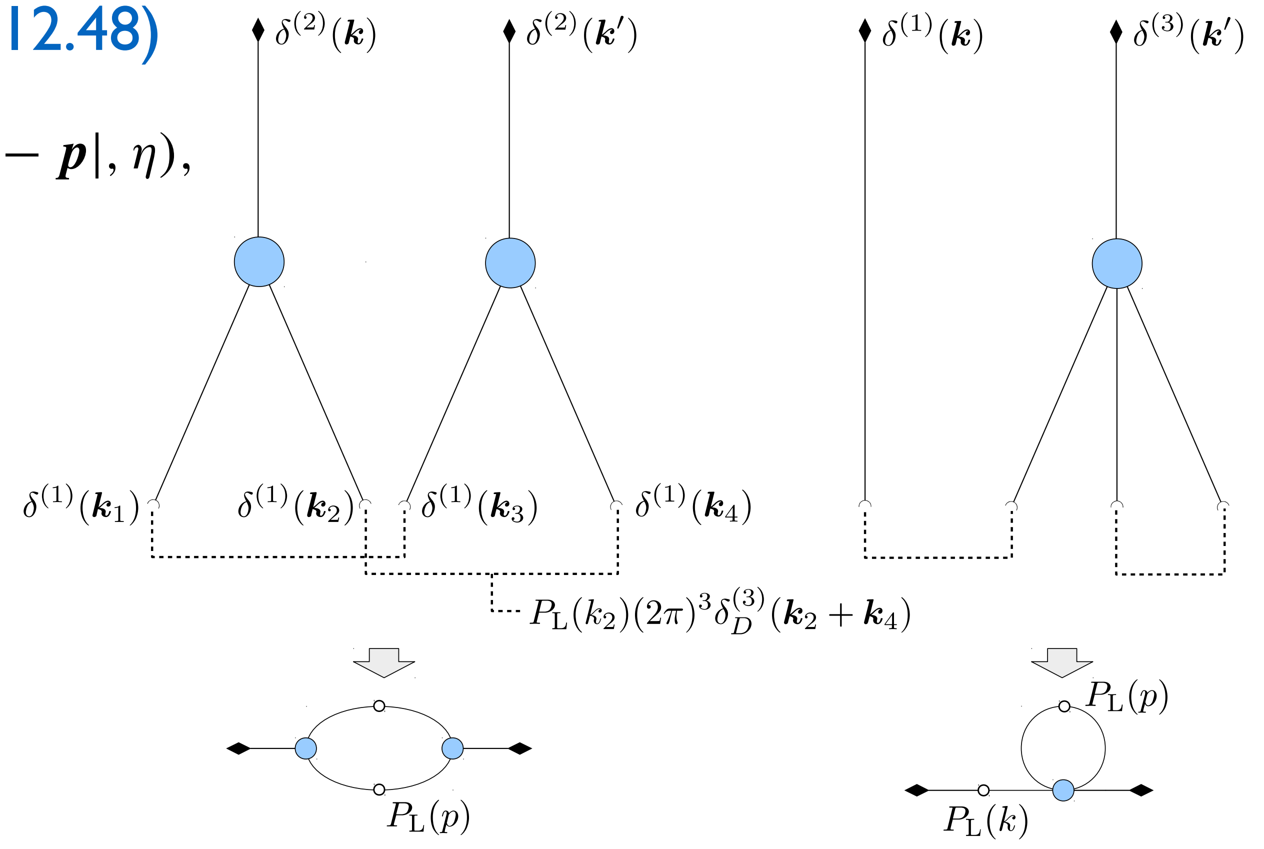
$$P(k,\eta) = P_{\rm L}(k,\eta) + P^{\rm NLO}(k,\eta) + \cdots\,,$$

$$P^{\rm NLO}(k,\eta) = P^{(22)}(k,\eta) + 2P^{(13)}(k,\eta),$$

Eq. (12.48)

$$P^{(22)}(k,\eta) = 2\int \frac{d^3p}{(2\pi)^3}\left[F_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\right]^2 P_{\rm L}(p,\eta)P_{\rm L}(|\boldsymbol{k}-\boldsymbol{p}|,\eta),$$

