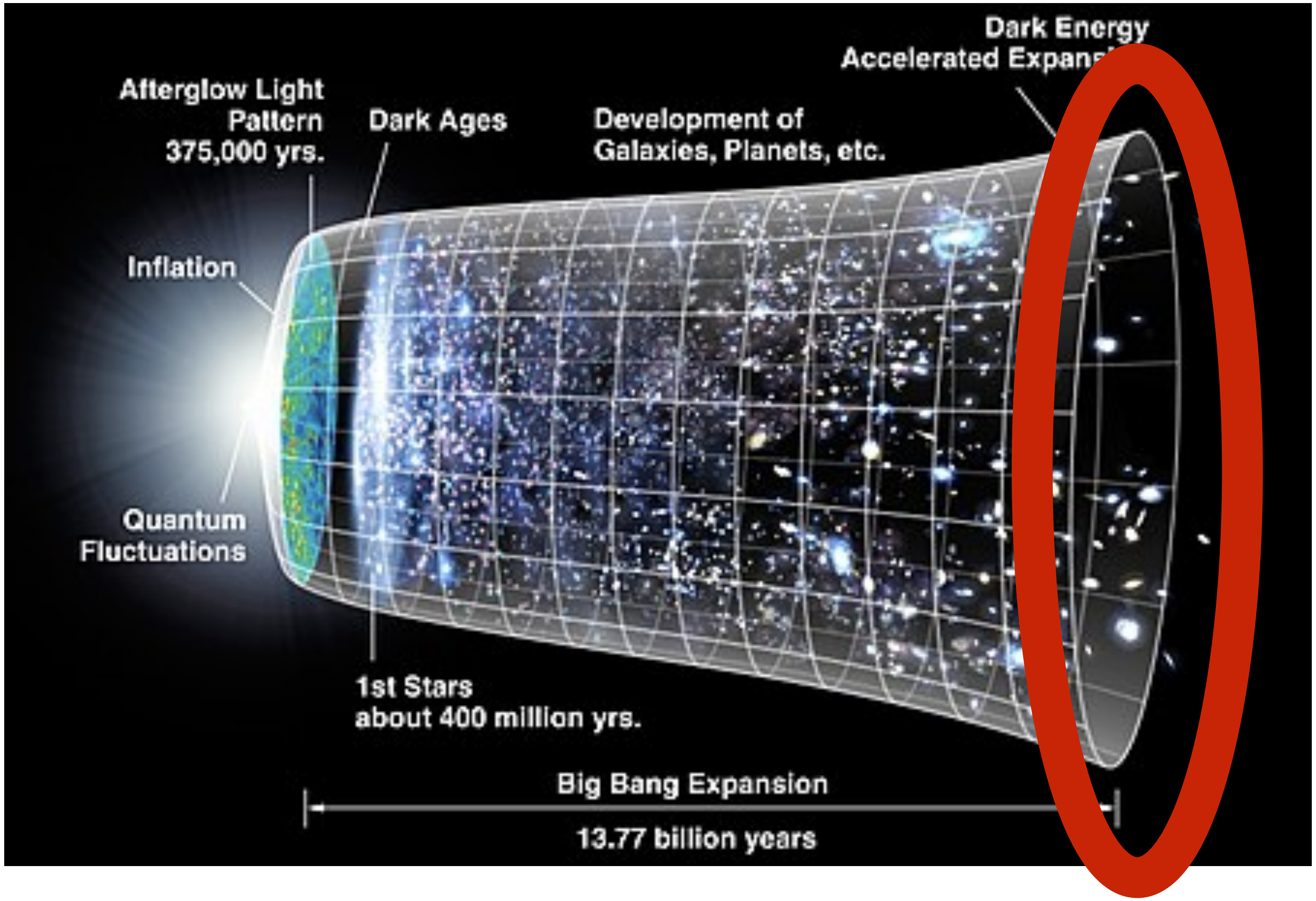
Cosmology from: gravitational waves

Chiara Caprini
CNRS (APC Paris)

LECTURE 4
GW emission from binaries: generalities

How can GW help to probe cosmology?



late
universe

The GW signal from binaries travels through the FLRW spacetime: it can be used to probe the late-time dynamics and the content of the universe

test of accelerated expansion, test of GR at large scales

Generation of GWs in linearised theory

Once again, we start with the GW propagation equation

$$H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 16\pi G\, a^3\, \Pi_r(\mathbf{k},\eta)$$

