Cosmology from: gravitational waves

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LECTURE 2
SGWB from the early universe: generalities

SGWB from the early universe: generalities

GW propagation equation in FLRW cosmology

Here there should be also scalar and vector perturbations

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

$$\partial_i h_{ij} = h_{ii} = 0$$

Source: tensor anisotropic stress

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\boxed{\Pi_{ij}(\mathbf{x},t)}$$

Perfect fluid

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

In the cosmological context: energy momentum tensor of the matter content of the universe (background + perturbations)

$$\delta T_{ij} = \bar{p}\,\delta g_{ij} + a^2[\delta p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + 2\partial_{(i}v_{j)}\boxed{+\,\Pi_{ij}}]$$

$$(\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0)$$

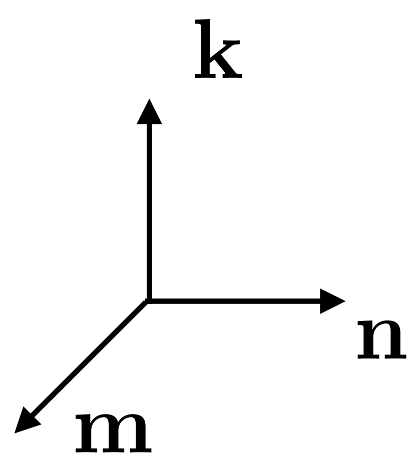
NO GWs FROM THE HOMOGENEOUS MATTER COMPONENT

SGWB from the early universe: generalities

Fourier decomposition, and polarisation components +, ×

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

$$e^+_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{m}_j - \hat{n}_i\,\hat{n}_j$$
$$e^\times_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{n}_j + \hat{n}_i\,\hat{m}_j$$



The same decomposition can be applied to $\Pi_{ij}(\mathbf{x},t)$

Free wave traveling in the z direction

$$\mathbf{k} = \omega\hat{z}$$

$$h_{ij}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega(t-z)\right]$$

SGWB from the early universe: generalities

The equation decouples for each polarisation mode.
In terms of conformal time and comoving wavenumber it becomes:

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

Solution of the homogeneous equation

$$H_r(\mathbf{k},\eta) = a\,h_r(\mathbf{k},\eta) \qquad H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 0$$

Power-law scale factor (it covers matter and radiation domination, and De Sitter inflation) $\quad a''/a \simeq \mathcal{H}^2$

CASE 1: Sub-Hubble modes, relevant for propagation after the source stops

$$k^2 \gg \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\,e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\,e^{-ik\eta}$$

In this limit, GWs are plane waves with redshifting amplitude

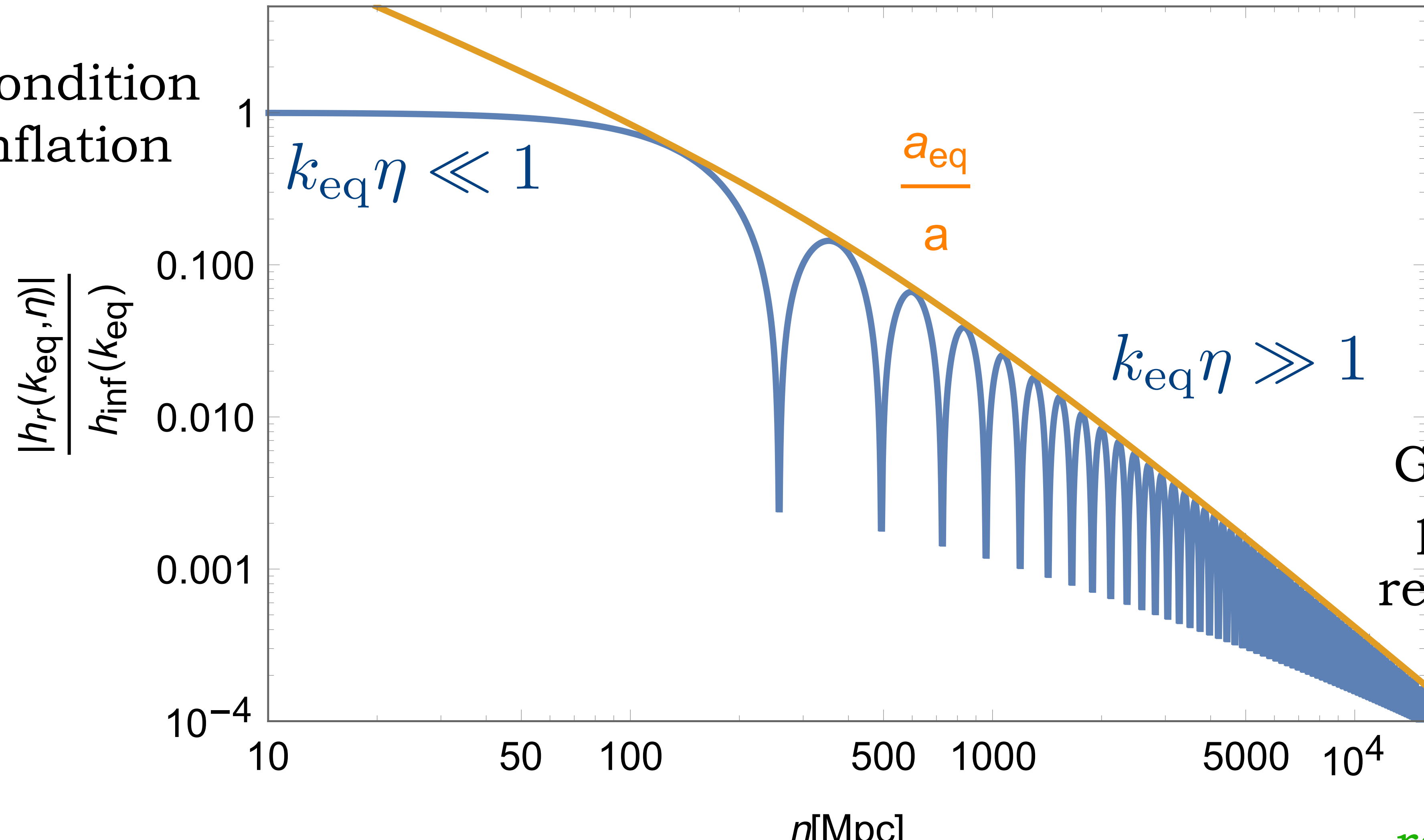
Solution of the homogeneous equation

CASE 2: Super-Hubble modes, relevant for inflationary tensor perturbations

$$k^2 \ll \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = A_r(\mathbf{k}) + B_r(\mathbf{k}) \int^{\eta} \frac{d\eta'}{a^2(\eta')}$$

Full solution with inflationary initial conditions
horizon re-entry at the radiation-matter transition

Initial condition from inflation



GWs oscillating as plane waves and redshifting with the expansion

Exercise: reproduce this plot

SGWB from the early universe: generalities

We now analyse what can be said **in general** about the SGWB signal generated by a sourcing process occurring at a **given time t$_*$** in a phase of **standard cosmic expansion** of the universe (not inflation) and therefore operating causally

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

↑

What this source could actually be in the primordial universe? See next lecture

Why sources in the early universe produce SGWBs?

A GW source acting at time t_* in the early universe cannot produce a signal correlated on length/time scales larger than the causal horizon at that time

$$\ell_* \leq H_*^{-1}$$

ℓ_* characteristic length-scale of the source
(typical size of variation of the tensor anisotropic stresses)

Why sources in the early universe produce SGWBs?

