Exercises IV Joint ICTP School on Cosmology

lecture 03

January 20, Valerie Domcke

1 - Wronskian condition (3 pt). Consider a real scalar quantum field $\hat{\phi}(\tau,\vec{x})$ and its conjugate momentum $\hat{\pi}(\tau,\vec{x})$. Using the Fourier decomposition of both ϕ and ϕ, show that the quantum uncertainty relation

$$[\hat{\phi}(\tau,\vec{x}),\hat{\pi}(\tau,\vec{y})] = i\hbar\delta(\vec{x}-\vec{y})\,, \tag{1}$$

implies the Wronskian condition

$$\frac{i}{\hbar}(\phi_k^*\pi_k - \pi_k^*\phi_k) = 1\,. \tag{2}$$

2 - Mukhanov equation (4pt). Consider the Mukhanov equation

$$v_k'' + (k^2 - \frac{z''}{z})v_k = 0\,, \tag{3}$$

with $z = a\dot{\phi}/H$. Use your favourit computer programm to plot the solution of this equation in a de Sitter background from $\tau_0 = -100/k$ to $\tau = -0.1/k$, with the initial condition $|v_k(\tau_0)| = 1/\sqrt{2k}$ and $v_k'(\tau_0)$ determined by (2). Describe and interpret the two qualitatively different regimes you observe.

3 - Tensor perturbations (8 pt)

(a) The second order action for the tensor perturbation of the metric is given by

$$S_{\rm EH}^{(2)} = \frac{M_P^2}{8}\int d\tau d^3x a^2(\tau)\left[(h_{ij}')^2 - (\partial h_{ij})^2\right]\,. \tag{4}$$

Show that the equation of motion for the Mukhanov Sasaki variable $v_k^{(s)} = \frac{a}{2}M_P h_k^{(s)}$ in Fourier space is given by

$$(v_k^{(s)})'' + \left(k^2 - \frac{a''}{a}\right)v_k^{(s)} = 0\,. \tag{5}$$

Note: use the Fourier expansion $h_{ij}(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\sum_{s=+,\times}\epsilon_{ij}^s(\vec{k})h_{\vec{k}}^s(\tau)e^{-\vec{k}\vec{x}}$ with the polarization tensor ϵ_{ij} obeying $k^i\epsilon_{ij} = 0$ and $\epsilon_{ij}^s\epsilon_{ij}^{s'} = 2\delta_{ss'}$. Here s denotes the two possible polarizations of the tensor perturbation. (2pt)

(b) Following the procedure for scalar perturbations in the lecture, determine the Bunch Davies intitial condition for $v_k^{(s)}$ at $\tau \to -\infty$. (1 pt)

(c) Solve Eq. (5) with this initial condition. (1pt)

(d) Determine the two point function $\langle h_k^{(s)} h_{k'}^{(s')} \rangle$ for super horizon modes, $k \ll aH$. (2pt)

(e) Assuming statistical isotropy and homogeneity, the two point function is given by

$$\langle h_k^{(s)} h_{k'}^{(s')} \rangle = (2\pi)^3 \delta_{ss'} \delta(\vec{k} - \vec{k}') P_h(k) \,. \tag{6}$$

Determine $P_h(k)$. (2 pt)

(f) Compute the relative strength of the scalar and tensor perturbations, determined by the tensor-to-scalar ration $r \equiv \Delta_t^2 / \Delta_{\mathcal{R}}^2$. CMB experiments have measured $\Delta_{\mathcal{R}}^2$ but only provide an upper bound on r. What does this imply for inflation models? (2 pt)