Large-scale Structure: the numerical version

Lecture 1:
LSS basics. Growth function and power spectrum.

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ICTP Trieste/SAIFR Cosmology School

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MICHIGAN

Max-Planck-Institut für Astrophysik

Alexander von Humboldt Stiftung / Foundation

Ann Arbor, Michigan

University of Michigan

Michigan Stadium (115,000)

LSA LEINWEBER CENTER
UNIVERSITY OF MICHIGAN

LCTP focuses on:

1. Particle theory
2. Particle pheno
3. **Cosmology**

Fairly old research group photo

Also: working with three Brazilian students at the moment
(U. Andrade, R. van Mertens; O. Alves ←@Michigan)

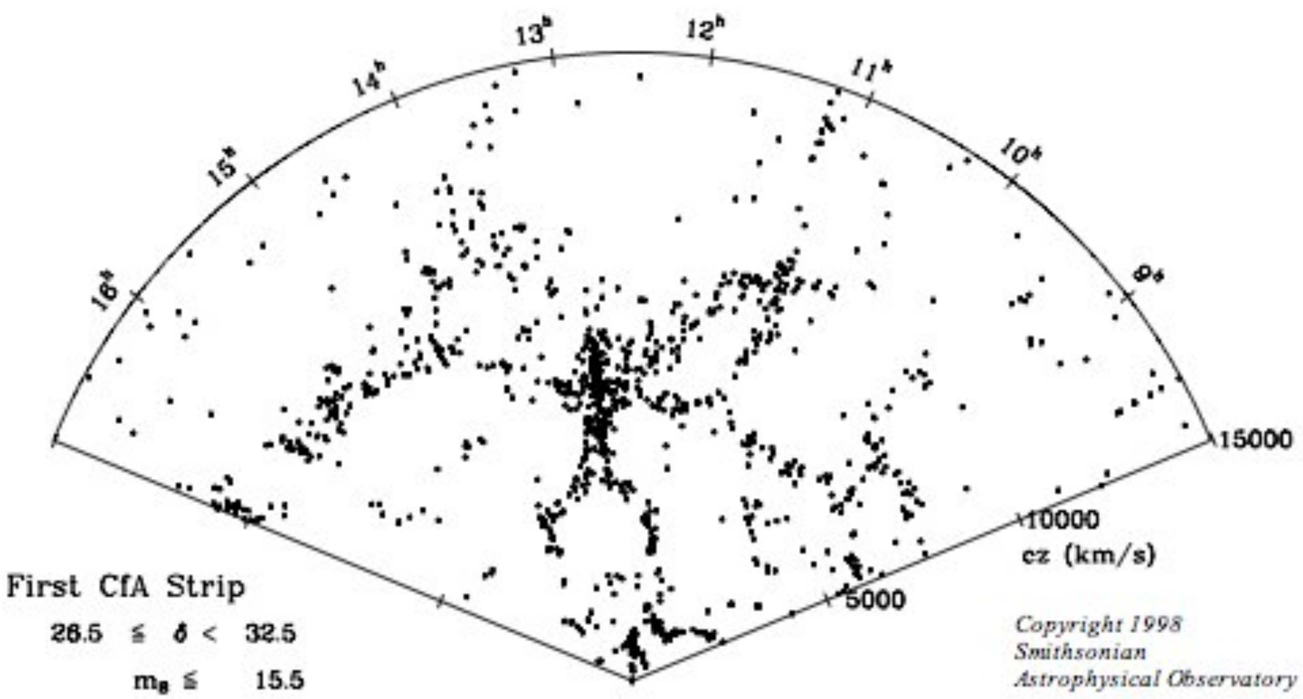
Emphasis of my lectures

1. Concepts made simple/intuitive
2. Developing experience with numerical work

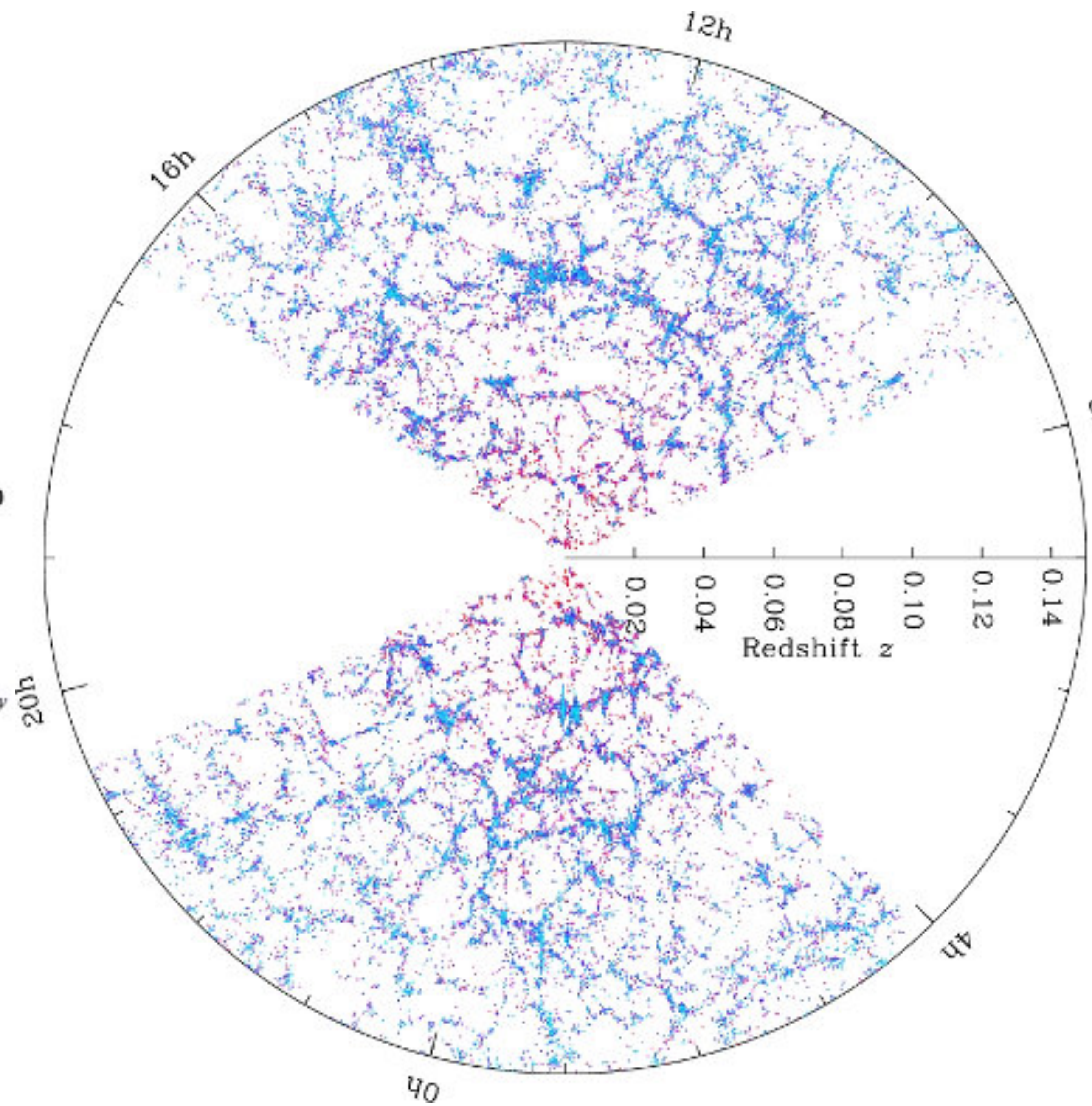
I want to enable to you to code up calculations
FROM SCRATCH
and produce useful, research-ready quantities.

$\Longrightarrow$ Probably the **least theoretical** lectures in this school
Probably the **most numerical** lectures in this school

How do we describe the large-scale structure and constrain cosmological model?



Harvard-Cfa survey;
de Lapparent, Geller & Huchra (1986)



SDSS/BOSS survey
SDSS/BOSS collaboration

Ongoing or upcoming LSS experiments:

