Cosmology from: gravitational waves

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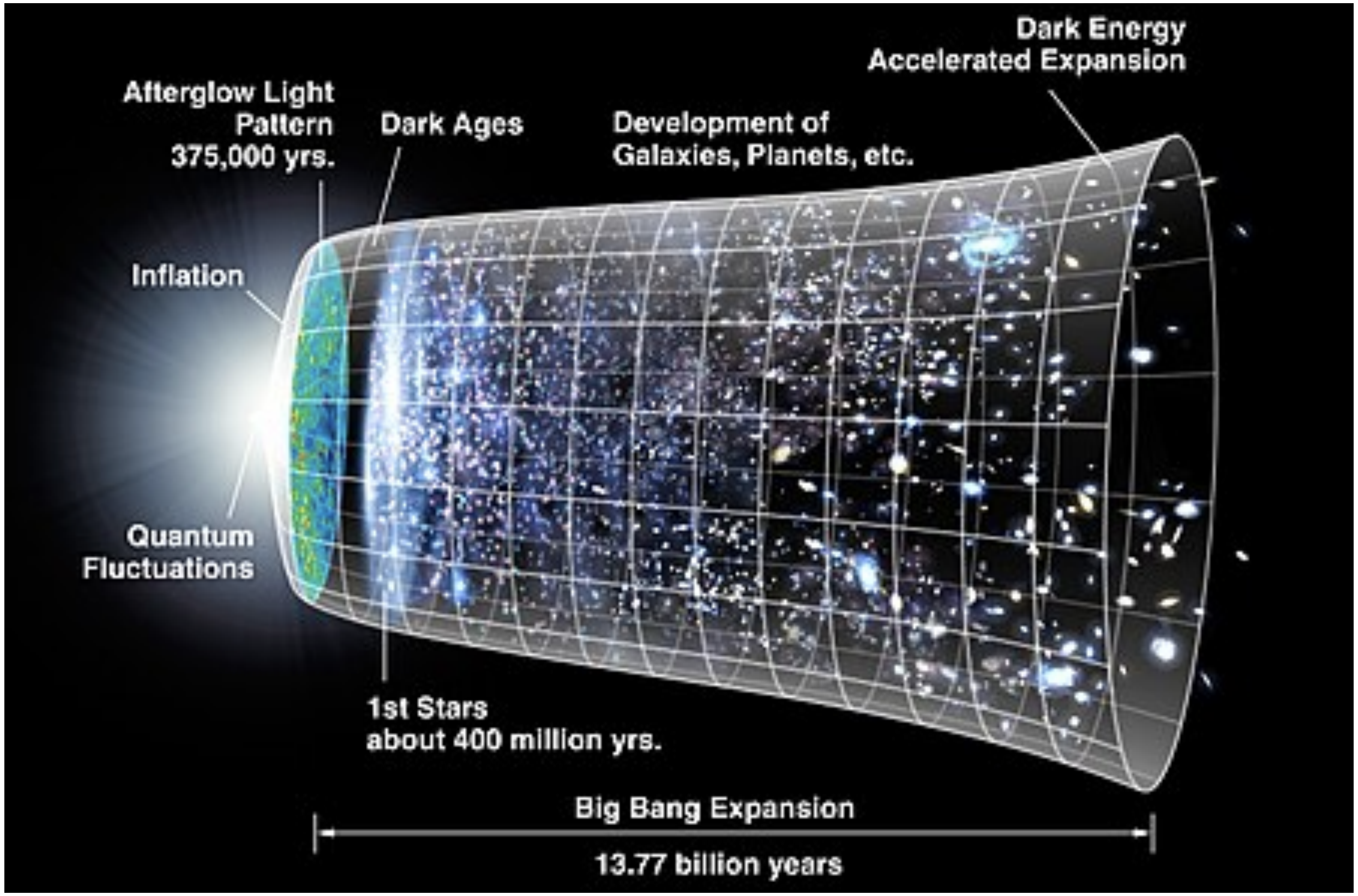
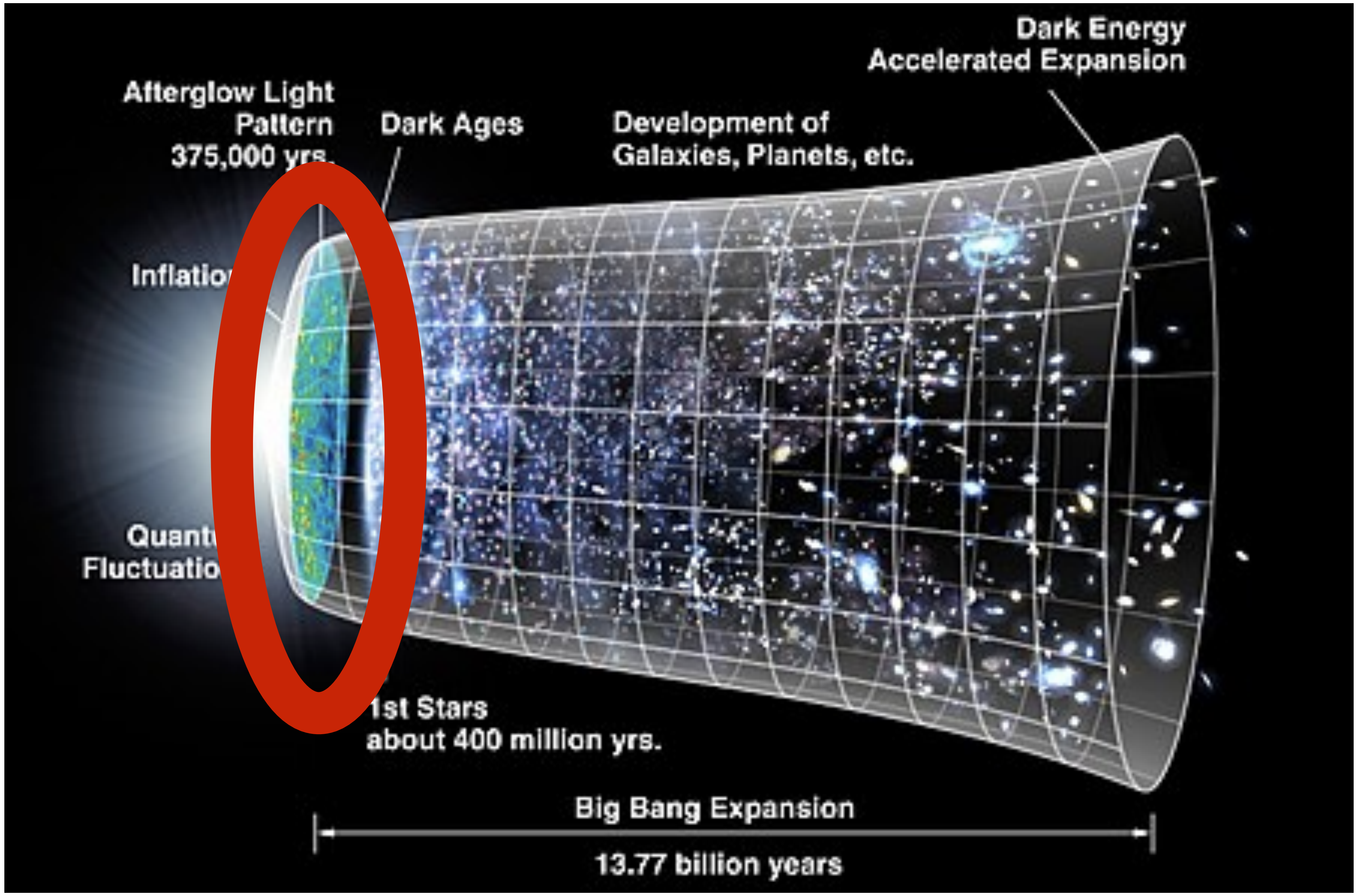
How can GW help to probe cosmology?



