Exercises IV Joint ICTP School on Cosmology

lecture 04

January 21, Valerie Domcke

1 - Inflation models (II) (4pt). Consider the inflation model specified by $V(\phi) = \frac{1}{2}m^2\phi^2$.

(a) Compute the amplitude of the scalar perturbations $\Delta^2_{\mathcal{R}}$ at $N_{cmb} = 60$ e-folds before the end of inflation (2 pt).

(b) Measurements of the CMB anisotropies yield $\Delta^2_{\mathcal{R}} \sim 10^{-9}$. Use this to determine m (1pt).

(c) Compute the scalar spectran tilt n_s and the tensor-to-scalar ratio r. Compare your result to Fig. 8 of the Planck publication arxiv:1807.06211. (1pt)

2 - Continuity equation (10 pt). Consider the $\nu = 0$ component of energy momentum conservation,

$$0 = \nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha\,, \tag{1}$$

up to first order in perturbations of the metric and the stress energy tensor. Work in the approximation of a flat universe ($k = 0$) and consider the Universe filled with a single fluid with $p = \omega\rho$. In Newtonian gauge, the Christoffel symbols read

$$\Gamma^0_{00} = \mathcal{H} + \Psi'\,, \qquad \Gamma^0_{i0} = \partial_i \Psi\,, \qquad \Gamma^i_{00} = \delta^{ij}\partial_j \Psi\,,$$
$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - [\Phi' + 2\mathcal{H}(\Psi + \Phi)]\delta_{ij}\,, \qquad \Gamma^i_{j0} = (\mathcal{H} - \Phi')\delta^i_j\,, \qquad \Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi\,,$$

with the comoving Hubble parameter $\mathcal{H} = aH$ and prime denoting the derivative with respect to conformal time. $\mu, \nu = 0..3$ are Lorentz indices, $i, j = 1..3$ are spatial indices.

(a) Write Eq. (1) in terms of T^0_0, T^i_0 and T^i_j and drop all terms with are second order (or higher) in the perturbations (2 pt)

(b) Substituting the relevant Christoffel symbols, show that (2pt)

$$\partial_0 T^0_0 + \partial_i T^i_0 + (3\mathcal{H} - 3\Phi')T^0_0 - 3(\mathcal{H} - \Phi')T^i_i = 0 \tag{2}$$

(c) Expand T^0_0, T^i_0 and T^i_j to first order (see lecture). Inserting this into Eq. (2) show that (5 pt)

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}) \tag{3}$$

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta p) + (\bar{\rho} + \bar{p})(3\Phi' - \partial_i v^i) \tag{4}$$

(d) Show that this implies (1pt)

$$\left(\frac{\delta\rho}{\bar{\rho}}\right)' = -3\mathcal{H}(\frac{\delta p}{\bar{\rho}} - \frac{\bar{p}}{\bar{\rho}^2}\delta\rho) + (1 + \frac{\bar{p}}{\bar{\rho}})(3\Phi' - \partial_i v^i) \tag{5}$$