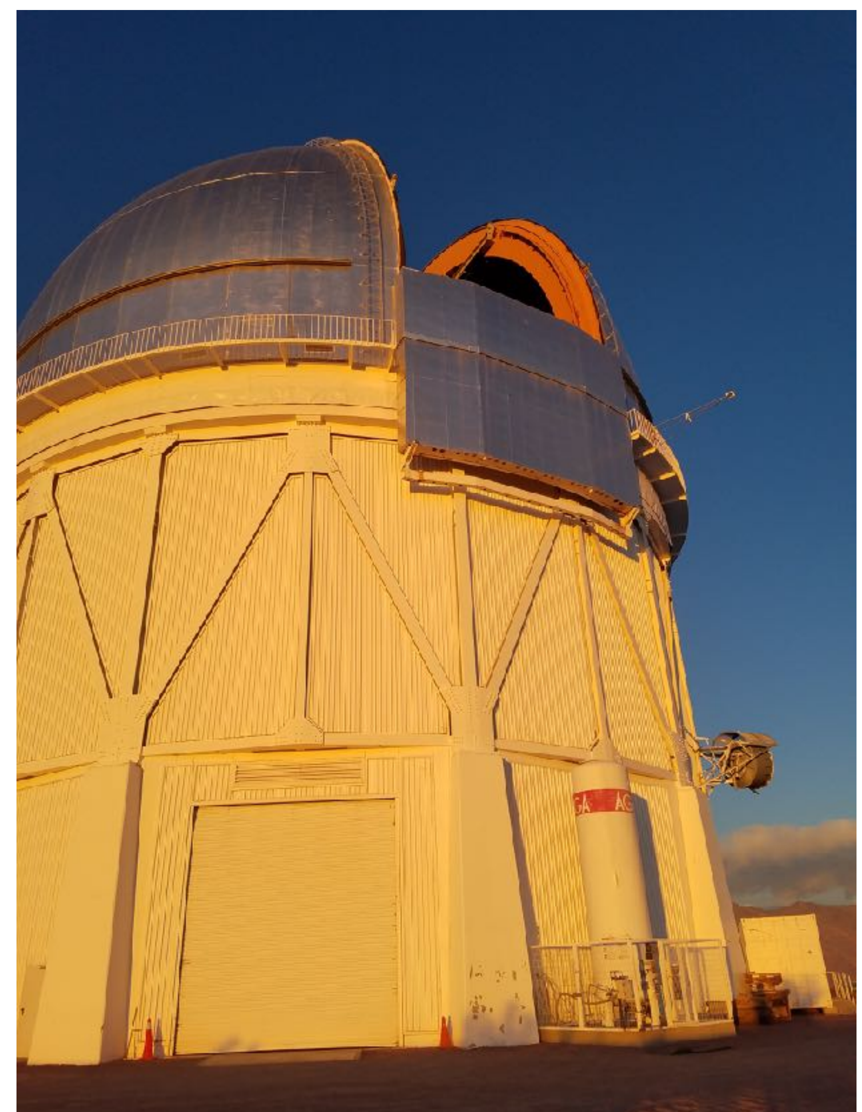
- **Ground photometric:**
  - Kilo-Degree Survey (KiDS)
  - Dark Energy Survey (DES)
  - Hyper Supreme Cam (HSC)
  - Large Synoptic Survey Telescope (LSST)
- **Ground spectroscopic:**
  - Hobby Eberly Telescope DE Experiment (HETDEX)
  - Prime Focus Spectrograph (PFS)
  - Dark Energy Spectroscopic Instrument (DESI)
- **Space:**
  - Euclid
  - Wide Field InfraRed Space Telescope (WFIRST)

Dark Energy Survey

- New camera on 4m telescope in Chile
- Observations 2013-2019
- ~700 scientists worldwide
- Analyses in progress (first major results Aug 2017)

Dark Energy Survey (DES)

Cerro Tololo, Chile



Blanco Telescope

Overdensity delta

The basic* "observable" quantity is the overdensity

(* this doesn't capture the overall number of galaxies, which is described separately by the mass function)

$$\delta(\mathbf{x}, t) \equiv \frac{\rho(\mathbf{x}, t) - \bar{\rho}}{\bar{\rho}}$$

Note that it satisfies $-1 \leq \delta \leq \infty$.

[A corollary of that is that, for large delta $\gtrsim 1$, the distribution of delta is always non-Gaussian.]

We will specialize in small fluctuations ($|\delta| \ll 1$) from here on.

In that limit, the statistical distribution of delta may or may not be Gaussian:

- Standard inflationary theory (single scalar field, always slow-rolls, in Einstein gravity, on a beautiful day in June) predicts tiny (nonzero but basically unobservable) Non-G - so basically the "sky is Gaussian"
- Searches for so-called primordial NG are at the forefront of research in cosmology - finding one would be finding Holy Grail

Fourier-space overdensity

$$\delta_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}} \int \delta(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r}$$

Note:

- $\delta(\mathbf{r})$ is dimensionless, but then
- $\delta_{\mathbf{k}}$ is not, and has units of $L^{3/2}$, or $[k]^{-3/2}$

$\Rightarrow$ so $k^3 |\delta_{\mathbf{k}}|^2$ is dimensionless

$\Rightarrow$ more on that a bit later

Two-point correlation function

Consider a point process (with point particles in 3D space) with underlying constant density n.

Probability of finding a particle in volume dV is

$$dP = n dV$$

Probability of finding *two* particles in respective volumes dV_1, dV_2 is

$$dP = n^2 (1 + \xi(r_{12}))\, dV_1 dV_2$$

where ksi (r12) is the **excess probability for clustering**.
It is the two-point correlation function.

Related quantities can be written as a function of ksi. For example,

$$dP(2|1) = n(1 + \xi(r_{12}))\, dV_2$$

$$\langle N \rangle = nV + n \int \xi(r) dV.$$

Two-point correlation function ξ(r)

Intuitive understanding of 2pt correlation function:

It is the excess probability for clustering.

"If I am sitting on one galaxy, I am more (or less) likely to find one nearby, then if I were sitting in space (between galaxies)"

What functional form does ξ(r) take?