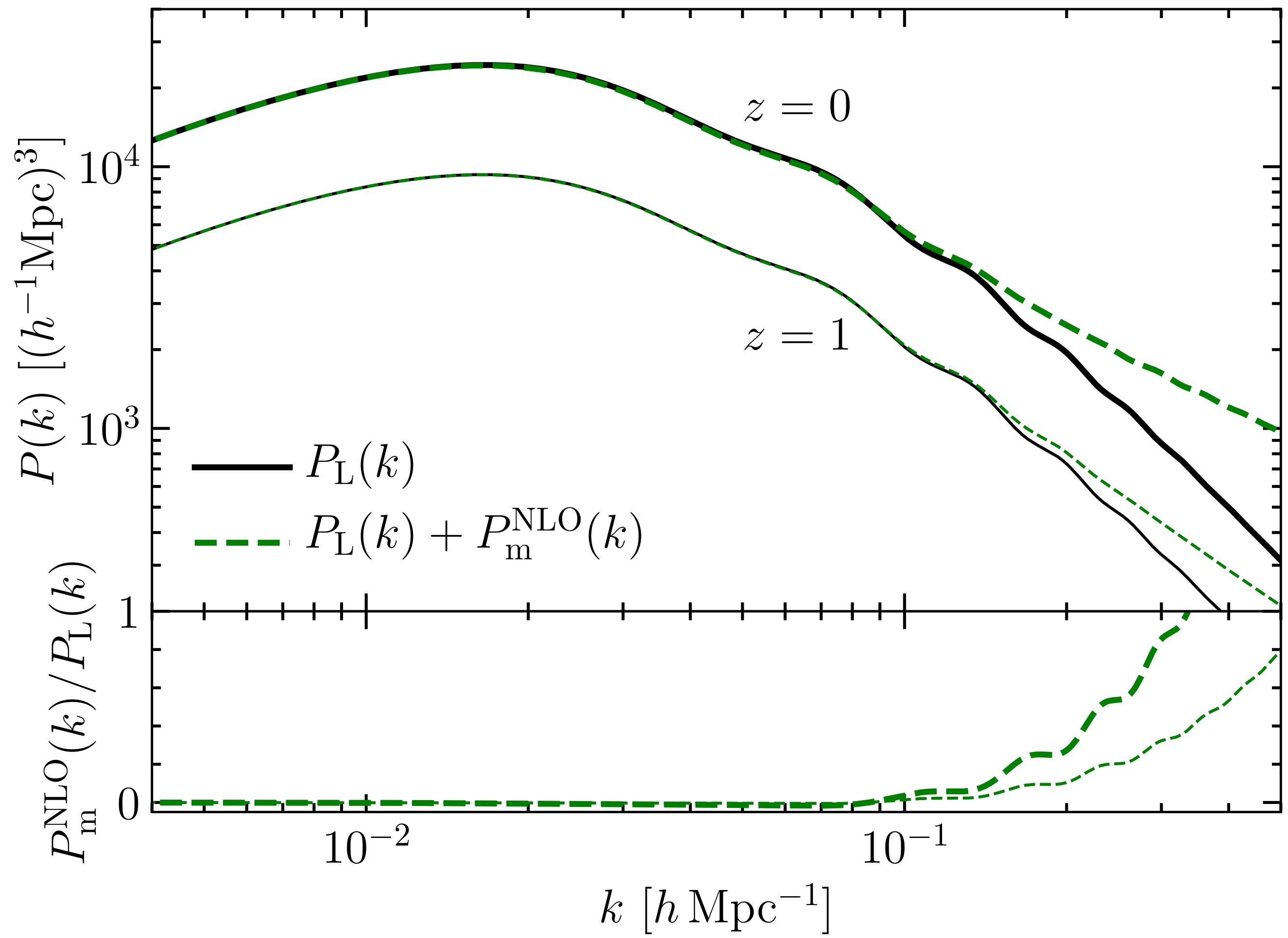
$$P^{(13)}(k,\eta) = 3P_{\rm L}(k,\eta)\int \frac{d^3p}{(2\pi)^3} F_3(\boldsymbol{p},-\boldsymbol{p},\boldsymbol{k})P_{\rm L}(p,\eta).$$



Matter power spectrum

- Then let computer do the work…

$$P(k,\eta) = P_{\rm L}(k,\eta) + P^{\rm NLO}(k,\eta) + \cdots\,,$$
$$P^{\rm NLO}(k,\eta) = P^{(22)}(k,\eta) + 2P^{(13)}(k,\eta),$$



Bispectrum

- The bispectrum, or three-point function of δ_0 vanishes, but not that of the evolved field δ_m, thanks to nonlinear evolution:

$$\langle\delta_{\rm m}(\boldsymbol{k}_1,\eta)\delta_{\rm m}(\boldsymbol{k}_2,\eta)\delta_{\rm m}(\boldsymbol{k}_3,\eta)\rangle = (2\pi)^3\delta_{\rm D}^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \times \left[2F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)P_{\rm L}(k_1,\eta)P_{\rm L}(k_2,\eta)+2\text{ perm.}\right]$$

Eq. (12.51)

At leading order; there are also “next-to-leading” (NLO) contributions - try writing down the diagram for the leading three-point function as well as the NLO one!

Beyond the fluid approximation

- So far, did well-defined perturbation theory, but of the *wrong equation:* collisionless matter is not a fluid
- Rather, the correct equation is the collisionless Boltzmann equation
- What is the error we are making?
- Recall that we neglected the velocity dispersion, or stress tensor σ_m, which adds force term to the Euler equation, $\rho_{\rm m}^{-1}\partial_j\sigma_{\rm m}^{ij}$

$$\frac{1}{m}\left\langle p^i\,p^j\right\rangle_{f_{\rm m}} = \rho_{\rm m}u_{\rm m}^i u_{\rm m}^j + \sigma_{\rm m}^{ij}.$$

- What is the effect of the stress tensor? Can we incorporate it?

Beyond the fluid approximation

- Idea: treat stress tensor as effective quantity, and parametrize it, at the background and perturbation level:

$$\sigma^{ij}_{\rm m}(\boldsymbol{x},\eta) = \bar{\sigma}_{\rm m}(\eta)\delta^{ij}\left[1 + c_\sigma(\eta)\delta_{\rm m}(\boldsymbol{x},\eta) + \ldots\right]$$

- We can't predict the coefficients from within the fluid picture - leave them free for now

$$\bar{\sigma}_{\rm m}(\eta),\ \ c_\sigma(\eta)$$

Beyond the fluid approximation

$$\sigma_{\rm m}^{ij}(\boldsymbol{x},\eta) = \bar{\sigma}_{\rm m}(\eta)\delta^{ij}\left[1 + c_\sigma(\eta)\delta_{\rm m}(\boldsymbol{x},\eta) + \ldots\right]$$

- Insert into Euler equation:

$$u_{\rm m}^{i}{}' + aHu_{\rm m}^i + \partial^i\Psi + \frac{c_\sigma\bar{\sigma}_{\rm m}}{\bar{\rho}_{\rm m}}\partial^i\delta_{\rm m} = \text{(unchanged 2nd-order terms)}$$

Notice that constant, background stress has no dynamical effect.