How can GW help to probe cosmology?

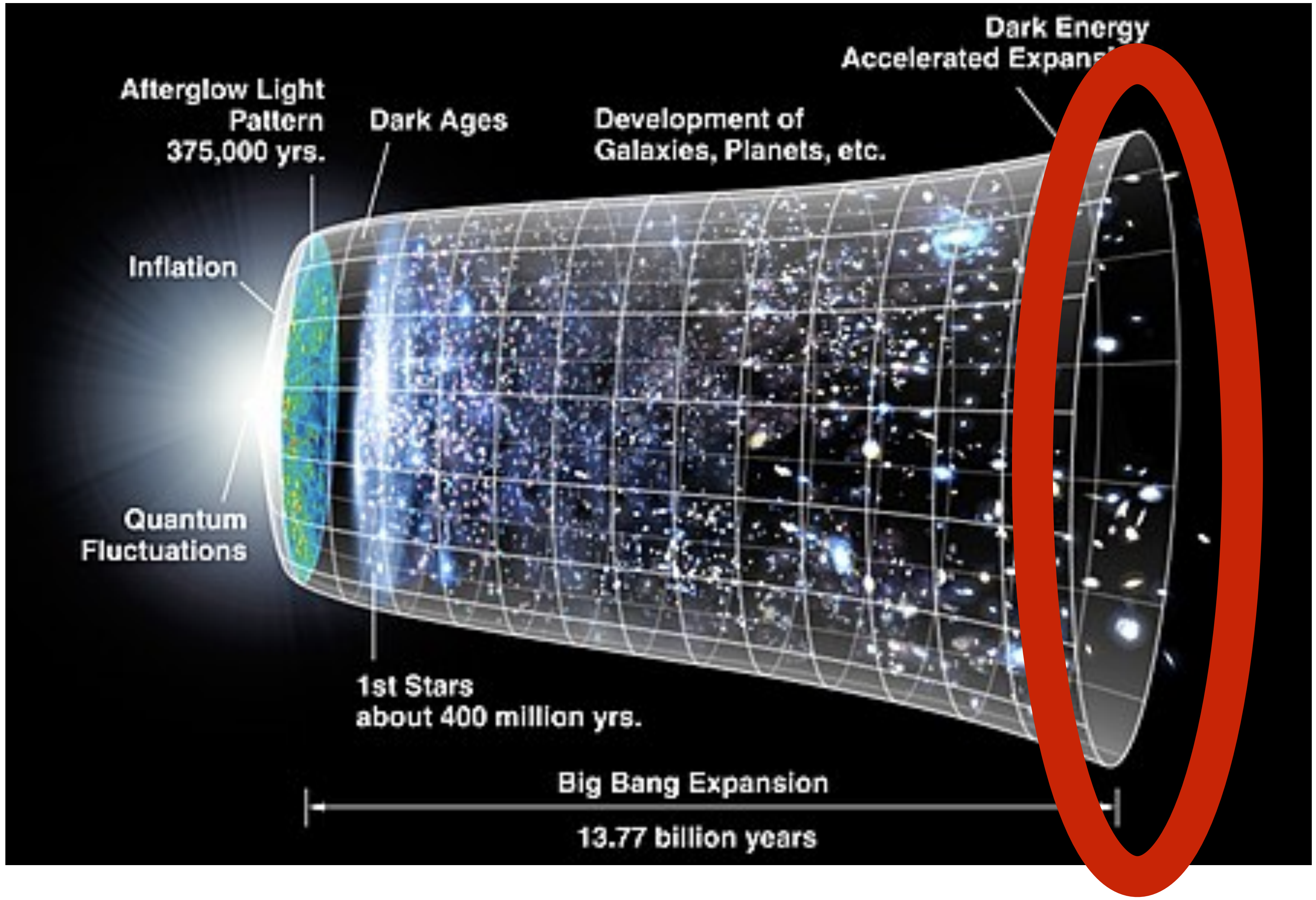
early universe



GW can bring information from the early universe: phenomena occurring in the early universe can produce stochastic GW backgrounds (SGWB)

test of the early universe and high energy phenomena

How can GW help to probe cosmology?