Cosmological theory actually most straightforwardly predicts the Fourier transform of the 2pt function - the power spectrum P(k). Nevertheless, this phenomenological fit worked well since 1980s:

$$\xi(r) = \left(\frac{r}{r_0}\right)^{\gamma}$$

with $\gamma \approx -1.8$ and $r_0 = 5$ Mpc (galaxies) or 20 Mpc (clusters).

Definition of $\boldsymbol{\xi}$(r)

$$\xi(r) = \frac{\langle\, [\, \rho(\mathbf{x}+\mathbf{r}) - \langle\rho\rangle\,]\,[\,\rho(\mathbf{x}) - \langle\rho\rangle\,]\,\rangle_{\mathbf{x}}}{\langle\rho\rangle^2}$$

$$= \langle\, \delta(\mathbf{x}+\mathbf{r})\,\delta(\mathbf{x})\,\rangle_{\mathbf{x}}$$

How to measure $\boldsymbol{\xi}$(r)

So measure points separated by a distance r, and average over the locations of all such points (vector **x**).

To do this precisely and accurately, you need an *estimator*. More about that in a bit.

Two subtle points

$$\xi(r) = \frac{\langle\, [\, \rho(\mathbf{x} + \mathbf{r}) - \langle \rho \rangle\,]\, [\, \rho(\mathbf{x}) - \langle \rho \rangle\,]\, \rangle_{\mathbf{x}}}{\langle \rho \rangle^2}$$

- The average above is nominally over all fixed spatial positions x in many *realizations* of the universe (!). We obviously don't have access to multiple universes, so we reinterpret the average as that over all locations x in *our* (one) universe. In doing that we have assumed the ergodic theorem. See Thorne-Blandford "Modern Classical Physics" for more on ergodic theorem and its proof.

- The two point function above is formally $\boldsymbol{\xi}(\mathbf{r})$ (note, vector **r**). We usually convert it to $\boldsymbol{\xi}$(r) (scalar r) by assuming the *principle of homogeneity* - that the universe is the same at every location in space. As far as we know (and as most inflationary models predict), the universe is homogeneous on largest scales, though testing that is cutting-edge area of research).

Three-point correlation function ζ(r)

You know the drill! Definition:

$$dP = n^3[1 + \xi(r_{12}) + \xi(r_{13}) + \xi(r_{23}) + \underline{\zeta(r_{123})}]\, dV_1 dV_2 dV_3$$

Explicitly:

$$\zeta(r, s, |\mathbf{r} - \mathbf{s}|) = \frac{\langle\, [\,\rho(\mathbf{x} + \mathbf{r}) - \langle\rho\rangle\,]\, [\,\rho(\mathbf{x} + \mathbf{s}) - \langle\rho\rangle\,]\, [\,\rho(\mathbf{x}) - \langle\rho\rangle\,]\,\rangle_{\mathbf{x}}}{\langle\rho\rangle^3}$$

$$= \langle\, \delta(\mathbf{x} + \mathbf{r})\, \delta(\mathbf{x} + \mathbf{s})\, \delta(\mathbf{x})\,\rangle_{\mathbf{x}}$$

Note, it is a function of a *triangle* configuration.
(source of a lot of complexity/complication in working with 3pt fun)

Fourier trans. of the 3-pt function is called the bispectrum.
We won't cover it here but it is another hot research topic.
Bispectrum, which is ≈0 in the CMB, can be large (and a huge pain to measure and model) in the LSS.

Growth of linear perturbations

Fabian's and Valerie's lectures covered perturbation theory, how perturbations are created evolve, adiabatic and issocurvature modes. Here we start with one important result:

Specializing in adiabatic modes, late times (c_s is tiny), sub-horizon modes, and assuming GR:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M(t)\delta = 0$$

This is the key equation that describes the growth of linear, sub-horizon fluctuations in General relativity.

Growth of linear perturbations

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M(t)\delta = 0$$

1. Radiation domination

$a(t) \propto t^{1/2}$, H(t)=dot(a)/a = 1/(2t), ignore $4\pi G\boldsymbol{\rho}_M$ term

get ddot(δ)+dot(δ)/t=0, or

$$\delta(t) = A_1 + A_2 \ln(t) \quad \text{(RD)}$$

2. Matter domination

$a(t) \propto t^{2/3}$, H(t)=dot(a)/a = 2/(3t), $4\pi G\boldsymbol{\rho}_M=(3/2)H^2$,

sol of the $\delta \propto t^n$ get n(n-1)+(4/3)n–2/3=0, or

$$\delta(t) = B_1 t^{2/3} + B_2 t^{-1} \simeq a(t) \quad \text{(MD)}$$

3. Lambda domination

$a(t) \propto \exp(H_\Lambda t)$, $H(t)=H_\Lambda$=const, ignore $4\pi G\boldsymbol{\rho}_M$ term

get ddot(δ)+$2H_\Lambda$dot(δ)=0, or

$$\delta(t) = C_1 + C_2 \exp(-2H_\Lambda t) \simeq const \quad (\Lambda\text{D})$$