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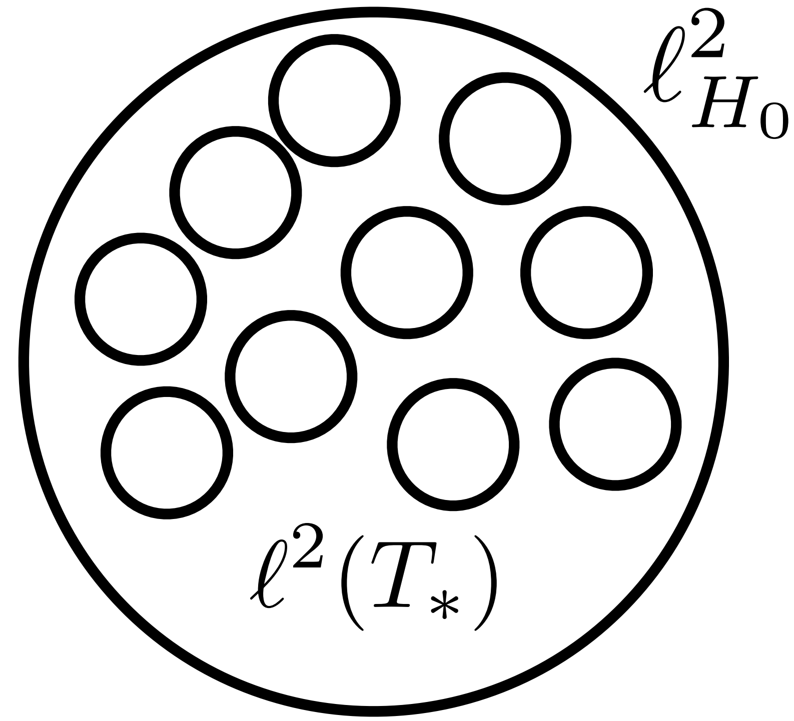


Angular size on the sky today of a region in which the SGWB signal is correlated

$$\Theta_* = \frac{\ell_*}{d_A(z_*)}$$

Angular diameter distance

Number of uncorrelated regions accessible today $\sim \Theta_*^{-2}$

Suppose a GW detector angular resolution of 10 deg $\longrightarrow$ $z_* \lesssim 17$

$$\Theta(z_* = 1090) \simeq 0.9\,\text{deg} \qquad \Theta(T_* = 100\,\text{GeV}) \simeq 10^{-12}\text{deg}$$

Only the statistical properties of the signal can be accessed

Why sources in the early universe produce SGWBs?

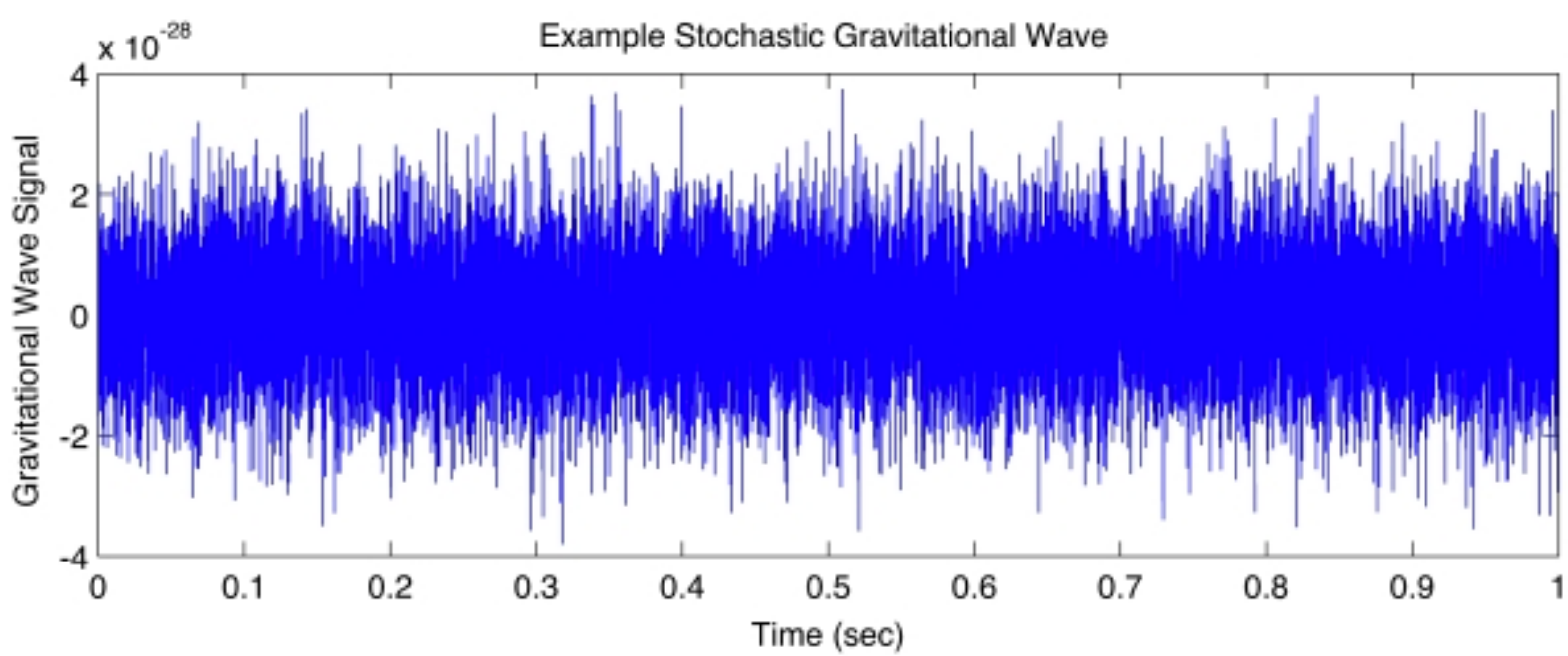
- We access today the GW signal from many independent horizon volumes: $h_{ij}(\mathbf{x},t)$ must be treated as a random variable

- The universe is homogeneous and isotropic, so the GW source is operating everywhere at the same time with the same average properties ("a-causal" initial conditions from inflation)

- Under the ergodic hypothesis, the ensemble average can be substituted with volume / time averages: we identify this average with the volume / time one necessary to define the GW energy momentum tensor (lecture one)

- Notable exception: *SGWB from inflation* (intrinsic quantum fluctuations that become classical (stochastic) outside the horizon)

LIGO website



Why sources in the early universe produce SGWBs?

The SGWB is in general homogenous and isotropic, unpolarised and gaussian

There are exceptions!

As the FLRW space-time

$$\langle h_{ij}(\mathbf{x},\eta_1)\,h_{lm}(\mathbf{y},\eta_2)\rangle = F_{ijlm}(|\mathbf{x}-\mathbf{y}|,\eta_1,\eta_2)$$

If the sourcing process preserves parity

$$\langle h_{+2}(\mathbf{k},\eta)h_{+2}(\mathbf{k},\eta) - h_{-2}(\mathbf{k},\eta)h_{-2}(\mathbf{k},\eta)\rangle = \langle h_{+}(\mathbf{k},\eta)h_{\times}(\mathbf{k},\eta)\rangle = 0$$

Helicity basis $\quad e_{ij}^{\pm 2} = \dfrac{e_{ij}^{+} \pm i\, e_{ij}^{\times}}{2}$

Central limit theorem: the signal comes from the superposition of many independent regions

Characterisation of a SGWB

Power spectrum of the GW amplitude $h_c(k,t)$

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,h_c^2(k,\eta)$$

Statistical homogeneity and isotropy

Unpolarised

Gaussianity: the two-point correlation function is enough to fully describe the SGWB

$$\langle h_{ij}(\mathbf{x},\eta)\, h_{ij}(\mathbf{x},\eta)\rangle = 2\int_0^{+\infty}\frac{dk}{k}\,h_c^2(k,\eta)$$

Related to the variance of the GW amplitude in real space

For *freely propagating sub-Hubble modes*, and taking the time-average:

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{1}{a^2(\eta)}\left[\langle A_r(\mathbf{k})\,A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\,B_p^*(\mathbf{q})\rangle\right]$$

$$h_c(k,\eta) \propto \frac{1}{a^2(\eta)}$$

Characterisation of a SGWB

Power spectrum of the GW energy density $\frac{d\rho_{\rm GW}}{d\log k}$

$$\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \frac{\langle h'_{ij}(\mathbf{x},\eta)\,h'_{ij}(\mathbf{x},\eta)\rangle}{32\pi G\,a^2(\eta)} = \int_0^{+\infty}\frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}$$

$$\langle h'_r(\mathbf{k},\eta)\,{h'_p}^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,{h'_c}^2(k,\eta)$$