late universe

The GW signal from binaries travels through the FLRW spacetime: it can be used to probe the late-time dynamics and the content of the universe

test of accelerated expansion, test of GR at large scales

Summary of the course

- LECTURE 1: GW definition, GW energy momentum tensor, GW in FLRW space-time, GW equation of motion
- LECTURE 2: SGWB from the early universe: generalities
- LECTURE 3: SGWB from the early universe: examples of sources
- LECTURE 4: GW emission from binaires: generalities
- LECTURE 5: GW emission from binaries: probe of the universe expansion

C.C. and D.G. Figueroa, "Cosmological backgrounds of GWs", arXiv:1801.04268

What are gravitational waves?

- GWs emerge naturally in General Relativity:

  Newtonian theory + special relativity = a causal theory of gravitation

  There must be some form of radiation propagating information causally: GWs!

- "waves" in physics are propagating perturbations over a background. In General Relativity:

  1. take a background space-time metric (the gravitational field)
  2. define a small perturbation over this background metric
  3. insert it into the equations that describe the space-time dynamics (Einstein equations)
  4. (if everything goes well) one finds a dynamical solution for the perturbation which is propagating as a wave -> GWs!

Which background metric to choose?
Simplest choice: flat space-time

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection $$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}(\partial_\nu h^{\alpha}{}_{\mu} + \partial_\mu h^{\alpha}{}_{\nu} - \partial^{\alpha} h_{\mu\nu}) + \mathcal{O}(h^2)$$

Riemann tensor $$R^{\alpha}{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^{\alpha}{}_{\beta} + \partial_\beta\partial^{\alpha} h_{\mu\nu} - \partial_\nu\partial^{\alpha} h_{\mu\beta} - \partial_\beta\partial_\mu h^{\alpha}{}_{\nu})$$

Einstein tensor $$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^{\alpha}\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_\mu\nu\partial_\alpha\partial^{\beta}\bar{h}^{\alpha}{}_{\beta})$$

$$\Box \equiv \partial_\alpha\partial^\alpha \qquad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h$$

trace-reversed metric perturbation (OK - still small)

“Gravitational Waves”, M. Maggiore, Oxford University Press 2008

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection $$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}(\partial_\nu h^\alpha{}_\mu + \partial_\mu h^\alpha{}_\nu - \partial^\alpha h_{\mu\nu}) + \mathcal{O}(h^2)$$

Riemann tensor $$R^\alpha{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^\alpha{}_\beta + \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial^\alpha h_{\mu\beta} - \partial_\beta\partial_\mu h^\alpha{}_\nu)$$

Einstein tensor $$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^\alpha{}_\mu + \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_\mu\nu\partial_\alpha\partial^\beta\bar{h}^\alpha{}_\beta)$$

$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ ⟶ we would like to set $\partial^\mu \bar{h}_{\mu\nu}(x) = 0$

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

GR is invariant under general coordinate transformation

the linearised theory is invariant under
infinitesimal (slowly varying) coordinate transformation

$$x'^{\mu} \longrightarrow x^{\mu} + \xi^{\mu} \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$|\partial_\alpha \xi_\beta| \lesssim |h_{\alpha\beta}| \quad \longrightarrow \quad |h'_{\mu\nu}(x')| \ll 1$$

$$\partial^{\mu} \bar{h}_{\mu\nu}(x) \longrightarrow \partial'^{\mu} \bar{h}'_{\mu\nu}(x') = \partial^{\mu} \bar{h}_{\mu\nu}(x) - \Box \xi_\nu$$

By a suitable coordinate transformation, it is always possible to go to the LORENTZ GAUGE

$$\partial'^{\mu} \bar{h}'_{\mu\nu}(x') = 0$$

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE
THE FORM OF A WAVE EQUATION!