Growth of linear perturbations

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M(t)\delta = 0$$

$$\delta(t) \propto \begin{cases} \mathrm{RD:} & \ln(t) \\ \mathrm{MD:} & a(t) \\ \Lambda\mathrm{D:} & \mathrm{const} \end{cases}$$

- RD and MD growth trends have been confirmed for a long time by basic cosmological observations
- ΛD growth suppression, even though it started "yesterday", has clearly been observed in the cosmological data!

Growth of linear perturbations

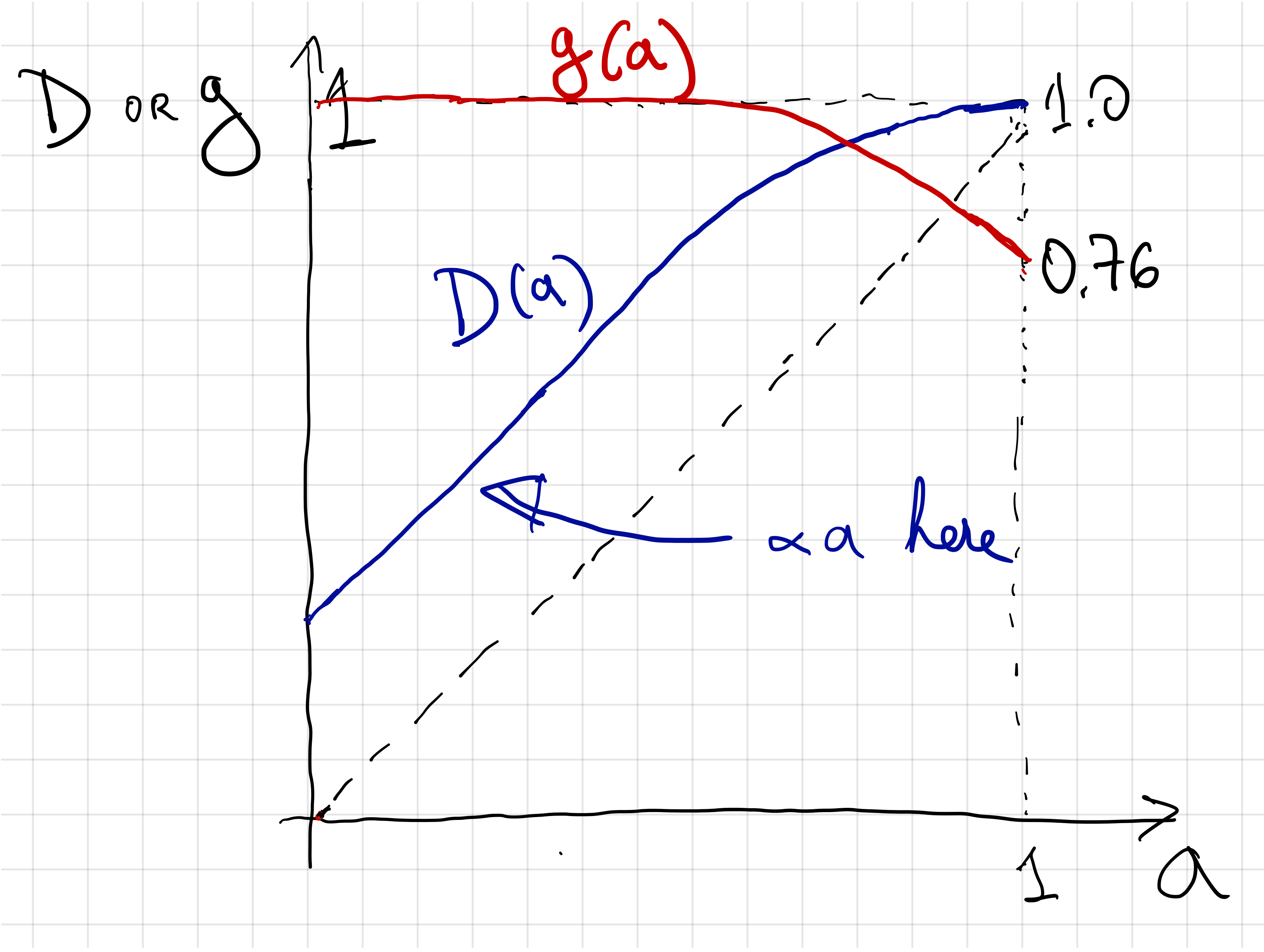
Calculating the general solutions

Linear growth rate D

$$D(a) = \frac{\delta(a)}{\delta(a=1)}$$

Growth suppression factor g (implicitly via D)