For *freely propagating sub-Hubble modes*, and taking the time-average:

$${h'_c}^2(k,\eta) \simeq k^2\,h_c^2(k,\eta)$$

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^2\,h_c^2(k,\eta)}{16\pi G\,a^2(\eta)}$$

Exercise: demonstrate this

$$\rho_{\rm GW} \propto \frac{1}{a(\eta)^4}$$

GW energy density scales like radiation for freely propagating sub-Hubble modes (free massless particles)

Evolution of the SGWB in the FLRW universe

GW energy density parameter

Evaluated today, for a source that operated at time η_*

$$h^2\,\Omega_{\rm GW}(k,\eta_0) = \frac{h^2\rho_*}{\rho_c}\left(\frac{a_*}{a_0}\right)^4\left(\frac{1}{\rho_*}\frac{d\rho_{\rm GW}}{d\log k}(k,\eta_*)\right)$$

characteristic frequency of the GW signal

$$f_* = \frac{1}{\ell_*} \geq H_*$$

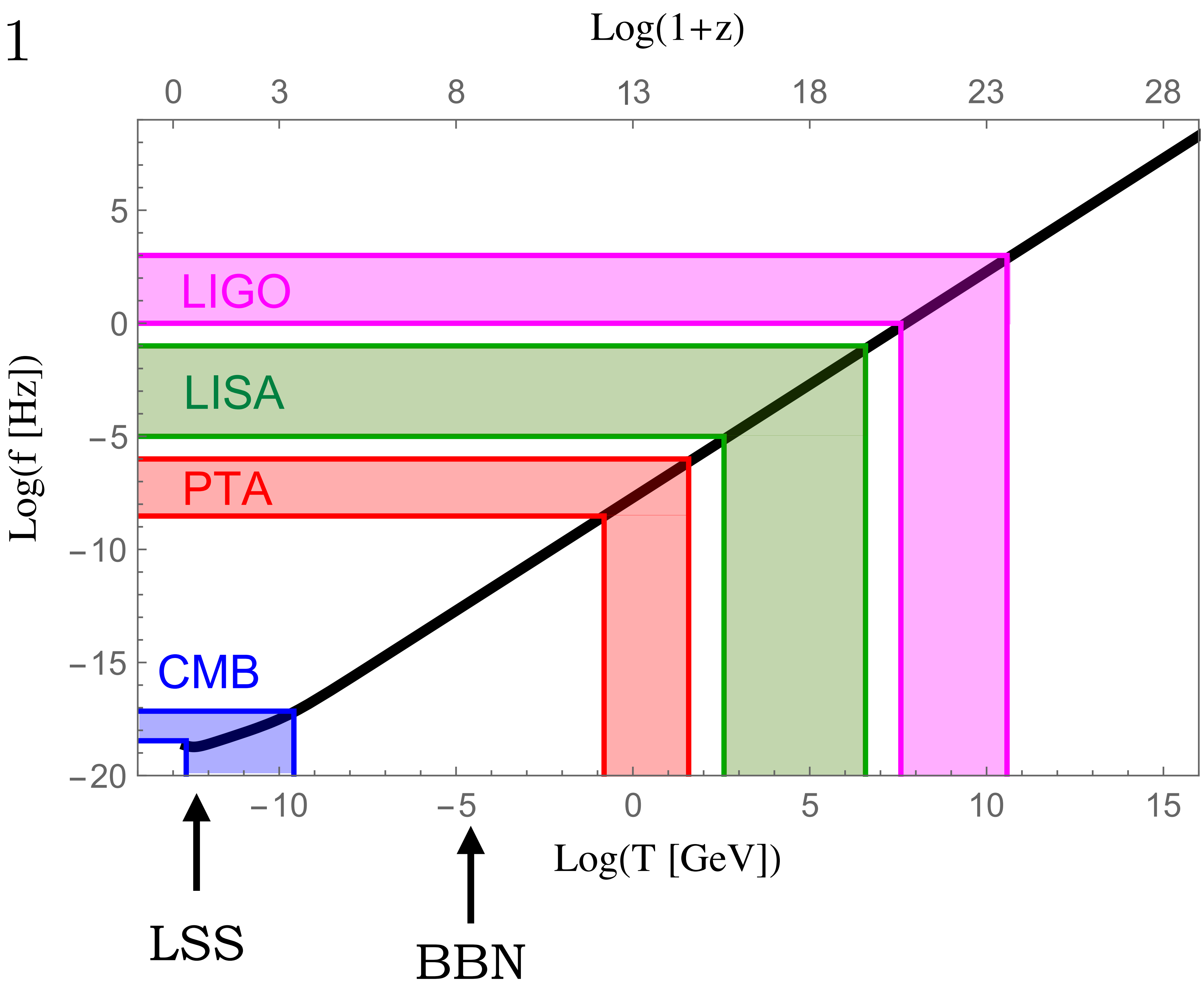
$$\epsilon_* = \ell_* H_*$$

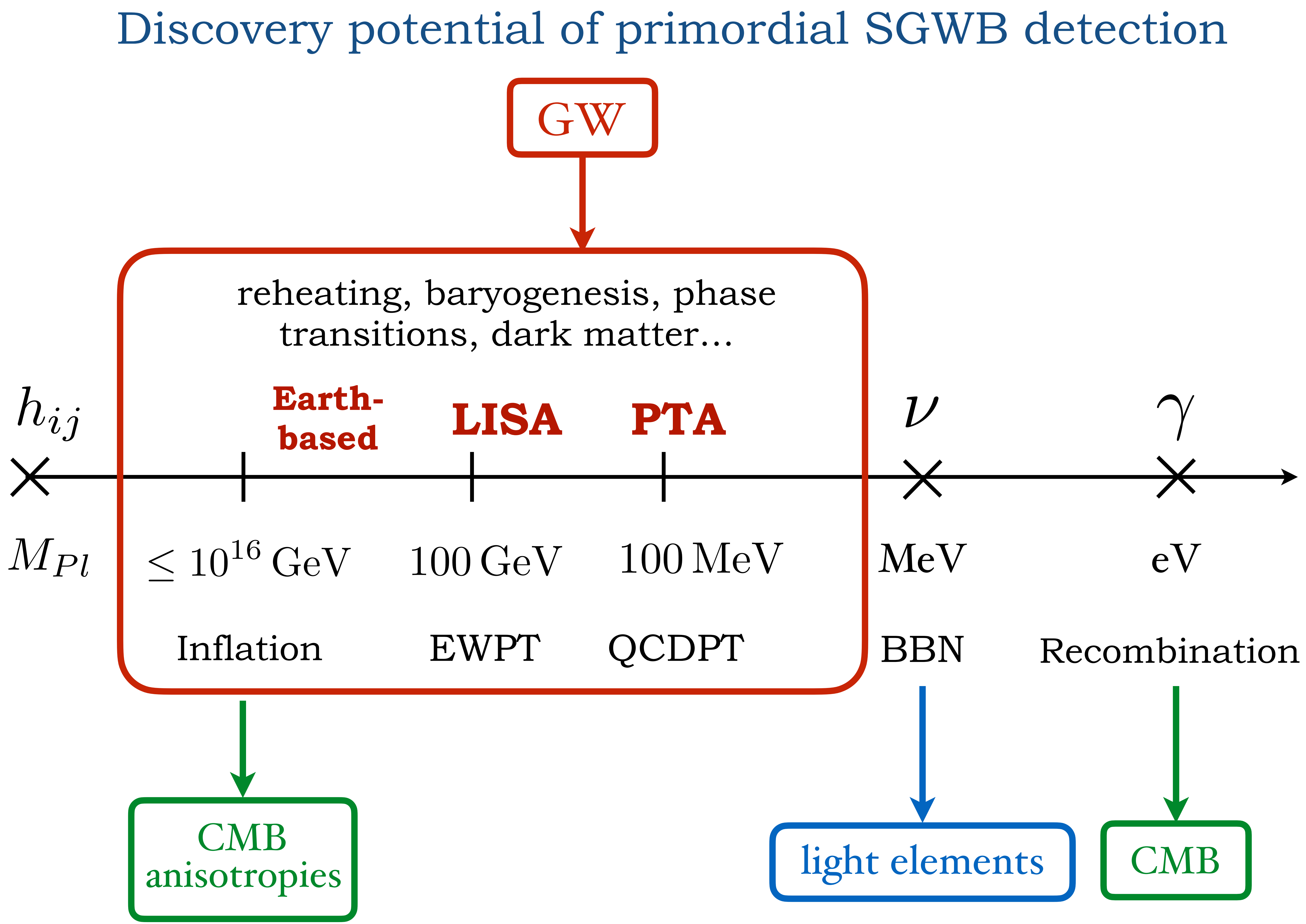
Ratio of the typical length-scale of the GW sourcing process (size of the anisotropic stresses) and the Hubble scale at the generation time