But we change point of view completely

- We suppose the source is close by and we can neglect the expansion of the universe
- We go back to linearised theory in flat spacetime
- The source is described by Newtonian gravity
- The gravitational field generated by the source is sufficiently weak not to perturb the Minkowski background

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \qquad\qquad \partial_\mu T^{\mu\nu} = 0$$

Generation of GWs in linearised theory

Self-gravitating 2-body system in Newtonian theory

Virial theorem $$E_{\rm kin} = -\frac{1}{2}U$$

The problem is equivalent to the motion of one body with reduced mass μ, in the gravitational field given by the sum of the masses

$$m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m}$$

$$\frac{1}{2}\mu v^2 = -\frac{1}{2}\frac{Gm\mu}{r}$$

Schwarzschild radius

$$R_S = 2\frac{Gm}{c^2}$$

$$\left(\frac{v}{c}\right)^2 = \frac{R_S}{2r} \ll 1$$

Weak gravitational field means low velocity for a self-gravitating system

“Gravitational Waves”, M. Maggiore, Oxford University Press 2008

Generation of GWs in linearised theory

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G \int d^3x' \, \frac{T_{\mu\nu}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Retarded time

We can go to TT gauge outside the source

$$h_{ij}(t,\mathbf{x}) = 4G\underbrace{\Lambda_{ij\ell m}(\hat{n})}\int d^3x' \, \frac{T^{\ell m}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

TT projector $\Lambda_{ij\ell m} = P_{i\ell}P_{jm} - \frac{1}{2}P_{ij}P_{\ell m}$

$$P_{i\ell} = \delta_{i\ell} - \hat{n}_i \hat{n}_\ell$$

Generation of GWs in linearised theory

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

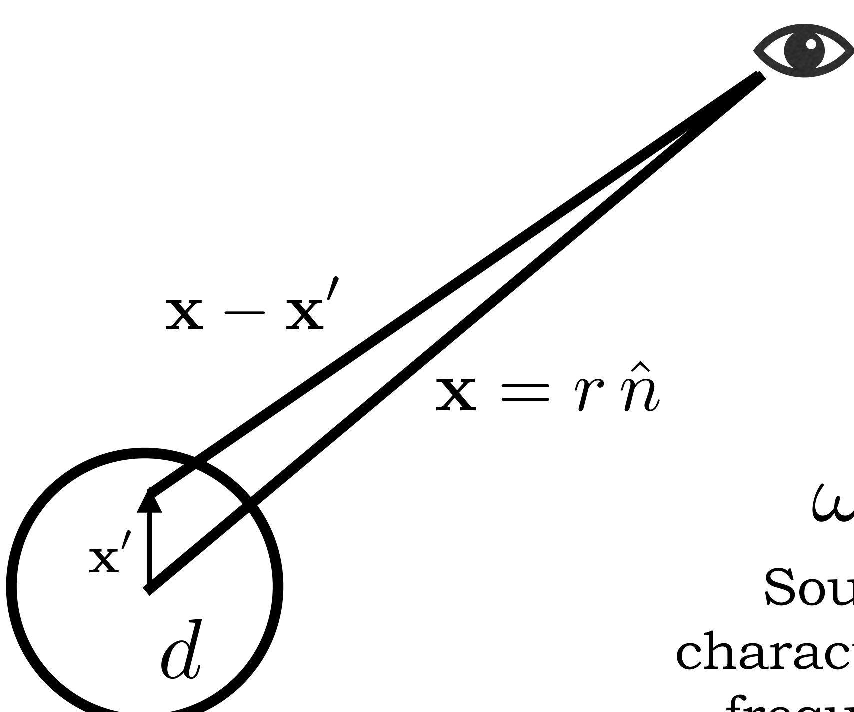
$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G \int d^3x' \, \frac{T_{\mu\nu}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Retarded time

We can go to TT gauge outside the source

$$h_{ij}(t,\mathbf{x}) = 4G\Lambda_{ij\ell m}(\hat{n}) \int d^3x' \, \frac{T^{\ell m}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Low velocity expansion in the radiation zone r » d



$$r - |\mathbf{x}-\mathbf{x}'| \sim \mathbf{x}' \cdot \hat{n}$$

$$T_{\ell m}(t - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}') \sim T_{\ell m}(t - r, \mathbf{x}') +$$
$$+ \underbrace{\partial_t T_{\ell m}}(t - r, \mathbf{x}')(\mathbf{x}' \cdot \hat{n})$$