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

From the Lorentz gauge condition $\partial^{\mu} \bar{h}_{\mu\nu}(x) = 0$ one gets

$$\partial^{\mu} T_{\mu\nu} = 0$$

The energy-momentum tensor of the source is conserved

the source does not loose energy and momentum by the GW emission

in linearised theory, the background space-time is flat, i.e. the source is described by Newtonian gravity

linearised theory does not describe how GW emission influences the source

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE THE FORM OF A WAVE EQUATION!

$$\Box \bar{h}_{\mu\nu} = -16\pi G \, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

$\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$ $\quad \partial^\mu \bar{h}_{\mu\nu}(x) = 0$ $\longrightarrow$ 6 radiative components

WAIT! ARE THESE ALL PHYSICAL?

$x'^\mu \longrightarrow x^\mu + \xi^\mu$ satisfying $\Box \xi_\mu = 0$ to remain in the Lorentz gauge

$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} + \xi_{\mu\nu}$ with $\xi_{\nu\mu} \equiv \eta_{\nu\mu} \partial^\alpha \xi_\alpha - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$

IF IN VACUUM: $\Box' \bar{h}'_{\mu\nu} \simeq \Box(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = 0$

$T_{\mu\nu} = 0$

GWs in linearised theory over Minkowski

Restricting to vacuum space-time, the residual coordinate freedom can be used to fix 4 constraints