$$f = f_*\frac{a_*}{a_0} = \frac{1.65\times 10^{-7}}{\epsilon_*}\left(\frac{g(T_*)}{100}\right)^{1/6}\frac{T_*}{\rm GeV}\,{\rm Hz}$$

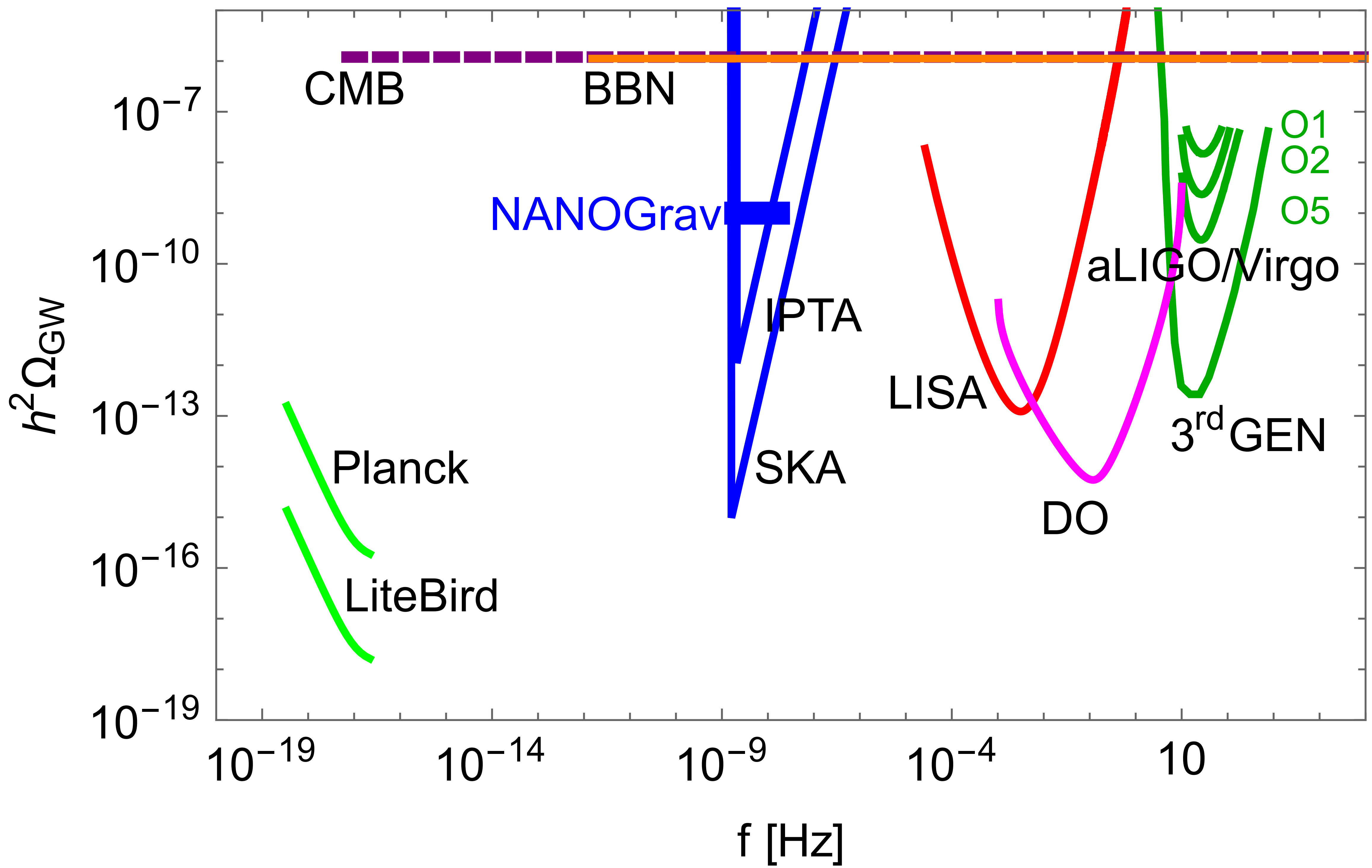
Characteristic frequency of the GW signal