ω_s

Source characteristic frequency

$$\lesssim \omega_s T_{\ell m} d \sim \frac{v}{d} T_{\ell m} d \sim v\, T_{\ell m}$$

Generation of GWs in linearised theory

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

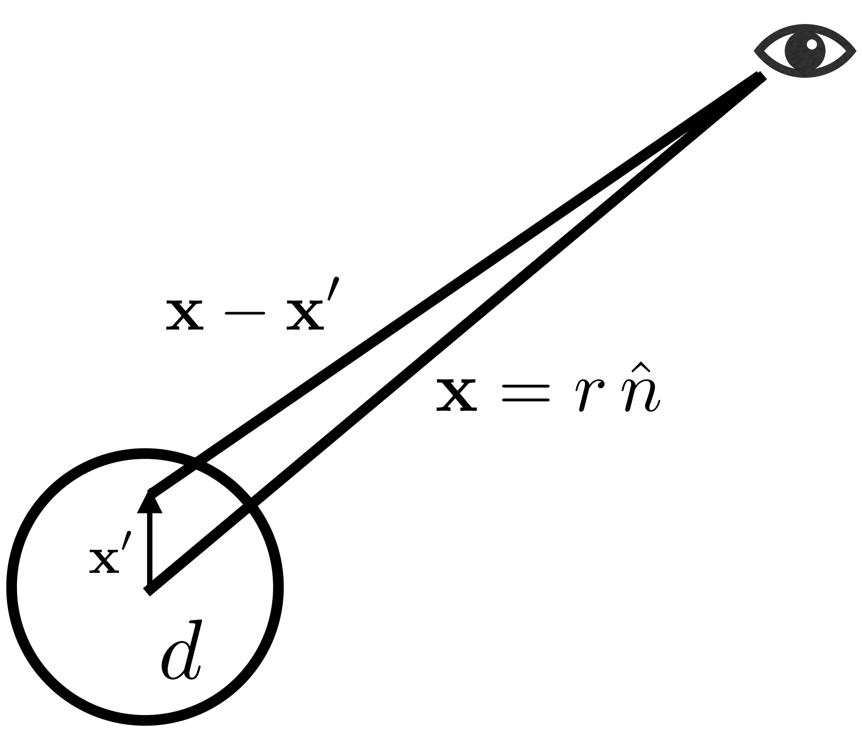
$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3x' \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Retarded time

We can go to TT gauge outside the source

$$h_{ij}(t, \mathbf{x}) = 4G\Lambda_{ij\ell m}(\hat{n}) \int d^3x' \frac{T^{\ell m}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Low velocity expansion in the radiation zone r » d



One cannot resolve the source with GWs

$$r \gg \lambda\!\!\!^{-} \gg d$$

$$v \sim \omega_s d$$

$$\lambda\!\!\!^{-} = \frac{c}{\omega} \sim \frac{c}{v}\, d \gg d$$

Generation of GWs in linearised theory

Keeping the lowest order $$h_{ij}(t,\mathbf{x}) \sim \frac{4G}{r}\Lambda_{ij\ell m}(\hat{n})\int d^3x' T^{\ell m}(t-r,\mathbf{x}')$$

Using energy momentum conservation, this can be rewritten $\partial_\mu T^{\mu\nu} = 0$

$$\int d^3x' T^{\ell m}(t-r,\mathbf{x}') = \frac{1}{2}\int d^3x'\, \partial_t^2 T^{00}(t-r,\mathbf{x}')\, x^\ell\, x^m = \frac{1}{2}\ddot{M}^{\ell m}$$

Second mass moment

$$M^{\ell m} - \frac{1}{3}\overbrace{\delta^{\ell m} M_{jj}}^{\text{trace}} = \int d^3x\, \rho(\mathbf{x},t)\left(x^\ell x^m - \frac{1}{3}x^2\delta^{\ell m}\right) = Q^{\ell m}$$

Mass quadrupole

at the lowest order in the low velocity expansion and in the radiation zone, GW emission arises from the quadrupole of the source

$$[h_{ij}(t,\mathbf{x})]_{\text{quad}} = \frac{2G}{r}\Lambda_{ij\ell m}(\hat{n})\ddot{Q}^{\ell m}(t-r)$$

C.W. Misner, K.S. Thorne, J.A. Wheeler, “Gravitation”, Freeman 1997 (chapter 35.15)

Generation of GWs in linearised theory

the gravitational interaction is weak: to produce a signal, one needs very massive objects moving relativistically.
Unfortunately these are far away, so the signal is anyway low!