$$D(a) \equiv \frac{a g(a)}{g(a=1)}$$



Growth of linear perturbations

Calculating the general solutions

Linear growth rate

$$D(a) = \frac{\delta(a)}{\delta(a=1)}$$

Growth suppression factor (implicitly via D)

$$D(a) \equiv \frac{a g(a)}{g(a=1)}$$

Then $\ddot{\delta} + 2H\dot{\delta} - 4\pi G \rho_M(t)\delta = 0$ becomes

$$2\frac{d^2 g}{d\ln a^2} + \left[5 - 3w(a)\Omega_{\rm DE}(a)\right]\frac{dg}{d\ln a} + 3\left[1 - w(a)\right]\Omega_{\rm DE}(a) g = 0$$

2nd order ODE, can easily solve on the computer.
Works for all w(a)CDM cosmological models!
(w(a) = time-dependent equation of state of DE)

The (matter) power spectrum

Remember the overdensity in real and Fourier space

$$\delta(\vec{r}) = \frac{\sqrt{V}}{(2\pi)^3} \int \delta_{\vec{k}}\, e^{-i\vec{k}\vec{r}}\, d^3k$$

$$\delta_{\vec{k}} = \frac{1}{\sqrt{V}} \int \delta(\vec{r})\, e^{i\vec{k}\vec{r}}\, d^3r$$

Consider shifting position **r** by some ***Δ*r**. Then

$$\delta_{\vec{k}} \to \delta_{\vec{k}}\, e^{i\vec{k}\Delta\vec{r}}$$

$$\langle \delta_{\vec{k}} \delta^*_{\vec{k}'} \rangle \to e^{i(\vec{k}-\vec{k}')\Delta\vec{r}} \langle \delta_{\vec{k}} \delta^*_{\vec{k}'} \rangle$$

For a homogeneous universe, this must be independent of ***Δ*r**!

Then we have our definition of power spectrum P(k):

$$\boxed{\langle \delta_{\vec{k}}\, \delta^*_{\vec{k}'} \rangle = (2\pi)^3\, \delta^{(3)}(\vec{k} - \vec{k}')\, P(k)}$$

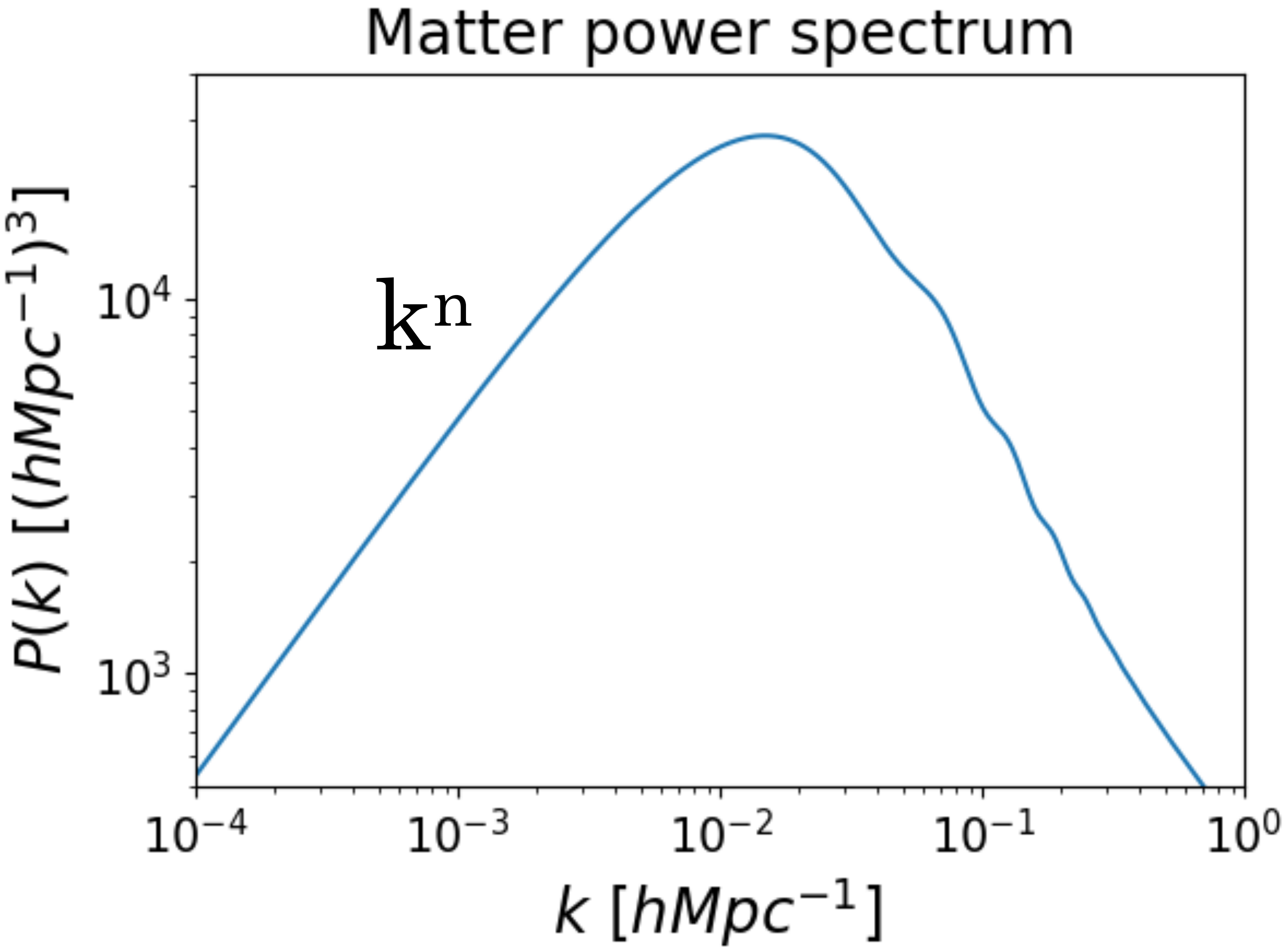
Note, this k is scalar (homogeneity)