$\epsilon_* = 1$

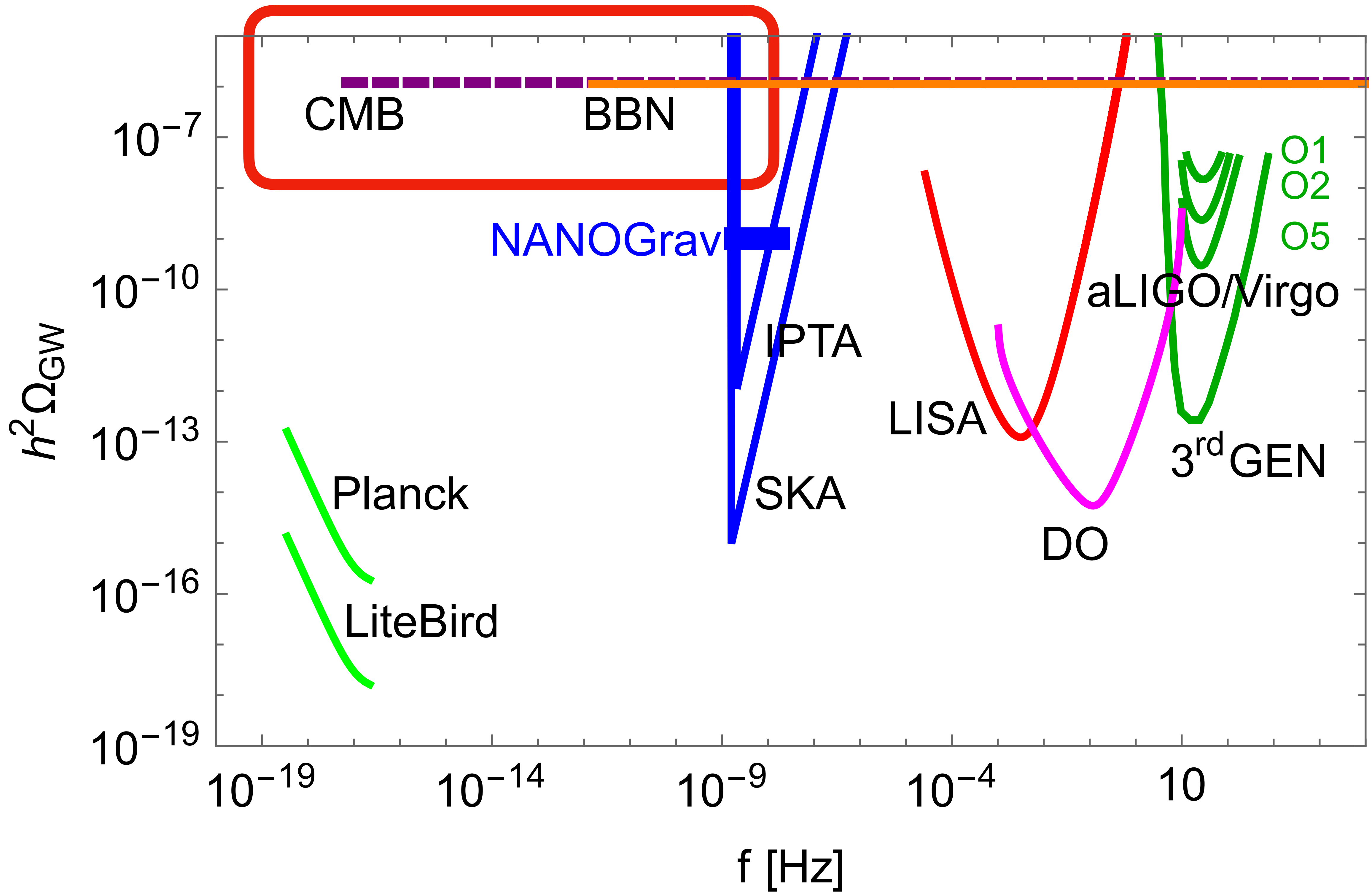



Discovery potential of primordial SGWB detection
GW
reheating, baryogenesis, phase transitions, dark matter...
h_{ij}
Earth-based
LISA
PTA
ν
γ
M_{Pl}
$\leq 10^{16}$ GeV
100 GeV
100 MeV
MeV
eV
Inflation
EWPT
QCDPT
BBN
Recombination
CMB anisotropies
light elements
CMB

What is/will be known about the SGWB



What is/will be known about the SGWB



What is/will be known about the SGWB

- GW contribute to the energy density in the universe and change its background evolution

$$H^2(T) = \frac{8\pi G}{3} \Sigma_i \rho_i(T)$$

- The abundances of elements produced at <u>Big Bang Nucleosynthesis (BBN)</u> depend on the relative abundance of neutrons and protons, which depends on the Hubble scale at T ~ MeV

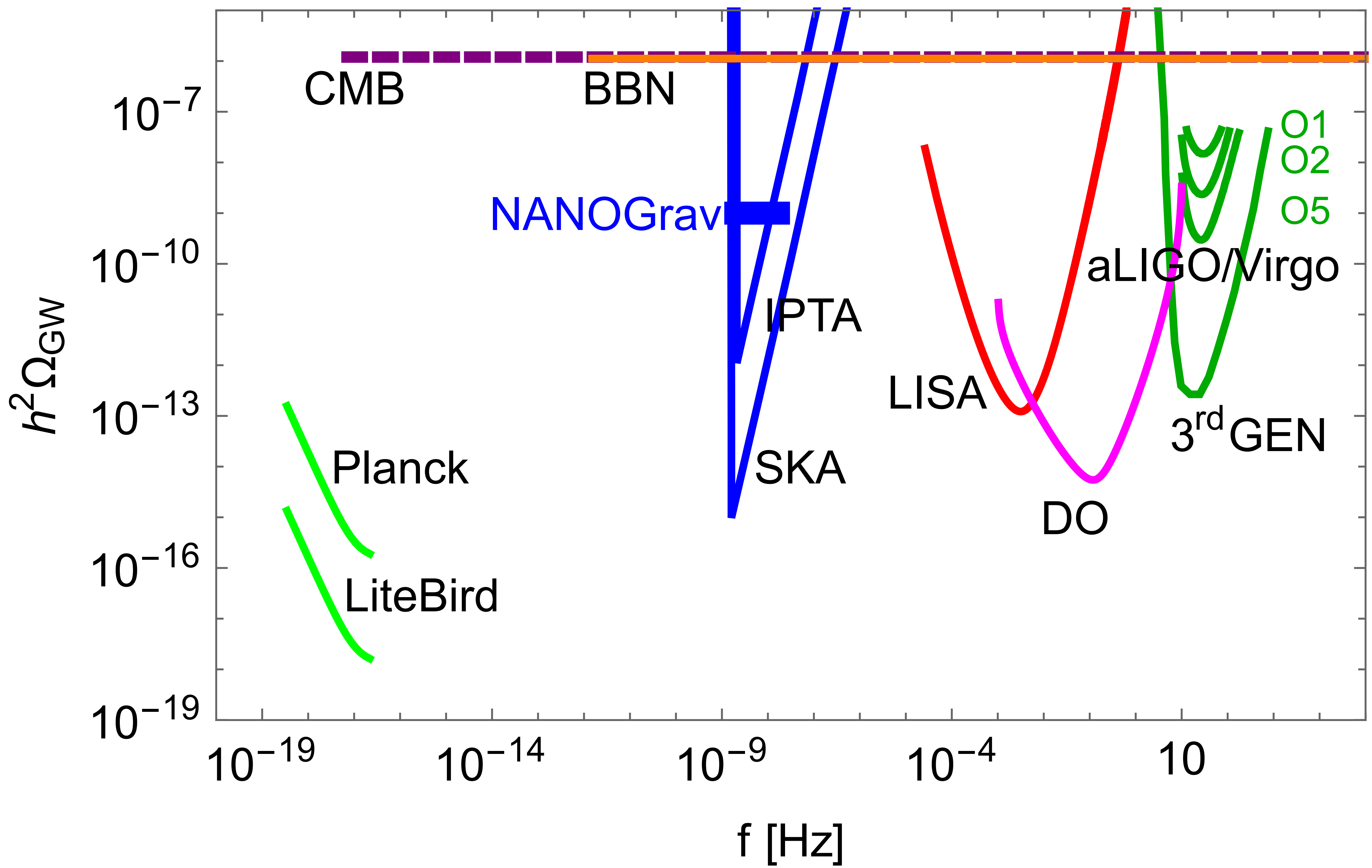
- <u>The Cosmic Microwave Background (CMB)</u> monopole and anisotropy spectrum depend on the Hubble scale at decoupling T ~ 0.3 eV, on the matter-radiation equality…

- Bounds on the *integrated GW energy density* at/previous to the BBN and CMB epochs

$$\left(\frac{\rho_{\rm GW}}{\rho_c}\right)_0 = \int \frac{df}{f}\,\Omega_{\rm GW}(f) = \Omega_\gamma^0 \left(\frac{g_S(T_0)}{g_S(T)}\right)^{4/3} \boxed{\left(\frac{\rho_{\rm GW}}{\rho_\gamma}\right)_T}$$