GW direct detection was a great scientific and technological achievement

$$\ddot{Q}_{k\ell} \sim M = 30 M_\odot \qquad r = 400\,\mathrm{Mpc} \qquad \longrightarrow \qquad h_{ij} \sim 10^{-20}$$

$$[h_{ij}(t,\mathbf{x})]_{\rm quad} = \frac{2G}{r}\Lambda_{ij\ell m}(\hat{n})\ddot{Q}^{\ell m}(t-r)$$

Generation of GWs in linearised theory

The quadrupole formula

Radiated GW energy energy per unit time and angle

$$\rho_{GW} = \frac{\langle \dot{h}_{ij}\dot{h}^{ij}\rangle}{32\pi G} \quad \longrightarrow \quad \frac{dE_{\rm GW}}{dt\, d\Omega} = \frac{r^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle$$

Total radiated power

$$[P]_{\rm quad} = \frac{G}{5}\langle \dddot{Q}_{ij}(t-r)\dddot{Q}_{ij}(t-r)\rangle$$

$$\int d\Omega \Lambda_{ij\ell m} = \frac{2\pi}{15}(11\delta_{i\ell}\delta_{jm} - 4\delta_{ij}\delta_{\ell m} + \delta_{im}\delta_{j\ell})$$

Generation of GWs in linearised theory

- Connection with previous lectures: early universe sources are relativistic and not gravitationally bound -> we need to calculate the full tensor anisotropic stress

- By expanding the tensor anisotropic stress in multipoles, one would find that the first moment is the quadrupole

- The multipole expansion of a classical radiation field has zero contribution from multipoles ℓ < S where S is the spin of the associated quantum mechanical particle

- Monopole $\ell = 0$ and dipole $\ell = 1$ radiation is absent for GWs

$$h^{00}(t, \mathbf{x}) \sim \frac{4G}{r} \int d^3x' T^{00} \sim \frac{4G}{r} M(t-r)$$

h^{00} is a static component, because of mass conservation for an isolated system $\dot{M} = 0$

$$h^{0i}(t, \mathbf{x}) \sim \frac{4G}{r} \int d^3x' T^{0i} \sim \frac{4G}{r} P^i(t-r)$$

h^{0i} a static component, because of momentum conservation for an isolated system $\dot{P}^i = 0$

Generation of GWs in linearised theory

- Connection with previous lectures: early universe sources are relativistic and not gravitationally bound -> we need to calculate the full tensor anisotropic stress
- By expanding the tensor anisotropic stress in multipoles, one would find that the first moment is the quadrupole
- The multipole expansion of a classical radiation field has zero contribution from multipoles $\ell < S$ where S is the spin of the associated quantum mechanical particle
- Monopole $\ell = 0$ and dipole $\ell = 1$ radiation is absent for GWs

Exercise: is this true in general?

h^{00} is a static component, because of mass conservation for an isolated system $\dot{M} = 0$

h^{0i} a static component, because of momentum conservation for an isolated system $\dot{P}^i = 0$

Generation of GWs in linearised theory

More explicit formulas

$$\Lambda_{ij\ell m}\ddot{M}_{\ell m} = (P\ddot{M}P)_{ij} - \frac{1}{2}P_{ij}\mathrm{tr}(P\ddot{M})$$

$$P_{i\ell} = \delta_{i\ell} - \hat{n}_i\hat{n}_\ell$$

The polarisation amplitudes for a wave propagating in the z direction

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} h_+(t,\hat{z}) &= \frac{G}{r}(\ddot{M}_{11} - \ddot{M}_{22})(t-r) \\ h_\times(t,\hat{z}) &= \frac{2G}{r}\ddot{M}_{21}(t-r) \end{aligned}$$

A rotation of the reference system allows to find the GW emission in a general direction

$$n_i = R_{ij}z'_j \qquad\qquad M'_{ij} = (R^T\, M\, R)_{ij}$$

In this case, the polarisation amplitudes depend on all mass moment components, and on the angles (φ,θ) of the direction n