Beyond the fluid approximation

$$\sigma_{\rm m}^{ij}(\boldsymbol{x},\eta) = \bar{\sigma}_{\rm m}(\eta)\delta^{ij}\left[1 + c_\sigma(\eta)\delta_{\rm m}(\boldsymbol{x},\eta) + \ldots\right]$$

- Insert into Euler equation, take divergence again:

$$\theta'_{\rm m} + aH\theta_{\rm m} + \nabla^2\Psi + \frac{c_\sigma\bar{\sigma}_{\rm m}}{\bar{\rho}_{\rm m}}\nabla^2\delta_{\rm m} = (\text{unchanged 2nd-order terms})$$

Notice that constant, background stress has no dynamical effect.

Beyond the fluid approximation

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Notice that constant, background stress has no dynamical effect.

- Additional contribution is suppressed on large scales: two additional derivatives, ~k^2 in Fourier space
- Hence, can take into account stress tensor at leading order by adding one term to equations, *at the price of an unknown, free coefficient* $\bar{\sigma}_{\rm m} c_\sigma$

Cold collisionless matter = *effective fluid*

- At leading order, this is just another linear term in the equations (but with more derivatives)
- By redefining coefficient, correction to final density field can be written as (with free coefficient C_s^2)

$$\delta^{(1)}(\boldsymbol{k},\eta) \to \left[1 - C_s^2(\eta)k^2\right] D_+(\eta)\delta_0(\boldsymbol{k}) \quad ; \quad P_{\rm NLO}(k) \to P_{\rm NLO}(k) - 2C_s^2(\eta)k^2 P_{\rm L}(k)$$

Similar size as $P_{\rm NLO}(k)$

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$$\delta^{(1)}(\boldsymbol{k},\eta) \to \left[1 - C_s^2(\eta)k^2\right] D_+(\eta)\delta_0(\boldsymbol{k}) \quad ; \quad P_{\mathrm{NLO}}(k) \to P_{\mathrm{NLO}}(k) - 2C_s^2(\eta)k^2 P_{\mathrm{L}}(k)$$

Similar size as $P_{\mathrm{NLO}}(k)$

- In fact, theoretical consistency forces us to introduce C_s^2 (~ eff. sound horizon) as *counterterm:*

$$P^{(13)}(k,\eta) = 3P_{\mathrm{L}}(k,\eta)\int \frac{d^3p}{(2\pi)^3} F_3(\boldsymbol{p},-\boldsymbol{p},\boldsymbol{k}) P_{\mathrm{L}}(p,\eta)$$

$$\propto k^2/p^2 \text{ for } p \gg k$$

Yes, P_{22} also leads to a counterterm, but that one is much smaller.

Effective Field Theory of Structure Formation

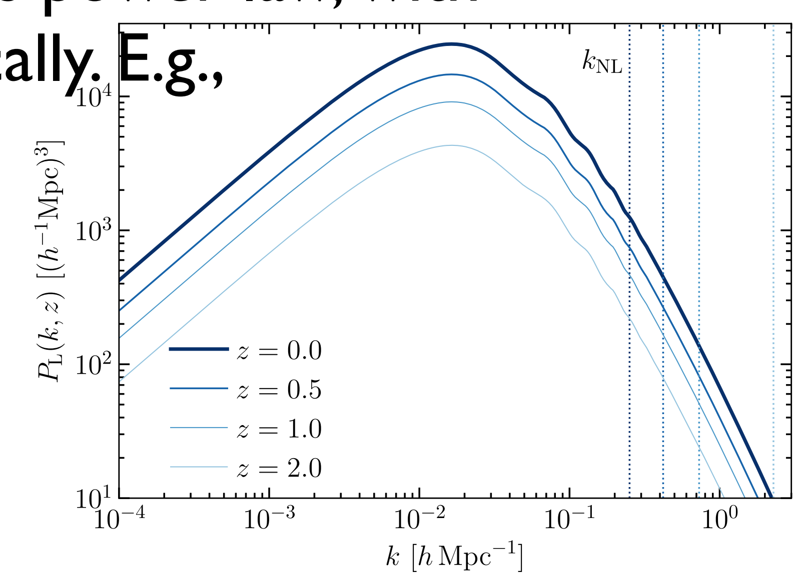
- Idea: allow for all counterterms in effective fluid equations consistent with symmetries: general covariance; mass and momentum conservation
- Order different contribution according to their scaling with k
- Only one relevant scale: k_{NL}, where (roughly) matter density field becomes fully nonlinear:

$$k_{\rm NL}^{-2} = \int \frac{d^3p}{(2\pi)^3} p^{-2} P_{\rm L}(p)$$

- For this ordering, we typically approximate $P_L(k) \sim k^n$ as power law, with $n \sim -1.5$, allowing us to compute loop integrals analytically. E.g.,

$$P_{\rm NLO}(k) \sim \left(\frac{k}{k_{\rm nl}}\right)^{3+n} P_{\rm L}(k)$$

$$P_{C_s^2}(k) \sim \left(\frac{k}{k_{\rm nl}}\right)^{2} P_{\rm L}(k)$$



Non-gravitational interactions of baryons

- So far, completely ignored non-gravitational interactions, while I argued that we are including baryons…
- Let's consider the effect of pressure then, assuming some relation *p=p(ρ)* (barotropic fluid). Pressure term in *baryon* Euler equation, at leading order:

$$\rho_{\rm b}^{-1}\partial_i\, p(\rho_{\rm b}) = \rho_{\rm b}^{-1}\partial_i\left[\frac{dp}{d\rho}\delta\rho_{\rm b}\right] = c_s^2\partial_i\delta_{\rm b}$$