CAREFUL! Plot wrong…

What is/will be known about the SGWB



Earth-based interferometers

aLIGO/aVirgo

arm length L = 4 km

frequency range of detection:
10 Hz < f < 5kHZ

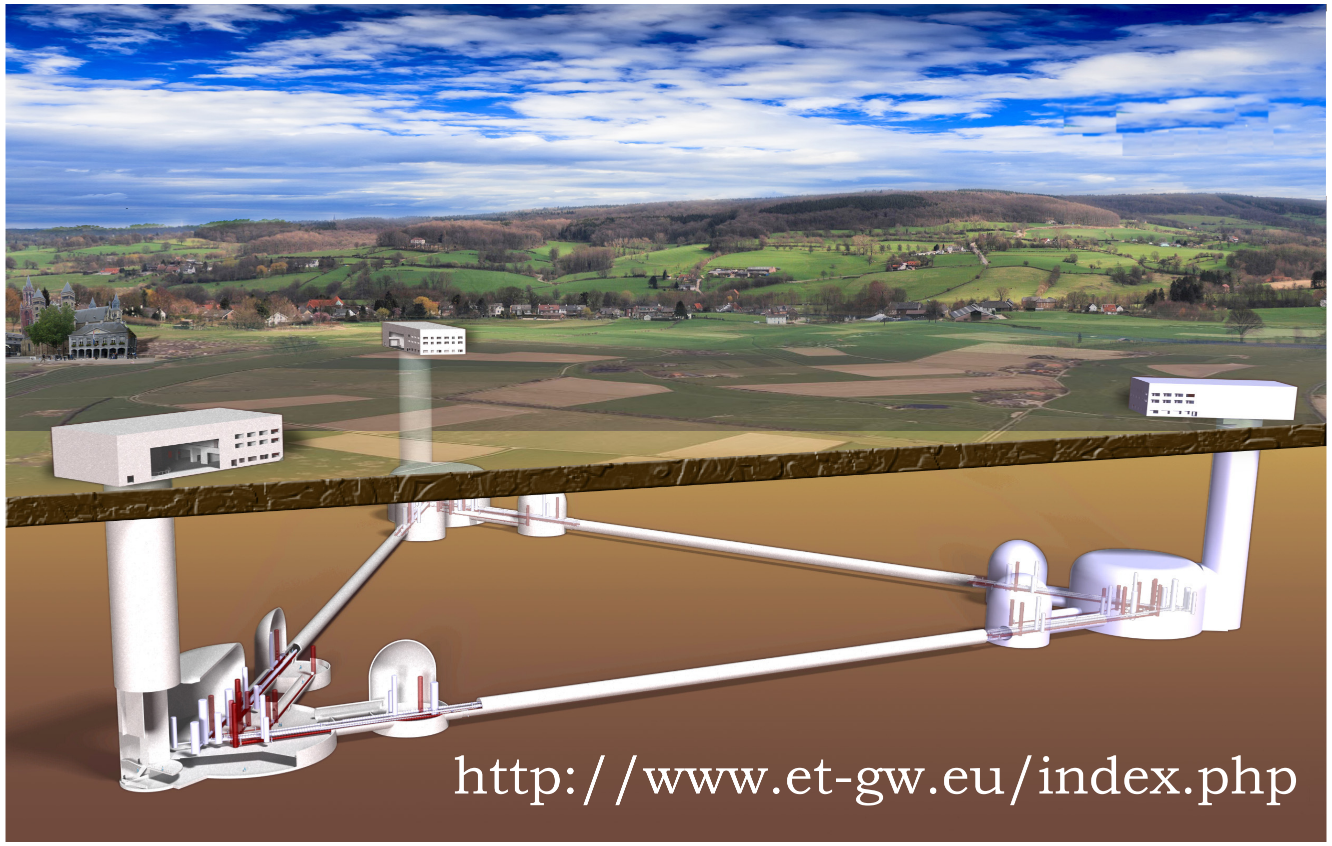
3rd generation (ET, CE...)

arm length L = 3 km

frequency range of detection:
1 Hz < f < 10^4 HZ

DETECTION TARGETS:

- Black hole coalescing binaries of masses few to hundred solar masses
- Neutron Star and NS-BH binaries / SN explosions
- Stochastic GW background



Space-based interferometers

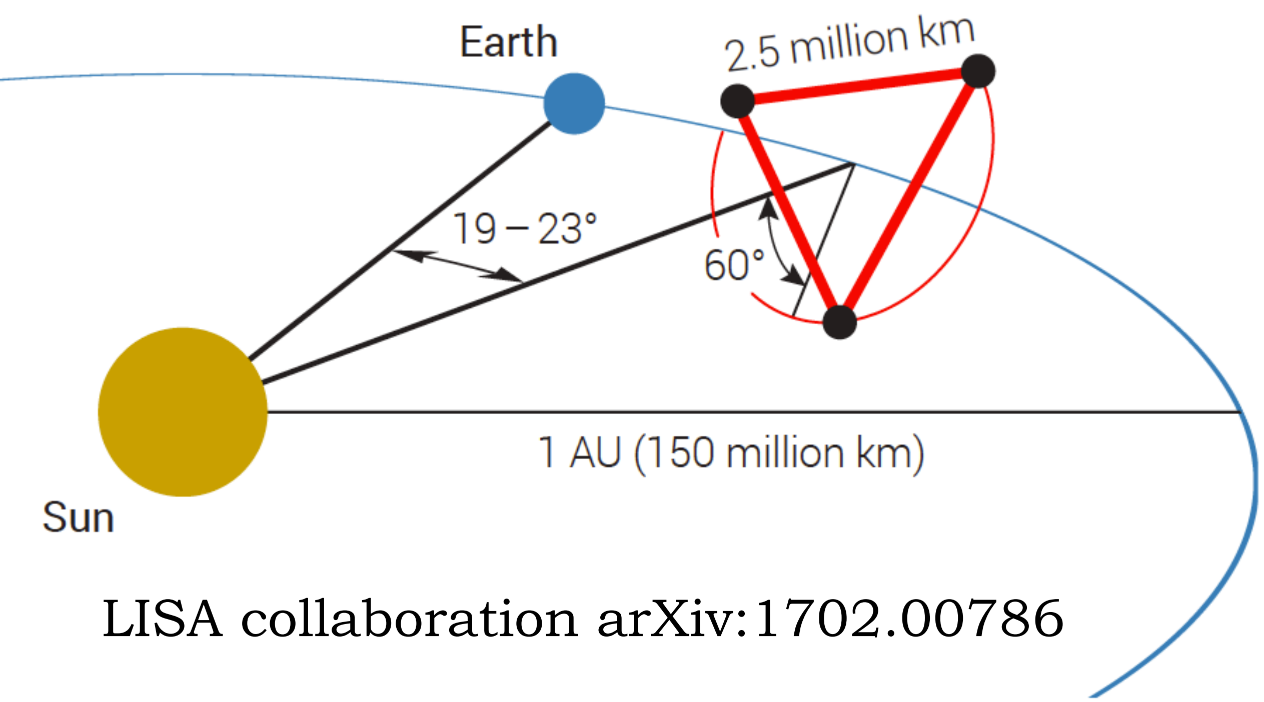
- no seismic noise
- much longer arms than on Earth

LISA: Laser Interferometer Space Antenna

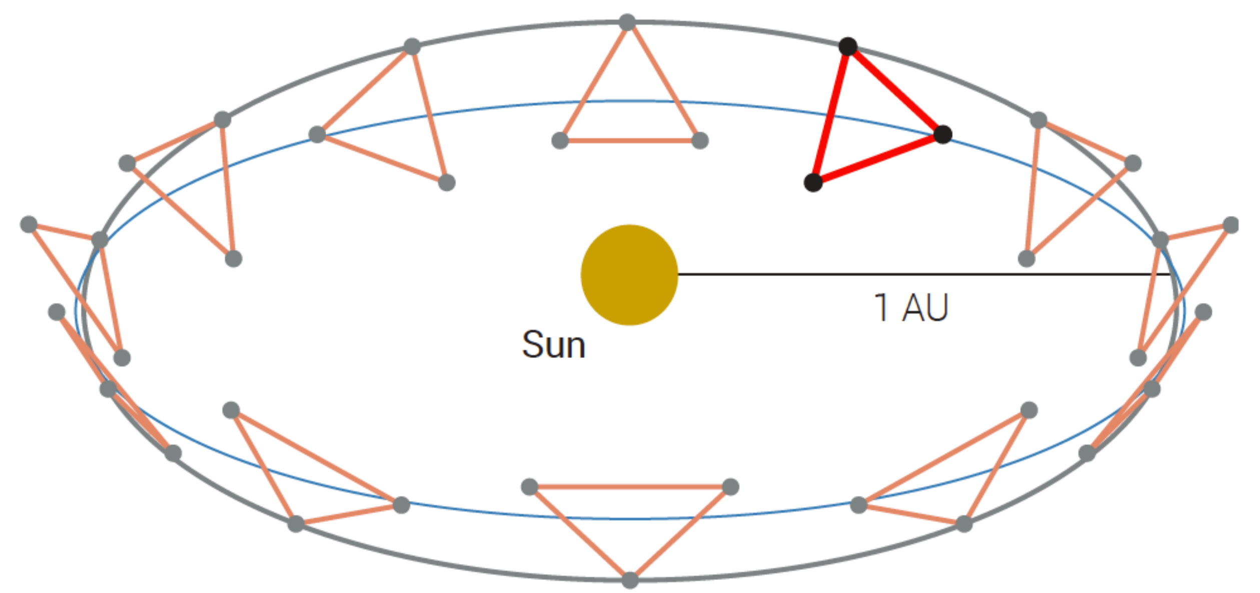
frequency range of detection:

$$10^{-4}\,\mathrm{Hz} < f < 1\,\mathrm{Hz}$$

- Launch in ~2034
- two masses in free fall per spacecraft
- 2.5 million km arms
- picometer displacement of masses