GWs from a binary system in circular orbit

Isolated system of two point masses moving on circular trajectories determined solely by their mutual interaction

$$\partial_\mu T^{\mu\nu} = 0$$

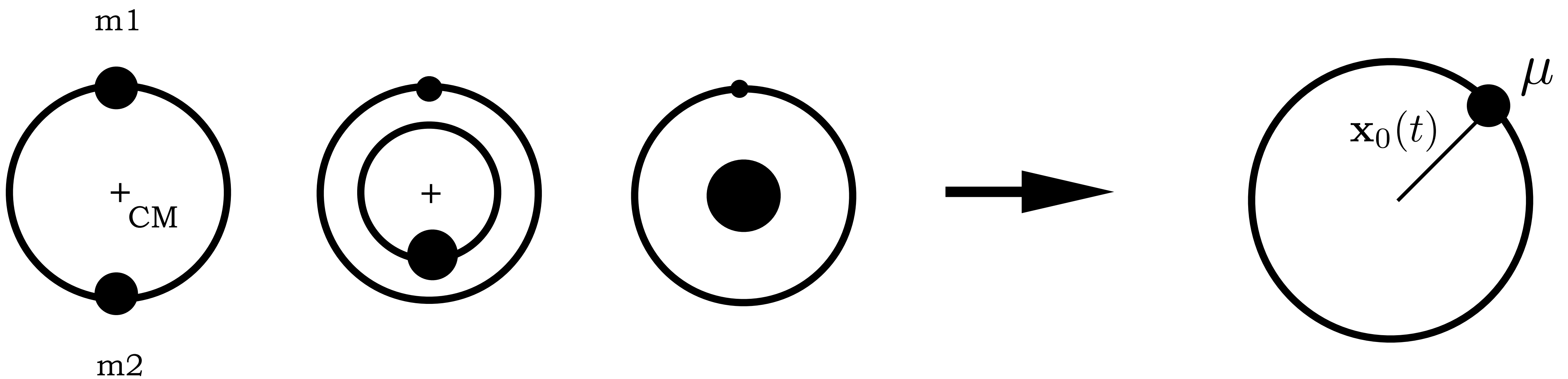
Otherwise, we cannot use linearised theory!

In the centre of mass reference frame, the second mass moment is the one of a single particle with reduced mass, and trajectory given by the relative trajectory of the two point masses

$$M^{ij} = \mu\, x_0^i(t)\, x_0^j(t)$$

$$\mu = \frac{m_1 m_2}{m}$$

$$\mathbf{x}_0(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$$



GWs from a binary system in circular orbit

Orbit in the (x,y) plane

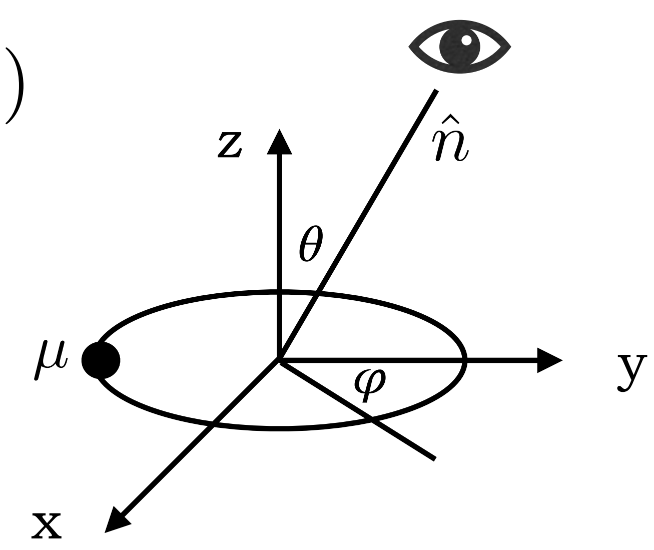
$$x_0(t) = R\cos(\omega_s t + \pi/2)$$
$$y_0(t) = R\sin(\omega_s t + \pi/2)$$
$$z_0(t) = 0$$

Double time derivative of the second mass moment

$$\ddot{M}_{11}(t) = -\ddot{M}_{22}(t) = 2\mu R^2\omega_s^2\cos(2\omega_s t)$$
$$\ddot{M}_{12}(t) = 2\mu R^2\omega_s^2\sin(2\omega_s t)$$

After performing the rotation of the reference system, the GW polarisation in arbitrary direction n are

$$h_+(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\varphi)$$
$$h_\times(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\,\cos\theta\,\sin(2\omega_s t_{\rm ret} + 2\varphi)$$