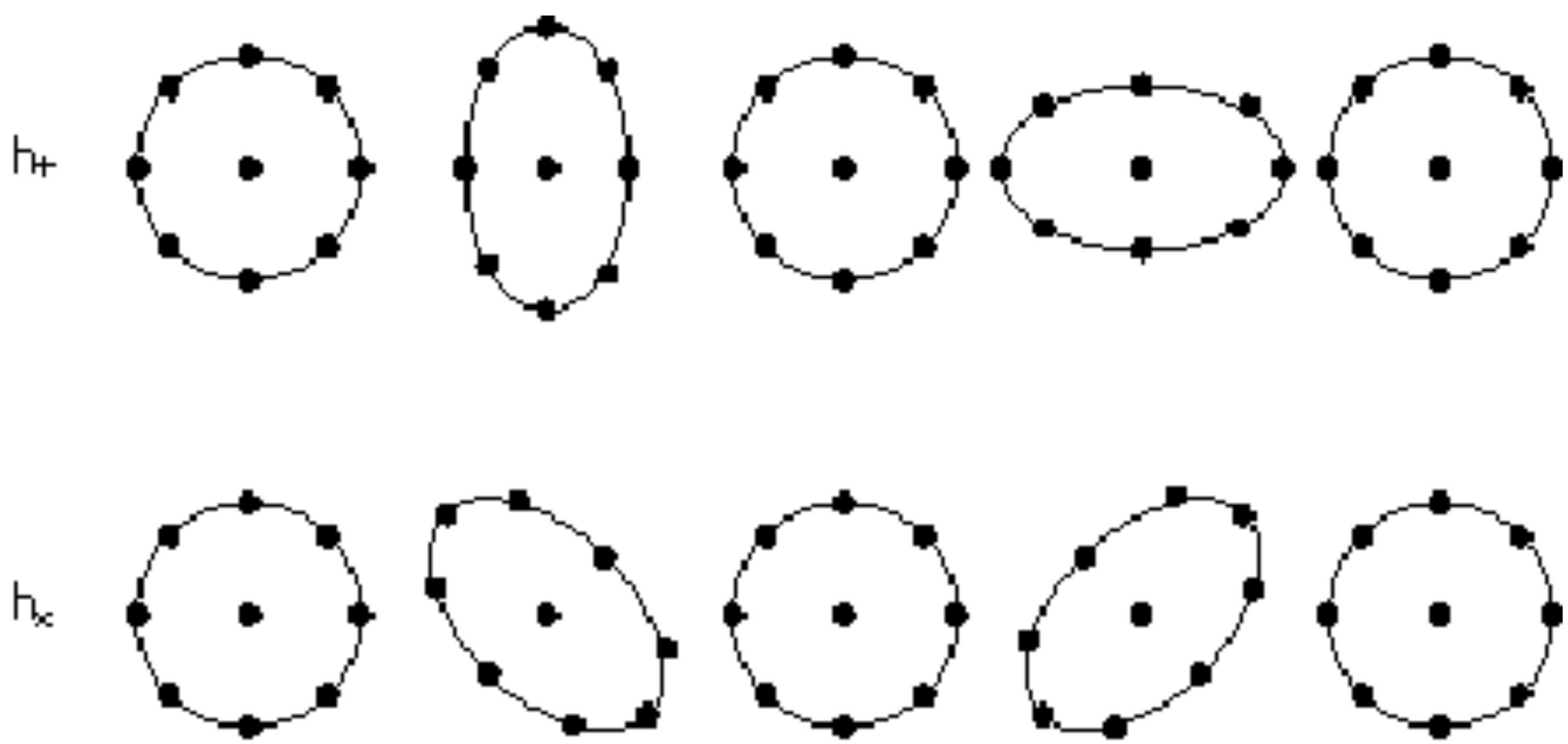
TRANSVERSE TRACELESS GAUGE

$$\bar{h}^{\mu}{}_{\mu} = 0 \qquad h_{0i} = 0 \qquad \partial^i h_{ij} = 0$$

$$h_{00} = 0$$ come for free

There are only 2 remaining *physical* degrees of freedom in the metric

$h_+, h_\times$



Exercise: solve Einstein equations and write the line element in TT gauge

Are we happy with this definition? So so…

To exhibit the two physical d.o.f. of GWs we had to restrict to vacuum

However, the fact that GWs have only two physical components is a manifestation of the intrinsic nature of the gravitational interaction, mediated by the graviton, a spin-two massless field that has only two independent helicity states

It should be true also in space-time with matter!

WHAT WE DO NEXT:

1. Drop the condition of empty space-time
2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations (scalar, vector, tensor)
3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations
4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs in non-vacuum space-times*

GWs in linearised theory over Minkowski, with matter

2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad\qquad |h_{\mu\nu}(x)| \ll 1$$

$$\begin{aligned}
h_{00} &= -2\phi \\
h_{0i} &= \partial_i B + S_i \quad (\partial_i S_i = 0) \\
h_{ij} &= -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + H_{ij} \\
&\quad (\partial_i F_i = 0,\ \partial_i H_{ij} = 0,\ H_{ii} = 0)
\end{aligned}$$

scalars	vectors	tensor
$\phi,\ B,\ \psi,\ E$	$S_i,\ F_i$	H_{ij}

E.E. Flanagan and S.A. Hughes, "The basics of GW theory", arXiv:gr-qc/0501041

"Space-time and geometry: an introduction to GWs", S. Carroll, Pearson Education Limited, 2014

"The Cosmic Microwave Background", R. Durrer, Cambridge University Press, 2008

GWs in linearised theory over Minkowski, with matter

3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations

$$x'^{\mu} \longrightarrow x^{\mu} + \xi^{\mu} \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i) \qquad \text{with} \quad \partial_i d_i = 0$$

Two scalars, one vector and one tensor physical variables

$$\Phi \equiv \phi + \dot{B} - \ddot{E}/2$$

$$\Theta \equiv -2\psi - \nabla^2 E/3$$

$$\Sigma_i \equiv S_i - \dot{F}_i \qquad \text{with} \quad \partial_i \Sigma_i = 0$$

$$H_{ij} \qquad \text{with} \quad \partial_i H_{ij} = 0 \;\; H_i^i = 0$$

gauge invariant

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

1. Drop the condition of empty space-time

$$\begin{aligned} T_{00} &= \rho \\ T_{0i} &= \partial_i u + u_i \quad (\partial_i u_i = 0) \\ T_{ij} &= p\delta_{ij} + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij} \\ & \quad (\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0) \end{aligned}$$

scalars	vectors	tensor
$\rho,\ u,\ p,\ \sigma$	$u_i,\ v_i$	Π_{ij}

energy-momentum conservation: four further constraints
(OK since we are still in linearised theory!)