Power spectrum P(k)

At large scales (k ≪0.01 h Mpc^{-1})

$$\boxed{P(k) \propto k^n \quad \text{with} \quad n \simeq 1} \quad (\text{Harrison, Zel'dovich, Peebles spectrum})$$

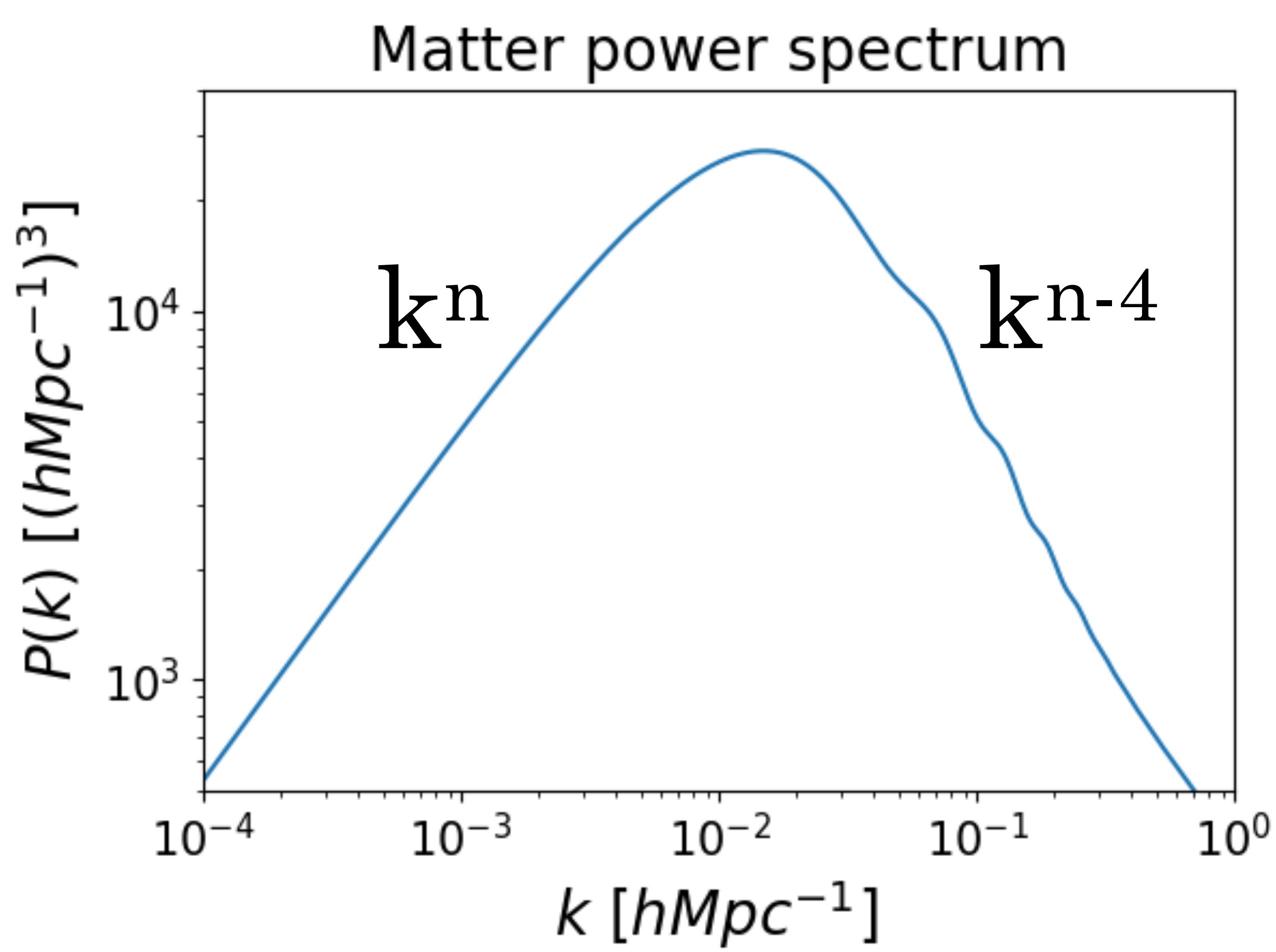
- See my lecture notes about the HZP 1969 argument that n≈1
- Inflation (1980⇒) predicts n = 1 – 6ε+2η, where ε, η are "small"
- WMAP, Planck (2000s⇒) measure n =0.965 ±0.004 (P18)