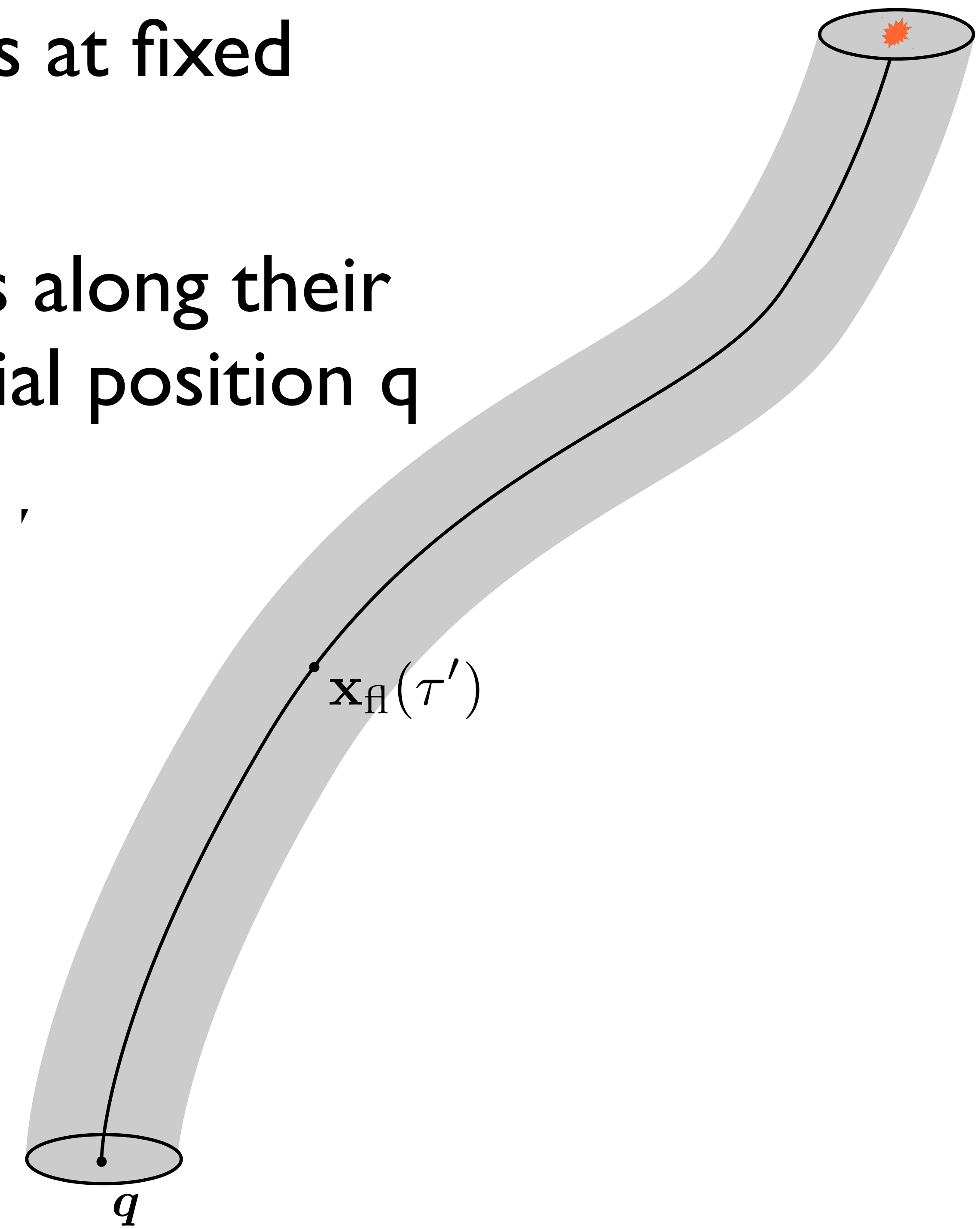
$$\text{with}\quad c_s^2 = \frac{dp}{d\rho};\qquad \delta\rho_{\rm b} = \bar{\rho}_{\rm b}\delta_{\rm b}$$

- Precisely the same shape as effective stress contribution! As long as we are interested only in total matter, we can combine the two into a single C_s^2.

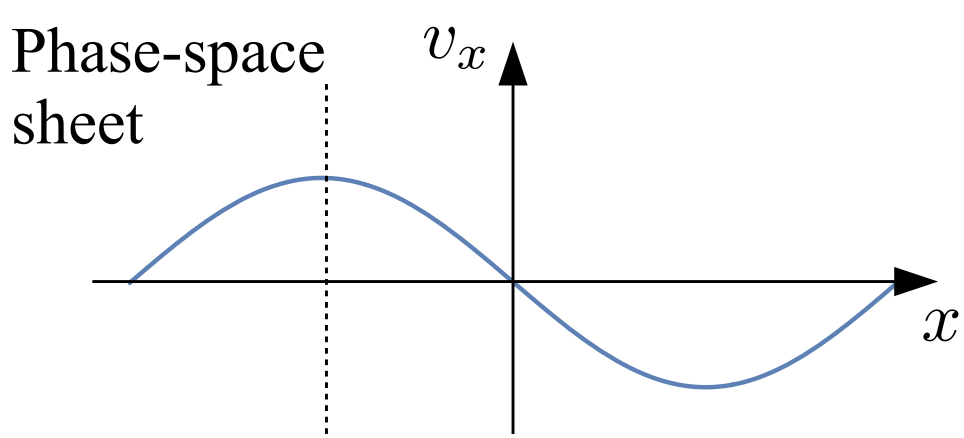
$$\frac{c_\sigma\bar{\sigma}_{\rm m}}{\bar{\rho}_{\rm m}}\partial^i\delta_{\rm m}$$

Alternative: Lagrangian approach to structure formation

- So far, worked with Eulerian fields at fixed spatial position x
- Alternative: follow mass elements along their trajectory, labeling them with initial position q



Works because dark matter is cold:
vanishing initial velocities



$$f_{\rm m}(\boldsymbol{x}, \boldsymbol{p}, t) = \frac{\rho_{\rm m}(\boldsymbol{x}, t)}{m}(2\pi)^3 \delta_{\rm D}^{(3)}\left(\boldsymbol{p} - m\boldsymbol{u}_{\rm m}(\boldsymbol{x}, t)\right)$$

