Exercises for CMB physics Lecture 4 (analytic solutions of the Boltzmann equation)

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Exercise 1:

We derived the following equation in the tight-coupling regime:

$$c_s^2 \frac{\partial}{\partial \eta}\left[c_s^{-2}\dot{\delta}_\gamma\right] + k^2 c_s^2 \delta_\gamma = 0, \qquad c_s^2(\eta) \equiv \frac{1}{3}\left(1 + \frac{3}{4}\frac{\overline{\rho}_b}{\overline{\rho}_\gamma}\right)^{-1}. \tag{1}$$

Our goal is to look for an approximate solution in the limit $aH \ll k$, using the fact that the sound speed varies on a timescale $\sim 1/aH \gg 1/k$. For reference, this is known as the WKB approximation.

We look for a solution of the form

$$\delta_\gamma(\eta) = A(\eta)\cos\left[k\int_0^\eta d\eta'\ c_s(\eta')\right]. \tag{2}$$

- Plugging into the ODE for δ_γ, derive the ODE satisfied by $A(\eta)$.
- Assuming $\dot{A}/A \sim aH$ and $\ddot{A}/\dot{A} \sim aH$, derive the leading-order equation for $A(\eta)$, i.e. to lowest order in aH/k.
- Find the analytic solution for your ODE for $A(\eta)$.

Exercise 2:

In class we only talked about the tight-coupling approximation, which is the zero-th order of an expansion in $k/|\dot{\tau}|, aH/|\dot{\tau}|$, where $\dot{\tau} \equiv -aN_e\sigma_{\rm T}$. Working in the limit $k \gg aH$, try to get to the next order in $k/\dot{\tau}$, by following the steps outlined below – note that this is a somewhat advanced calculation, so you should be very proud if you get throught it!

(i) From the baryon fluid equation, express $\vec{v}_b - \vec{v}_\gamma$ in terms of photon fluid variables, by seeking an expansion of the form

$$\vec{v}_b = \vec{v}_\gamma + \frac{k}{\dot{\tau}}\vec{V}^{(1)} + \left(\frac{k}{\dot{\tau}}\right)^2 \vec{V}^{(2)} + ... \tag{3}$$

[For reference, during the lecture we already derived $\vec{V}^{(1)}$.

(ii) Write down the fluid equation for the photon anisotropic stress, of the form

$$\dot{\sigma}_\gamma^{ij} + ... = -\#aN_e\sigma_{\rm T}\sigma_\gamma^{ij} \tag{4}$$

(iii) From this equation, solve for σ_γ^{ij} at first order in $k/\dot{\tau}$, and express it in terms of $\vec{v}_\gamma$. [during the lecture we simply assumed $\sigma_\gamma^{ij} = 0$]. You should get an expression of the form

$$\sigma_\gamma^{ij} \propto \frac{k}{\dot{\tau}}\left(\hat{k}^i v_\gamma^j - \frac{1}{3}\delta^{ij}\hat{k}\cdot\vec{v}_\gamma\right). \tag{5}$$

(iv) Plug back into the photon momentum equation. You should now have an equation for a damped harmonic oscillator.

Exercise 3:

Compute explicitly the following integrals, by using spherical polar coordinates with $\hat{k}$ as the z axis, and then show that they are indeed related to the spherical Bessel functions:

$$\int \frac{d^2\hat{n}}{4\pi} e^{i\kappa\hat{k}\cdot\hat{n}}, \qquad \int \frac{d^2\hat{n}}{4\pi}\ \hat{n}\ e^{i\kappa\hat{k}\cdot\hat{n}}. \tag{6}$$

Exercise 4:

Define the transfer function for the variable $X(\vec{k}, \eta, \hat{n})$ as the linear relation between X and the primordial curvature perturbation:

$$X(\vec{k}, \eta, \hat{n}) = \mathcal{T}(\eta, k, \hat{k} \cdot \hat{n})\zeta(\vec{k}). \tag{7}$$

We moreover defined, in the lecture,

$$\tilde{\Theta}_{\rm obs}(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \mathcal{T}_\Theta(k, \hat{k} \cdot \hat{n})\zeta(\vec{k}). \tag{8}$$

By using the line-of-sight integral (and expressing the integrand in Fourier space), explcitly relate $\mathcal{T}_\Theta(k, \hat{k} \cdot \hat{n})$ to the transfer functions of ϕ, ψ and S_Θ.