Power spectrum P(k)

At small scales (k ≫ 0.01 h Mpc^{-1}) $P(k) \propto k^{n-4}$

- So basically $P(k) = P(k)_{large\text{-}scales} \times T^2(k)$, with $T(k) \propto k^{-2}$
- T(k) is called the transfer function, and it "penalizes" modes that entered the horizon before matter-radiation equality (modes with k>0.01)
- The higher-k the mode is, the earlier (before MR eq) it entered the horizon and the less time it had to grow
- See my lecture notes for a more precise argument



What is the relation between P(k) and ξ(r)?

They are Fourier transforms of each other because δ_k and $\delta(r)$ are (Wiener-Khinchin theorem)

Proof:

$$P(k) = \langle \delta_{\vec{k}}\, \delta^*_{\vec{k}} \rangle \ (= \langle \delta_{\vec{k}}\, \delta_{-\vec{k}} \rangle)$$

$$= \frac{1}{V} \int \int \langle \delta(\vec{r}_1) \delta^*(\vec{r}_2) \rangle e^{-i\vec{k}\vec{r}_1} e^{i\vec{k}\vec{r}_2} d^3\vec{r}_1\, d^3\vec{r}_2$$

$\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$;
integrate $\int d^3\mathbf{r}_2 = V$ ↓

$$= \int \xi(r_{12}) e^{-i\vec{k}\vec{r}_{12}} d^3\vec{r}_{12}$$

$$= \int \xi(r) e^{-i\vec{k}\vec{r}} d^3\vec{r}$$

Of course that P(k) and ξ(r) contain the same information

However they are measured differently in practice; methods are subject to different statistical and systematic errors. Roughly speaking:

- ξ(r) is easier to measure, esp with “holes” in the survey footprint
- P(k) is closer to theory

Dimensionless power spectrum Δ^2(k)

Consider

$$\xi(r) = \frac{1}{(2\pi)^3}\int P(k) e^{i\vec{k}\vec{r}} d^3\vec{k} = \frac{1}{2\pi^2 r}\int_0^\infty P(k)\sin(kr)kdk$$

Now evaluate the *zero-lag correlation function*, or *variance*, **ξ**(r=0)

$$\xi(0) = \frac{1}{2\pi^2}\int_0^\infty P(k)\lim_{r\to 0}\frac{\sin(kr)}{kr}k^2 dk \equiv \int_0^\infty \Delta^2(k) d\ln k$$

where $\boxed{\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}}$ is also called the power spectrum and has the following nice features:

- It is dimensionless (computer coders, rejoice!)
- Physically, it is the contribution to variance per log wavenumber

Code-friendly power spectrum Δ^2(k)

$$\Delta^2(k,a) = A\,\frac{4}{25}\,\frac{1}{\Omega_M^2}\left(\frac{k}{k_{\rm piv}}\right)^{n-1}\left(\frac{k}{H_0}\right)^4 [ag(a)]^2\,T^2(k)\,T_{\rm nl}(k)$$

A is the primordial amplitude (dimensionless, $\sim 10^{-10}$)

Ω_M is the density of matter relative to critical

k_{piv} is some chosen pivot; modern convention is k_{piv}=0.05 Mpc^{-1}

H_0 is the Hubble constant

g(a) is the growth suppression factor

T(k) is the transfer function, accounts mainly for “turnover” in power spectrum due to radiation-matter transition.

T_{nl}(k) is the prescription for nonlinear clustering; super important on scales k $\gtrsim$ 0.1 h Mpc^{-1}