Exercises for CMB physics Lecture 2 (ionization history)

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Exercise 1:

• The probability of *not* scattering in dt is $1 - N_e \sigma_{\rm T} dt$. Split a small (but finite) time interval Δt in N sub-intervals $dt = \Delta t / N$, and show that the probability of not scattering in Δt is $\exp(-\Delta t N_e \sigma_{\rm T})$, by taking the limit $N \rightarrow \infty$. From this derive the probability of not scattering between t and t_0.

• Check explicitly that the visibility function integrates to unity.

Exercise 2:

Given the of blackbody photons per phase-space volume element is

$$\frac{d\mathcal{N}_\gamma}{d^3x d^3p} = \frac{2}{h^3} \frac{1}{\mathrm{e}^{p/T_\gamma} - 1}. \tag{1}$$

The total number density of CMB photons is then

$$N_\gamma = \int d^3p \frac{d\mathcal{N}_\gamma}{d^3x d^3p}. \tag{2}$$

• Compute the total number density of CMB photons at $T_\gamma = T_0 = 2.73$K, and confirm that $N_\gamma / N_{\rm H} \sim 10^9$.

• Compute the number density of CMB photons with energy above 13.6 eV, as a funtion of redshift, given that $T_\gamma = T_0 \times (1+z)$;. From this, estimate the redshift of recombination by requiring that there is more than one CMB photon with energy $E > 13.6$ eV per Hydrogen atom.

Exercise 3:

Derive the Saha equilibrium equation, by following these steps:
(i) Write down the number density of electrons per unit volume per momentum volume element (the phase-space density), assuming it is the Fermi-Dirac distribution with chemical potential μ_e.
(ii) Integrate over momenta to find the relationship between N_e and μ_e. A similar relation exists for $N_p, N_{\rm H^0}$.
(iii) Using $\mu_e + \mu_p = \mu_{\rm H^0}$, derive the Saha equation – you will need to also use the fact that $m_{\rm H^0} = m_e + m_p - E_I$.

Exercise 4:

• Give an analytic expression for $\tau_{\rm reio}(z_{\rm reio})$ for instantaneous reionization, neglecting the contribution of radiation and of the cosmological constant to the expansion rate (i.e. only accounting for matter). Invert this expression to find $z_{\rm reio}$ given $\tau_{\rm reio}$.

• Write the integral equation for $\tau_{\rm reio}(z)$ for a non-zero Ω_Λ, and plot $\tau_{\rm reio}(z_{\rm reio})$ by computing the integral numerically.
• Look up the latest *Planck* measurement of $\tau_{\rm reio}$ and from this infer $z_{\rm reio}$.