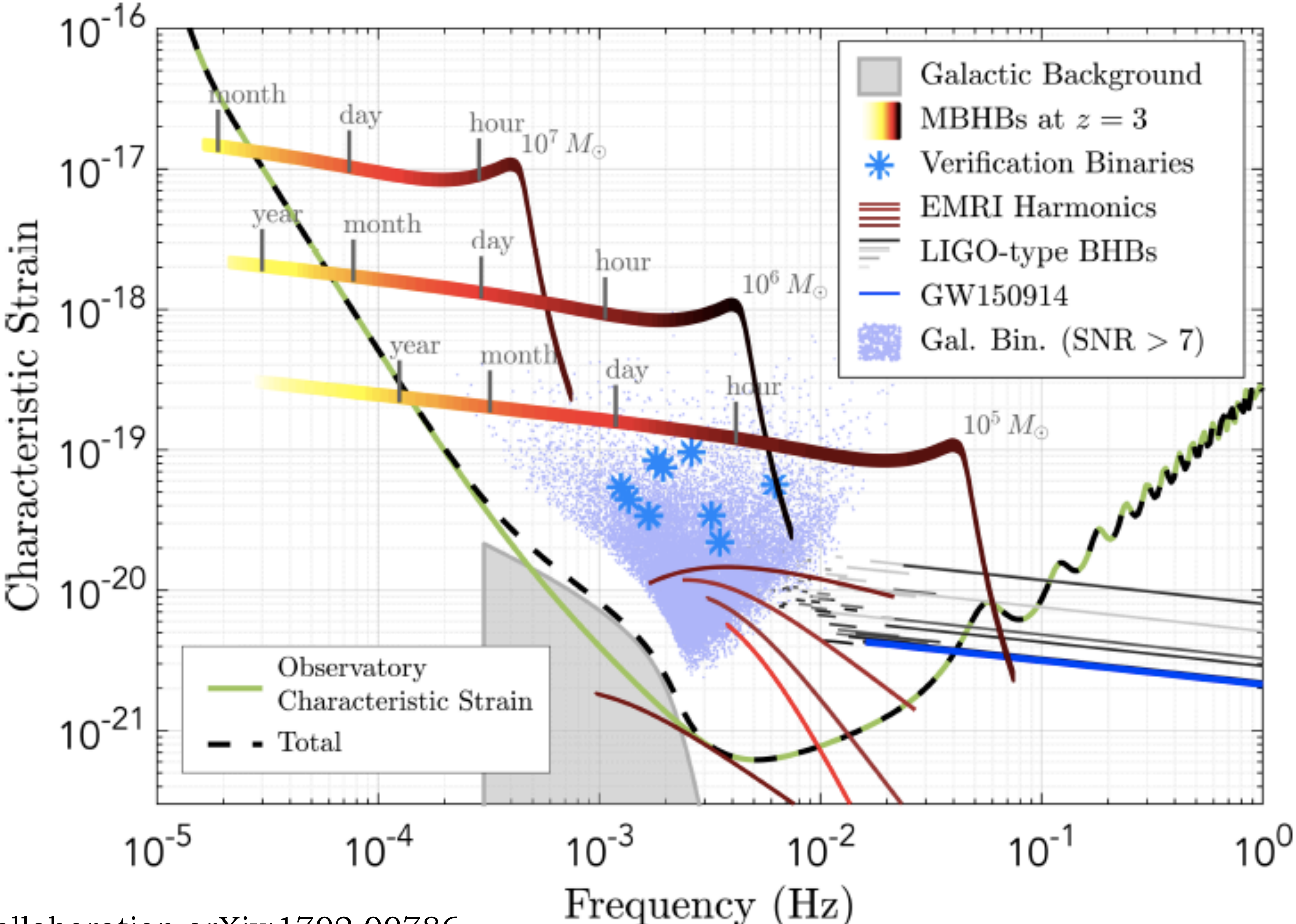
LISA collaboration arXiv:1702.00786



DECI-Hertz Observatories   Arm-length ~ 10^8 m

See e.g. arXiv:1908.11375

Space-based interferometers detection targets (LISA)