GWs from a binary system in circular orbit

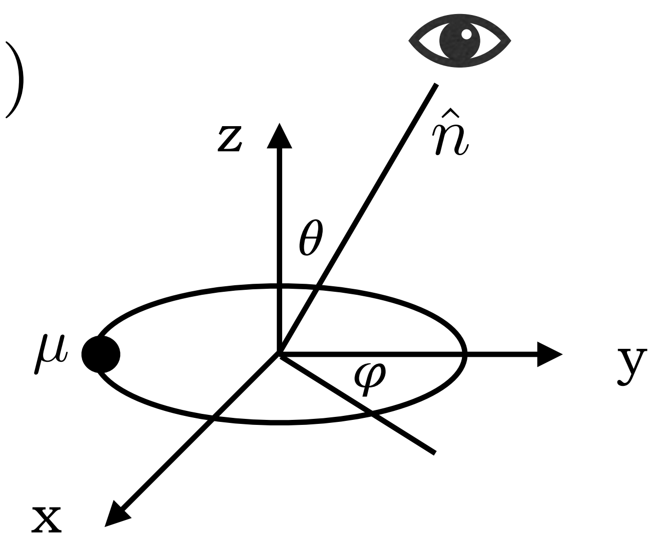
- A non-relativistic source performing harmonic oscillations with frequency ω_s emits monochromatic radiation with frequency 2 ω_s

- The dependence on φ can be reabsorbed in a redefinition of the origin of time

- From the degree of polarisation observed, one can derive the inclination of the orbit

$$h_+(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\varphi)$$

$$h_\times(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\,\cos\theta\,\sin(2\omega_s t_{\rm ret} + 2\varphi)$$



GWs from a binary system in circular orbit

Radiated power

$$\frac{dE_{\rm GW}}{dt\, d\Omega} = \frac{r^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle$$

$$\frac{dP_{\rm GW}}{d\Omega} = \frac{r^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$

$$\left[\frac{dP_{\rm GW}}{d\Omega}\right]_{\rm quad} = \frac{2G\mu^2 R^4\omega_s^6}{\pi}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]$$

$$[P_{\rm GW}]_{\rm quad} = \frac{32}{5}G\mu^2 R^4\omega_s^6$$

Inspiral of compact binaries in circular motion

Up to now we have assumed that the orbit is fixed
However, there is a way to account for
the back reaction of the GW on the emitting system in the context of linearised theory

$$E_{\rm orbit} = E_{\rm kin} + E_{\rm pot} = -\frac{G\mu m}{2R}$$

$$v^2 = \frac{Gm}{R}$$

Chirp mass

$$M_c \equiv \mu^{3/5} m^{2/5}$$

$$\omega_s^2 = \frac{Gm}{R^3} \longrightarrow [P_{\rm GW}]_{\rm quad} = \frac{32}{5}\frac{(G\,M_c\,\omega_s)^{10/3}}{G}$$

- The energy of the orbit must diminish because of the GW emission, so R must decrease
- If R decreases, $\omega_{\rm s}$ increases
- If $\omega_{\rm s}$ increases, the emitted power increases as well
- If the emitted power increases, R decreases further
- This runaway process leads to the coalescence of the binary system

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

Inspiral of compact binaries in circular motion

To account for the back reaction of the GW on the emitting system one postulates that
the energy lost by the source per unit time equals the power of the emitted GWs in the radiation zone, far away from the observer

NB! This is not so obvious beyond linear theory

$$-\frac{dE_{\mathrm{orbit}}}{dt} = [P_{\mathrm{GW}}]_{\mathrm{quad}}$$

$$\dot{f} = \frac{96\pi^{8/3}}{5}(G\,M_c)^{5/3} f^{11/3}$$

$$f = \frac{\omega_{\mathrm{GW}}}{2\pi} = \frac{\omega_s}{\pi}$$

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

Time to coalescence $\tau = t_{\mathrm{coal}} - t$

Inspiral of compact binaries in circular motion

To calculate the GW amplitudes, one needs to account for the fact that the orbit depends on time

$$x_0(t) = R(t)\cos(\Phi(t))$$
$$y_0(t) = R(t)\sin(\Phi(t))$$
$$z_0(t) = 0$$

$$\Phi(t) = \Phi_{\rm coal} + \int_{t_c}^{t} dt' \omega_s(t')$$

We remain in the approximation that the orbit is almost circular with slowly varying radius