<=> no velocity dispersion

Alternative: Lagrangian approach to structure formation

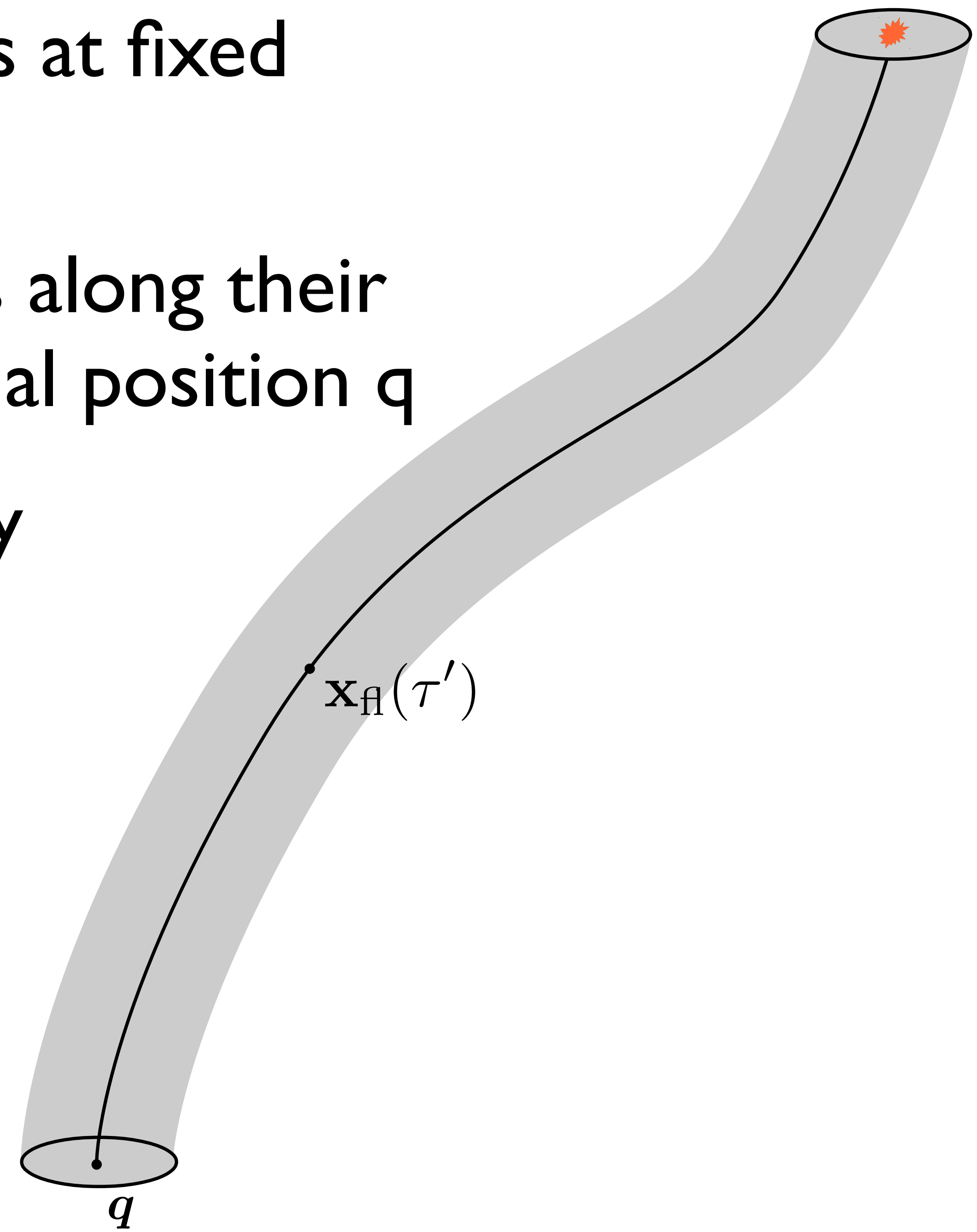
- So far, worked with Eulerian fields at fixed spatial position x
- Alternative: follow mass elements along their trajectory, labeling them with initial position q
- Time-dependent position given by

$$\boldsymbol{x}_{\text{fl}}(\boldsymbol{q},\eta) = \boldsymbol{q} + \boldsymbol{s}(\boldsymbol{q},\eta)$$

- Then use geodesic equations:

$$\frac{dx^i}{dt} = \frac{p^i}{am}$$

$$\frac{dp^i}{dt} = -Hp^i - \frac{m}{a}\partial_i\Psi$$



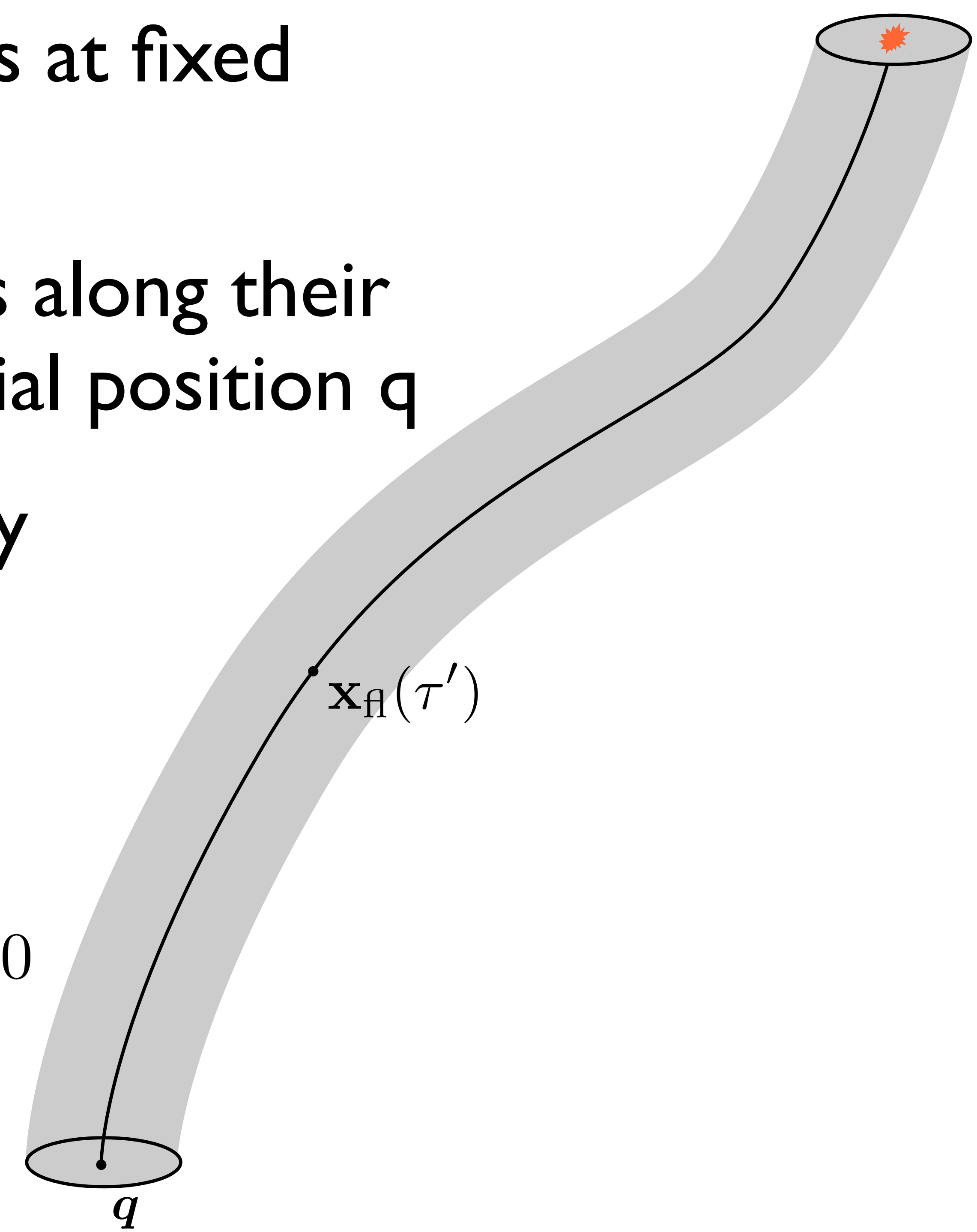
Alternative: Lagrangian approach to structure formation

- So far, worked with Eulerian fields at fixed spatial position x
- Alternative: follow mass elements along their trajectory, labeling them with initial position q
- Time-dependent position given by

$$\boldsymbol{x}_{\rm fl}(\boldsymbol{q},\eta) = \boldsymbol{q} + \boldsymbol{s}(\boldsymbol{q},\eta)$$

- Then use geodesic equations:

$$\boldsymbol{s}''(\boldsymbol{q},\eta) + aH\boldsymbol{s}'(\boldsymbol{q},\eta) + \boldsymbol{\nabla}_x\Psi(\boldsymbol{x}_{\rm fl}(\boldsymbol{q},\eta),\eta) = 0$$



Alternative: Lagrangian approach to structure formation

- At initial time, density perturbations were negligible, so a given element d^3q corresponds to equal mass everywhere. Hence, density is given directly by Jacobian:

$$\bar{\rho}_\mathrm{m}(\eta) d^3\boldsymbol{q} = \rho_\mathrm{m}(\boldsymbol{x},\eta) d^3\boldsymbol{x}$$

$$\Rightarrow \quad \frac{\rho_\mathrm{m}}{\bar{\rho}_\mathrm{m}} = 1 + \delta_\mathrm{m} = \left|\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{x}}\right| = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1} = |\delta_{ij} + \partial_{q,i} s_j(\boldsymbol{q},\eta)|$$

- Can insert δ_m into Poisson equation to obtain Ψ. Perturbation theory then proceeds by writing

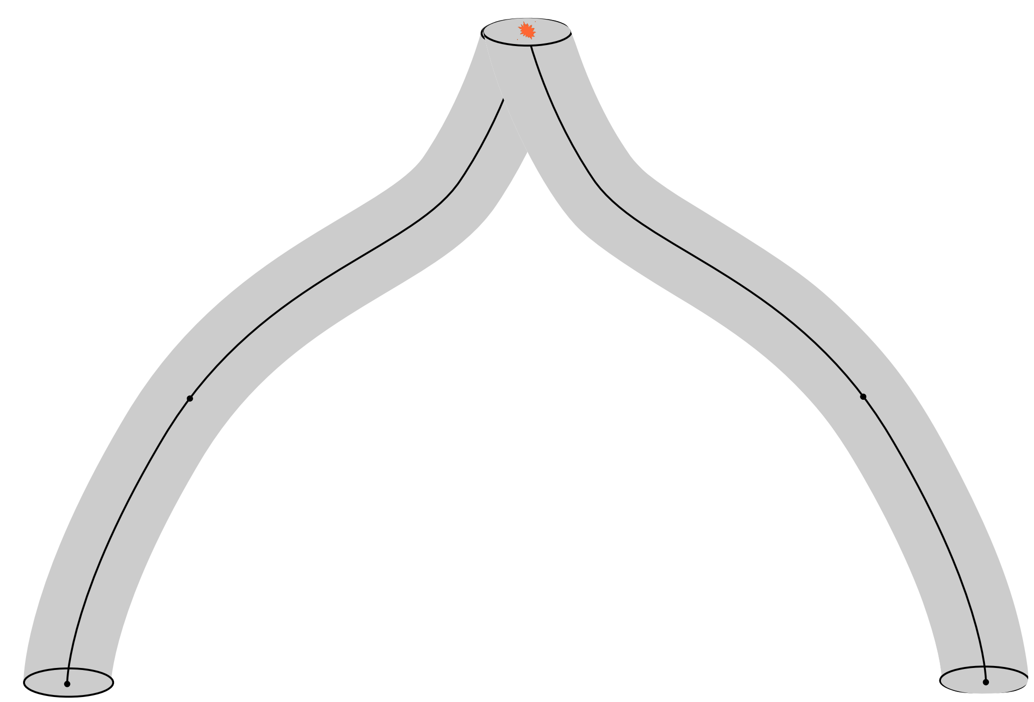
$$\boldsymbol{s} = \boldsymbol{s}^{(1)} + \boldsymbol{s}^{(2)} + \ldots$$