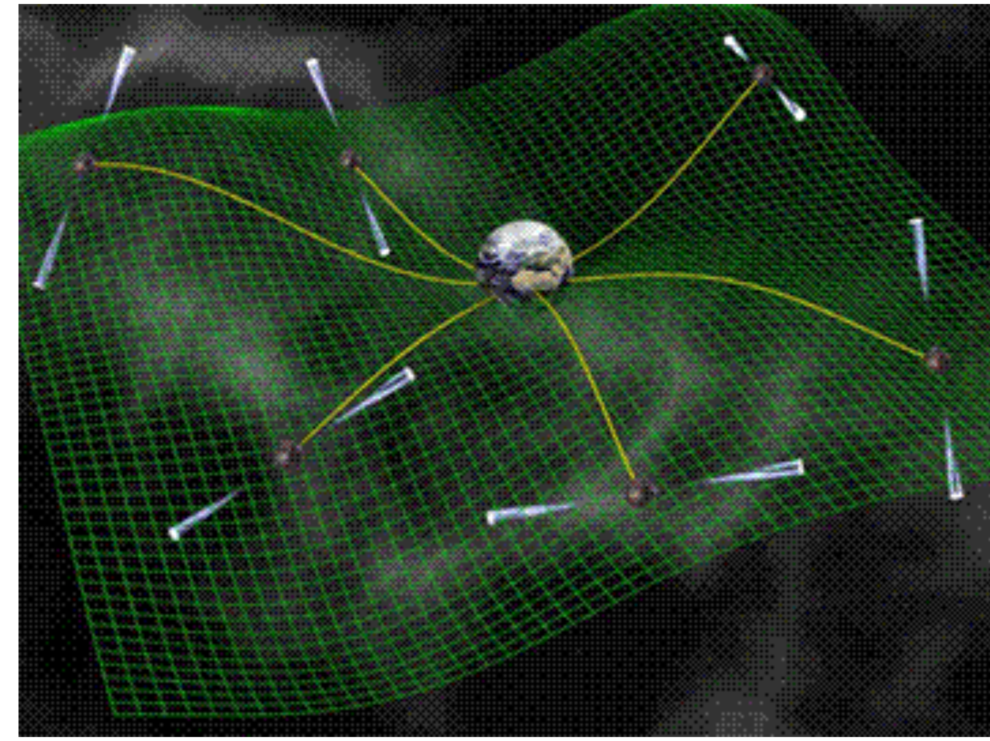


LISA collaboration arXiv:1702.00786

Pulsar timing array

EPTA, NANOGrav, PPTA, IPTA

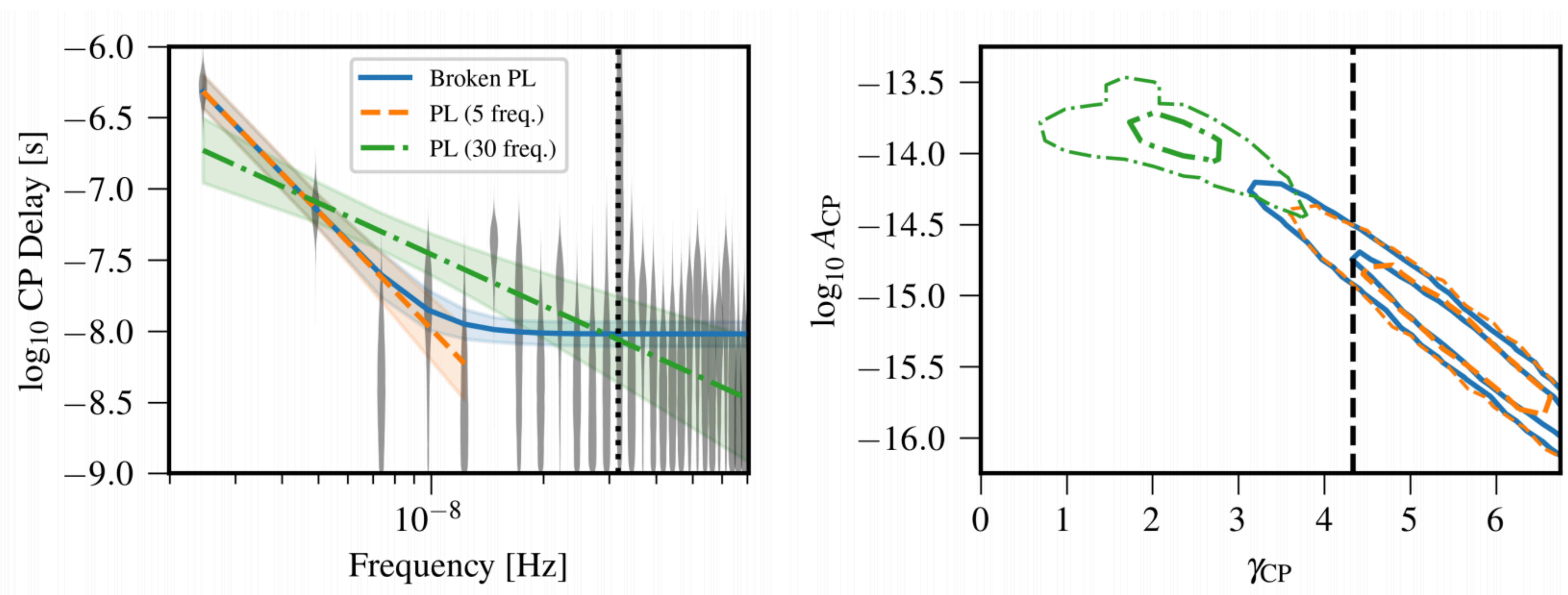
frequency range of detection: 10^{-9} Hz < f < 10^{-7} HZ



DETECTION TARGETS:

Individual emission and stochastic background from inspiralling SMBH binaries (masses of order 10^9 solar masses)

Recent NANOGrav result! First SGWB detection?



NANOGrav collaboration: arXiv:2009.04496

Cosmic microwave background

frequency range of detection: 10^{-18} Hz < f < 10^{-16} HZ

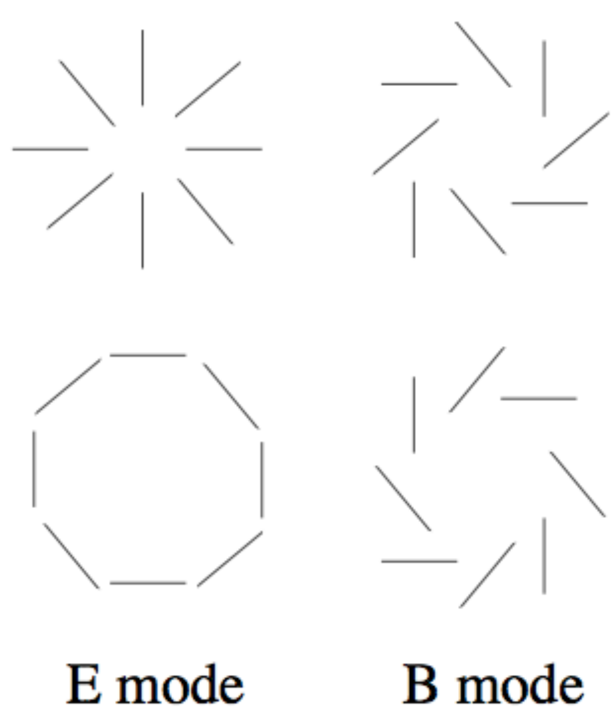
- temperature : limit by Planck

$$\frac{\delta T}{T} = -\int_{t_{\rm dec}}^{t_0} \dot{h}_{ij}\, n^i n^j\, dt$$

- polarisation: BB spectrum measured by BICEP2 and Planck generated at photon decoupling time, from Thomson scattering of electrons by a quadrupole temperature anisotropy in the photons

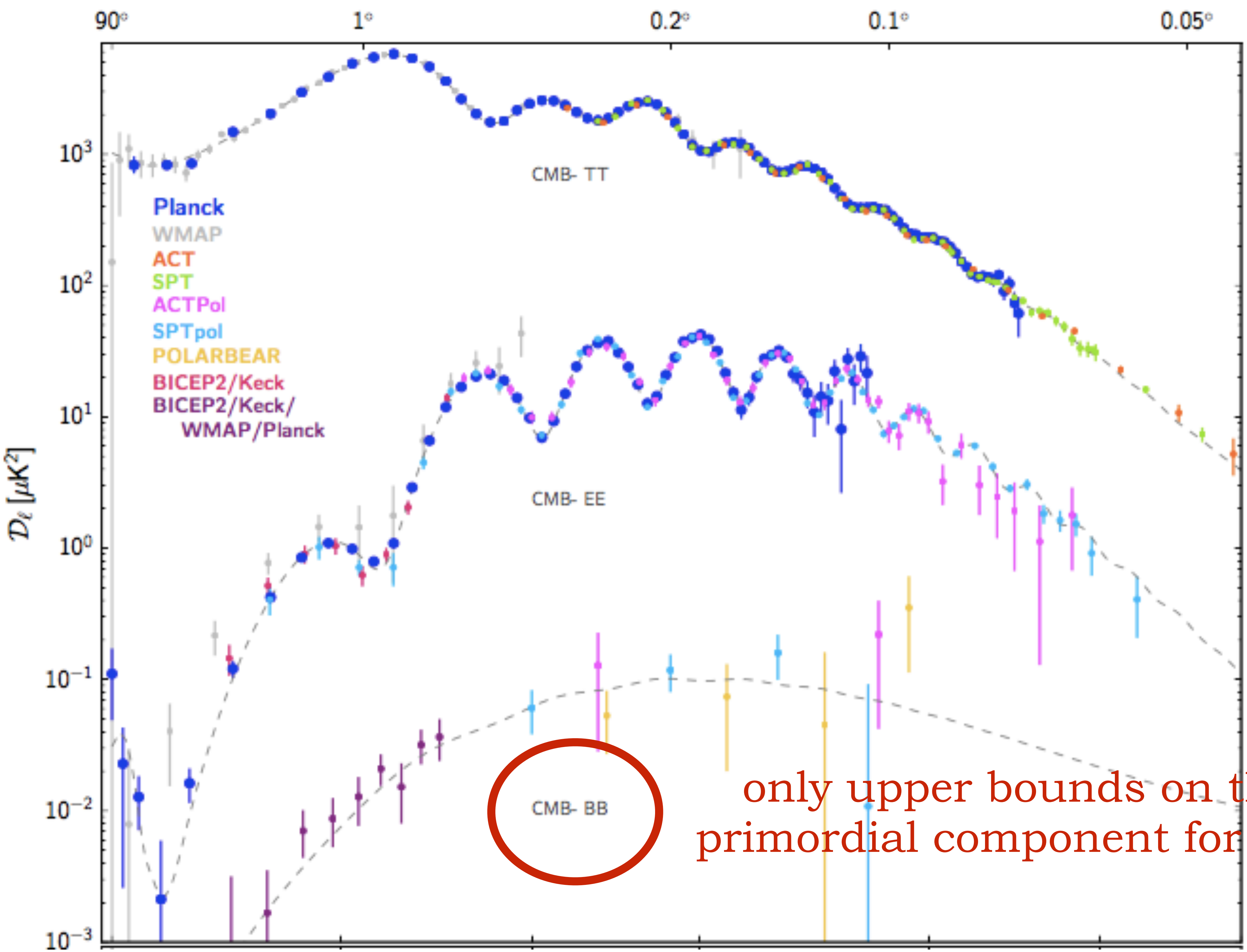
polarisation patterns

generated by primordial scalar and tensor perturbations



generated only by primordial tensor perturbations or by foregrounds

Cosmic microwave background



only upper bounds on the primordial component for now

Planck collaboration: arXiv:1807.06205