Exercises for CMB physics Lecture 3 (the Boltzmann equation)

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Exercise 1:

Prove that the photon momentum density is indeed $\vec{\mathcal{P}}_\gamma = \frac{4}{3}\overline{\rho}_\gamma \vec{v}_\gamma$, as stated during the lecture.

Exercise 2:

The goal of this exercise is to confirm the coefficient of the anisotropic stress. Recall that it is defined as

$$\sigma_\gamma^{ij} \equiv \int \frac{d^2\hat{n}}{4\pi} \left(3\hat{n}^i\hat{n}^j - \delta^{ij}\right)\Theta(\hat{n}). \tag{1}$$

We want to find the coefficient β such that

$$\Theta(\hat{n}) = \Theta_0 + \vec{v}_\gamma \cdot \hat{n} + \beta\ \hat{n}^i\hat{n}^j\sigma_\gamma^{ij} + \Theta_{\ell>2}(\hat{n}). \tag{2}$$

- Compute the coefficient α in this equation:

$$\int \frac{d^2\hat{n}}{4\pi}\hat{n}^i\hat{n}^j\hat{n}^k\hat{n}^l = \alpha(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). \tag{3}$$

Hint: contract this equation in two indices (e.g. kl), and use the fact that

$$\int \frac{d^2\hat{n}}{4\pi}\hat{n}^i\hat{n}^j = \frac{1}{3}\delta^{ij}. \tag{4}$$

- Find β by inserting Eq. (2) into Eq. (1), using the fact that

$$\int \frac{d^2\hat{n}}{4\pi}\Theta_{\ell>2}(\hat{n}) = 0. \tag{5}$$

Exercise 3:

Imagine that photons scatter with another particle χ (e.g. dark matter!) with some *chromatic* cross section. For instance, if the scatterer as an electric or magnetic dipole moment, the cross section would scale as E_γ^2. For simplicity, consider an isotropic cross section of the form

$$\frac{d\sigma}{d^2\hat{n}'} = \frac{\sigma_0}{4\pi}\left(\frac{E_\gamma}{m_\chi}\right)^2 \tag{6}$$

In this case, what would the Boltzmann equation look like? This question is more open, and is meant to make you think in more detail about the simplfications we made to the Boltzmann equation.

Exercise 4:

This exercise requires knowledge of general relativity, and is for the advanced students. It aims to guide you through the derivation of the relativistic version of the Boltzmann equation.

Consider the perturbed FLRW metric

$$ds^2 = a^2(\eta)\left[-d\eta^2(1+2\psi)d\eta^2 + (1-2\phi)d\vec{x}^2\right] = g_{\mu\nu}dx^\mu dx^\nu. \tag{7}$$

- "Comoving observers" are observers with fixed spatial coordinates. Find their 4-velocity u^μ, using the fact that $g_{\mu\nu}u^\mu u^\nu = -1$.

- Compute the Christoffel symbols $\Gamma^{\sigma}_{\mu\nu}$ of this metric.

- Write down the geodesic equations for photon momenta $p^\mu = dx^\mu/d\lambda$, where λ is an affine parameter.

- Write down the null-geodesic condition $g_{\mu\nu}p^\mu p^\nu = 0$ to relate p^i and p^0.

- Instead of the components p^μ, we may equivalently work with the variables $q \equiv -a g_{\mu\nu} u^\mu p^\nu$ and $\hat{n}^i \equiv p^i/\sqrt{\delta_{jk}p^j p^k}$, where u^μ is the 4-velocity of comoving observers. The variable q is such that $aq = -g_{\mu\nu}u^\mu p^\nu$ is the photon energy as observed by comoving observers. Prove that $dn^i/d\eta$ vanishes if $\phi, \psi = 0$ (i.e. it is linear in perturbations). Using your results from the previous two questions, derive the evolution of q along photon trajectories. You should arrive at

$$\frac{dq}{d\eta} = q(-n^i \nabla_i \psi + \dot{\phi}). \tag{8}$$

- Derive the left-hand-side of the Boltzmann equation, by using $(\eta, \vec{x}, q, \hat{n})$ as independent variables.