$$\dot{\omega}_s \ll \omega_s^2 \qquad |\dot{R}| \ll v$$

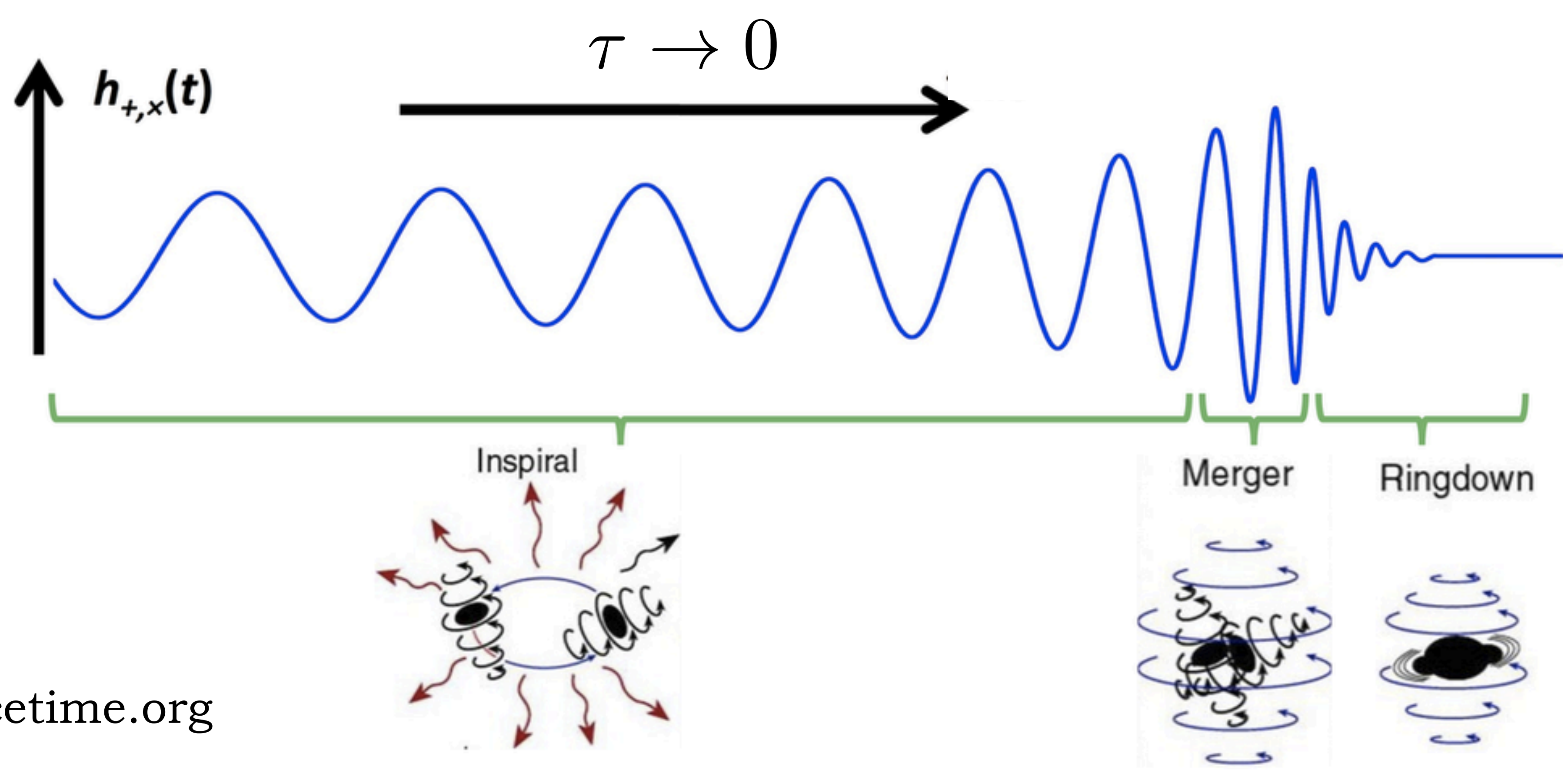
$$h_+(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_{\rm ret}))$$
$$h_\times(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\,\cos\theta\,\sin(2\Phi(t_{\rm ret}))$$