$$\partial_\mu T^{\mu\nu} = 0$$

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs in non-vacuum space-times*

Write Einstein equations in terms of the 6 gauge invariant variables

$$\nabla^2\Theta = -8\pi G\rho \qquad \nabla^2\Phi = 4\pi G\left(\rho + 3p - 3\dot{u}\right)$$

$$\nabla^2\Sigma_i = -16\pi G S_i \qquad \Box H_{ij} = -16\pi G\,\Pi_{ij}$$

Three Poisson-like equations, one wave equation

Only the TT metric components are radiative

Exercise: find the differences with cosmological perturbation theory

GW energy-momentum tensor and GW propagation

According to GR, any form of energy contributes to space-time curvature
Are GWs a source of space-time curvature?

- One needs to go beyond linearisation over Minkowski, otherwise one excludes from the beginning the presence of any background space-time curvature

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

- In this new setting, how to decide what is the background and what is the fluctuation?

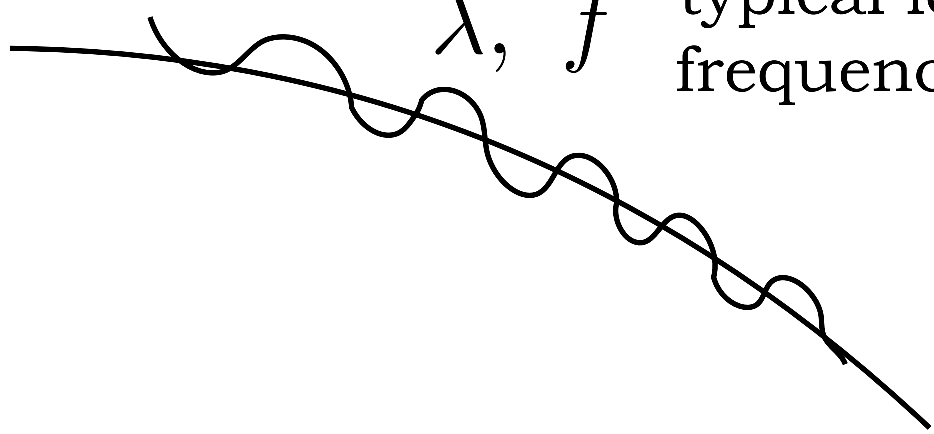
1. The background space-time has a clear symmetry (static, FLRW…)
2. It is possible to resort to a clear separation of scales/frequencies

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

E.E. Flanagan and S.A. Hughes, "The basics of GW theory", arXiv:gr-qc/0501041

R.A. Isaacson, Physical Review, Volume 166, number 5, pages 1263 and 1272, 1968

GW energy-momentum tensor and GW propagation



$\lambda, \; f$ typical length-scale / frequency of the GWs $\qquad \lambda = \dfrac{c}{f}$

$L_B, \; f_B$ typical scale of spatial variations / frequency of time variations of the background (needn't be related among themselves)

- There are two expansions in the game:

1. $|h_{\mu\nu}| \ll 1$

2. $\dfrac{\lambda}{L_B} \ll 1, \quad \dfrac{f_B}{f} \ll 1$

- In order to effectively implement the distinction among background and GWs, one needs to average physical quantities

$$\lambda \ll \bar{\ell} \ll L_B \qquad\qquad f_B \ll \bar{f} \ll f$$

Examples: GW detectors? GW from the early universe?