and solving equation for displacement order by order.

Alternative: Lagrangian approach to structure formation

- Note: two matter elements can end up at the same final position x!

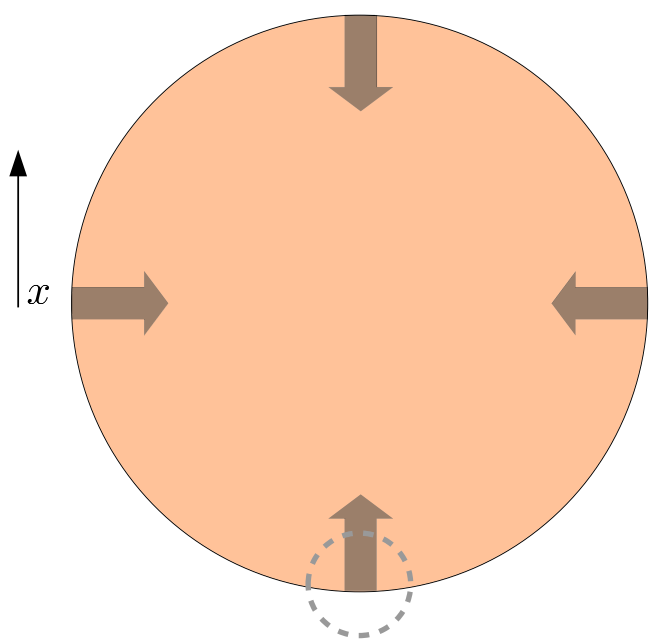
$$\boldsymbol{x}_{\mathrm{fl}}(\boldsymbol{q},\eta) = \boldsymbol{q} + \boldsymbol{s}(\boldsymbol{q},\eta)$$

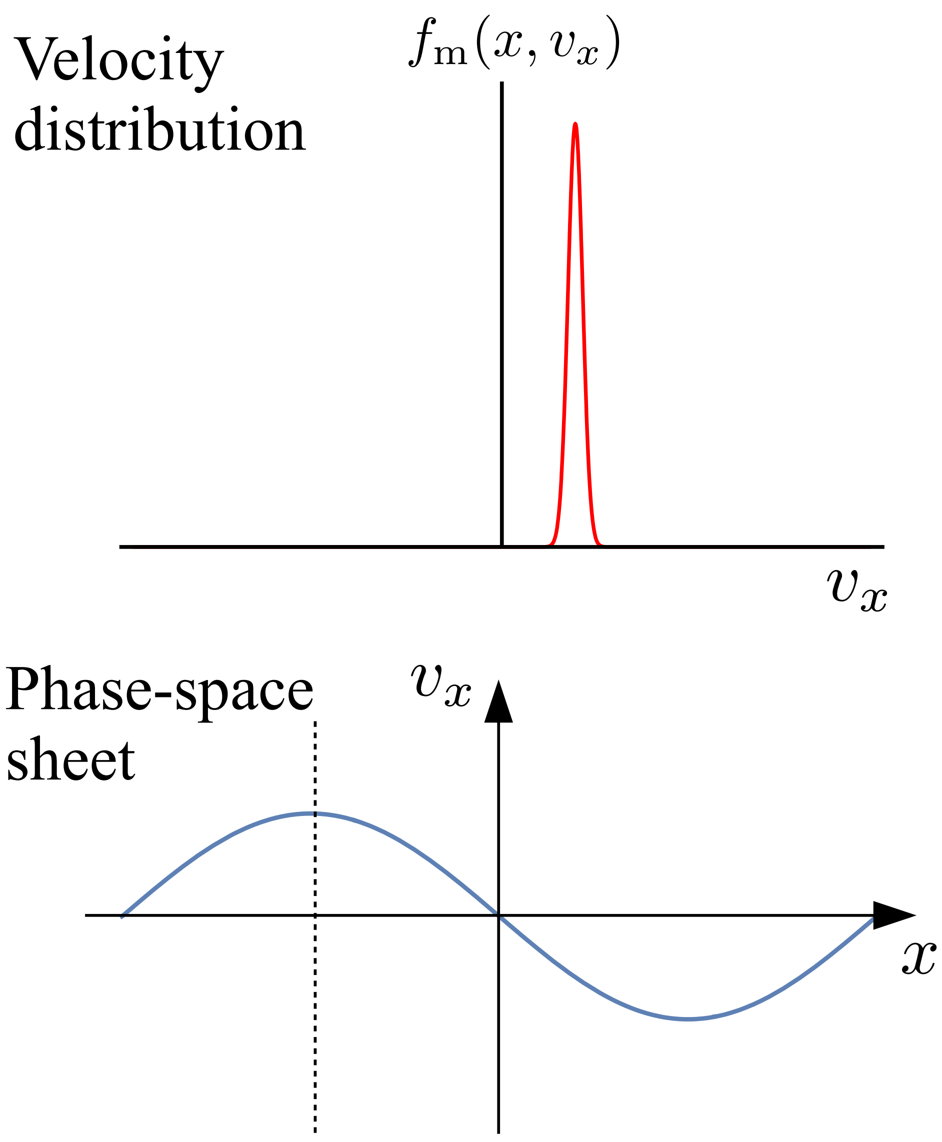


- Known as *shell crossing*
- Possible because (dark) matter is collisionless.
- This cannot happen in the fluid picture though.
- In this sense, Lagrangian approach is closer to the correct physics.

Phasespace view of structure formation

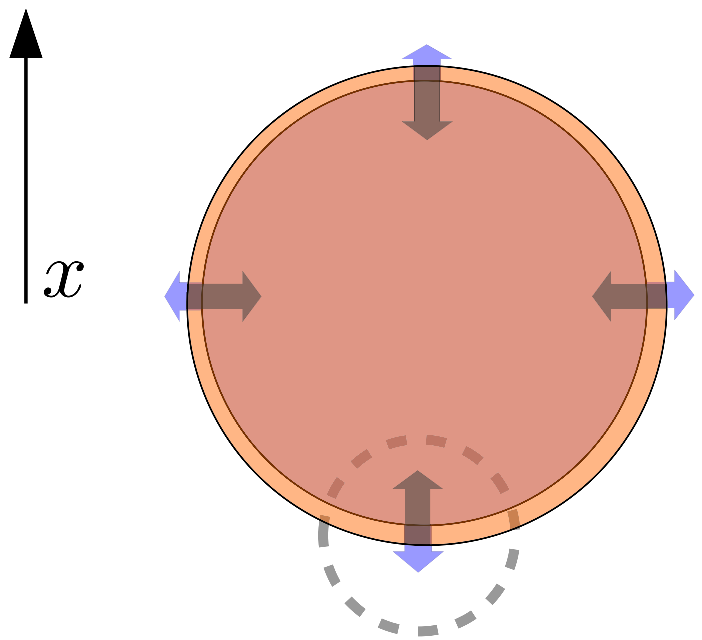
- Initial stages of collapse of overdense region

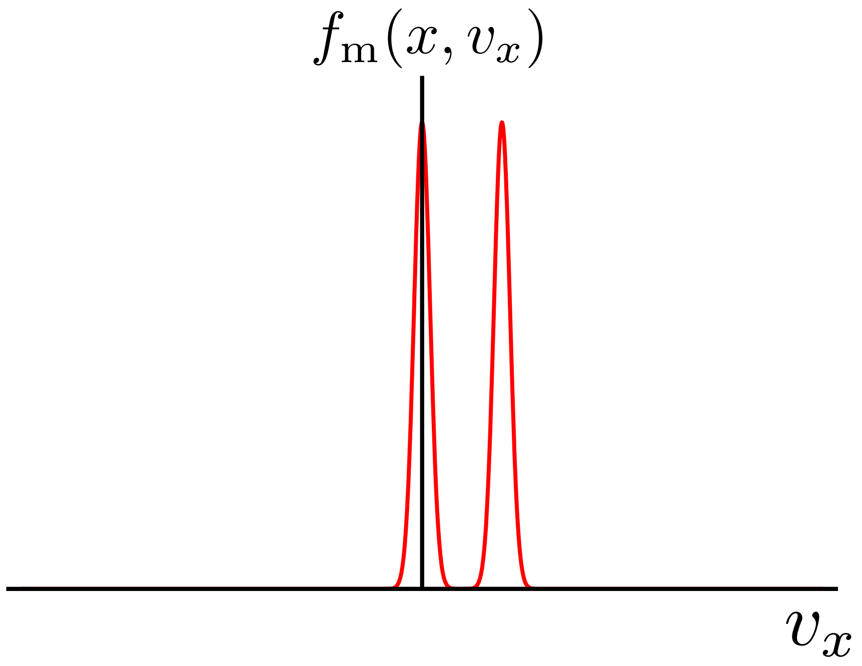


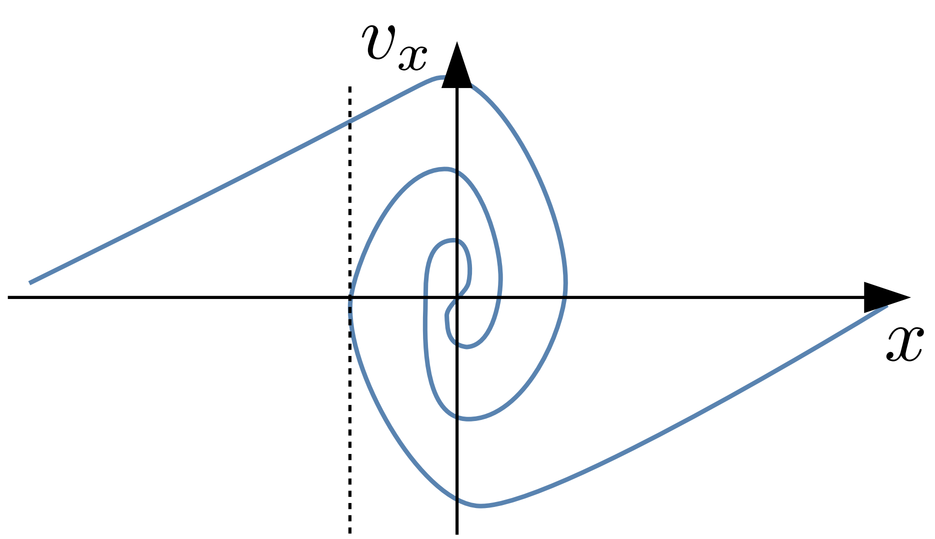


Phasespace view of structure formation

- Later stages of collapse of overdense region







Structure formation beyond perturbation theory

- In order to take this phasespace evolution into account properly, need to go beyond fluid picture and perturbation theory.
- Back to collisionless Boltzmann equation!