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$

The linear term averages to zero

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

The quadratic term can influence the background, as it contains both high and low modes

$$\langle ... \rangle = [...]^{\rm low}$$

Background Einstein equation

$$\bar{R}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\rm low} + 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{\rm low}$$

GWs sourcing the bckg curvature

Matter sourcing the bckg curvature

$$\mathcal{O}\left(\frac{1}{L_B}\right)^2$$

$$\mathcal{O}\left(\frac{h}{\lambda}\right)^2$$

$$h \lesssim \frac{\lambda}{L_B}$$

Necessary condition for GW to make sense

GW energy-momentum tensor and GW propagation

Rearranging Einstein equation at order zero, calculating $R^{(2)}_{\mu\nu}$ and performing the average leads to:

$$\underbrace{\bar{G}_{\mu\nu}}_{\text{Dynamics of the bckg space-time}} = \langle R_{\mu\nu} \rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R \rangle = 8\pi G(\underbrace{\langle T_{\mu\nu} \rangle}_{\text{Low-mode part of the matter component}} + \underbrace{T^{\mathrm{GW}}_{\mu\nu}}_{\text{GWs}})$$

Dynamics of the bckg space-time

Low-mode part of the matter component

GWs

not separately conserved!

GW energy-momentum tensor in the TT gauge (all gauge freedom removed)

$$T^{GW}_{\mu\nu} = -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(2)} \rangle = \frac{1}{32\pi G}\langle \nabla_\mu h_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \rangle$$

$$\rho_{GW} = \frac{\langle \dot{h}_{ij} \dot{h}^{ij} \rangle}{32\pi G}$$

C.W. Misner, K.S. Thorne, J.A. Wheeler, “Gravitation”, Freeman 1997 (chapter 35.15)

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$

The linear term averages to zero

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

The quadratic term can influence the background, as it contains both high and low modes

$\langle \ldots \rangle = [\ldots]^{\text{low}}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\text{high}} + 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{\text{high}}$$

$\mathcal{O}\left(\frac{h}{\lambda^2}\right)$

$\mathcal{O}\left(\frac{h}{\lambda}\right)^2$

Matter possibly sourcing GWs

Negligible
(non-linear interaction of the wave with itself)

GW propagation equation

1. The matter produces only smooth curvature of the background

$$[T_{\mu\nu}]^{\text{high}} = 0 \quad \longrightarrow \quad R^{(1)}_{\mu\nu} = 0$$

$$\bar{\nabla}_\sigma \bar{\nabla}^\sigma h_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\rho} h^{\alpha\rho} - \cancel{\bar{R}_{\alpha\mu}} h^{\alpha\nu} - \cancel{\bar{R}_{\alpha\nu}} h^{\alpha\mu} = 0$$

$$\longrightarrow \quad \mathcal{O}\left(\frac{h}{L_B^2}\right) \text{ or } \mathcal{O}\left(\frac{h^3}{\lambda^2}\right)$$

Effects due to the propagation of GWs on a curved background such as gravitational redshift and lensing

NOTE: if the background is flat space-time (a part from the curvature generated by the GWs themselves) one goes back to $\Box h_{\mu\nu} = 0$

GW propagation equation

2. The matter has high-mode components $\quad [T_{\mu\nu}]^{\text{high}} \neq 0$

$$R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu}R^{(1)} + h_{\mu\nu}\bar{R}) \simeq 8\pi G\,[T_{\mu\nu}]^{\text{high}}$$

Evolution of GWs on a curves but smooth / slowly evolving background

Possible source of GWs

In a FLRW universe, equation of sourcing and propagation of GWs

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

$$\partial_i h_{ij} = h_{ii} = 0$$

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

GW propagation equation

COMMENTS

- In the rest of the course, we will be dealing with solutions of the above equation
- It can be derived also from cosmological perturbation theory, here I presented the connection with a more general approach
- In cosmology, the FLRW space-time is homogeneous and isotropic, so tensor modes can be defined also when $\lambda \sim L_B$ (exemple: horizon re-entry after inflation), but one cannot say these are GWs, unless modes are well within the horizon ($\lambda \ll L_B$)