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}} \qquad \Phi(\tau) = -2\left(\frac{\tau}{5\,G\,M_c}\right)^{5/8} + \Phi_{\rm coal}$$

$$h_+(t,\theta,\varphi) = \frac{1}{r}(G\,M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(\tau))$$

$$h_\times(t,\theta,\varphi) = \frac{1}{r}(G\,M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}\cos\theta\,\sin(2\Phi(\tau))$$



soundofspacetime.org

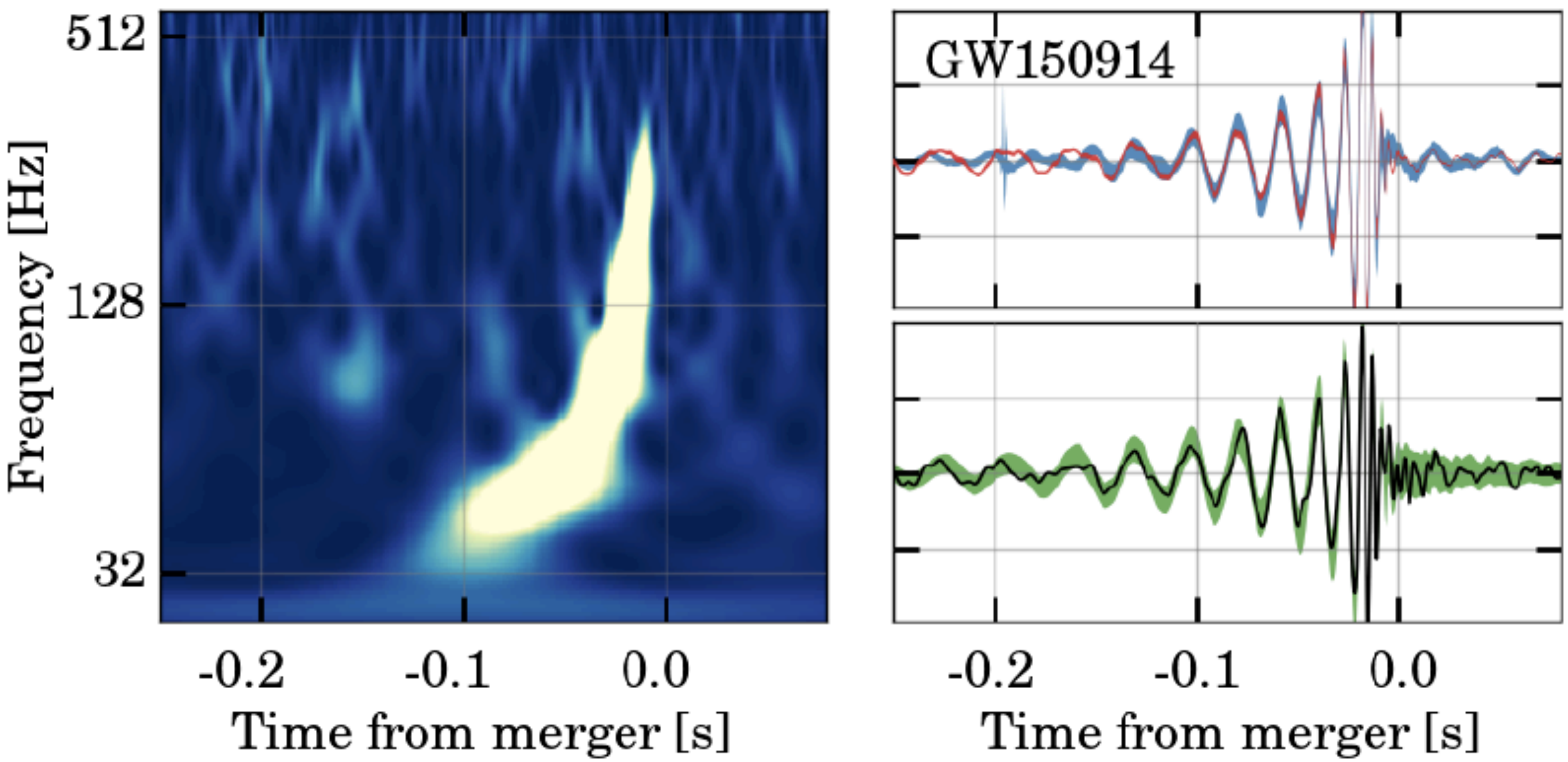
Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 0.2\,\mathrm{sec} \qquad \longrightarrow \qquad f = 37\,\mathrm{Hz}$$

$$M_c = 1.2\,\mathrm{M}_\odot \qquad \tau = 30\,\mathrm{sec} \qquad \longrightarrow \qquad f = 38\,\mathrm{Hz}$$

two examples of detection from Earth-based interferometers

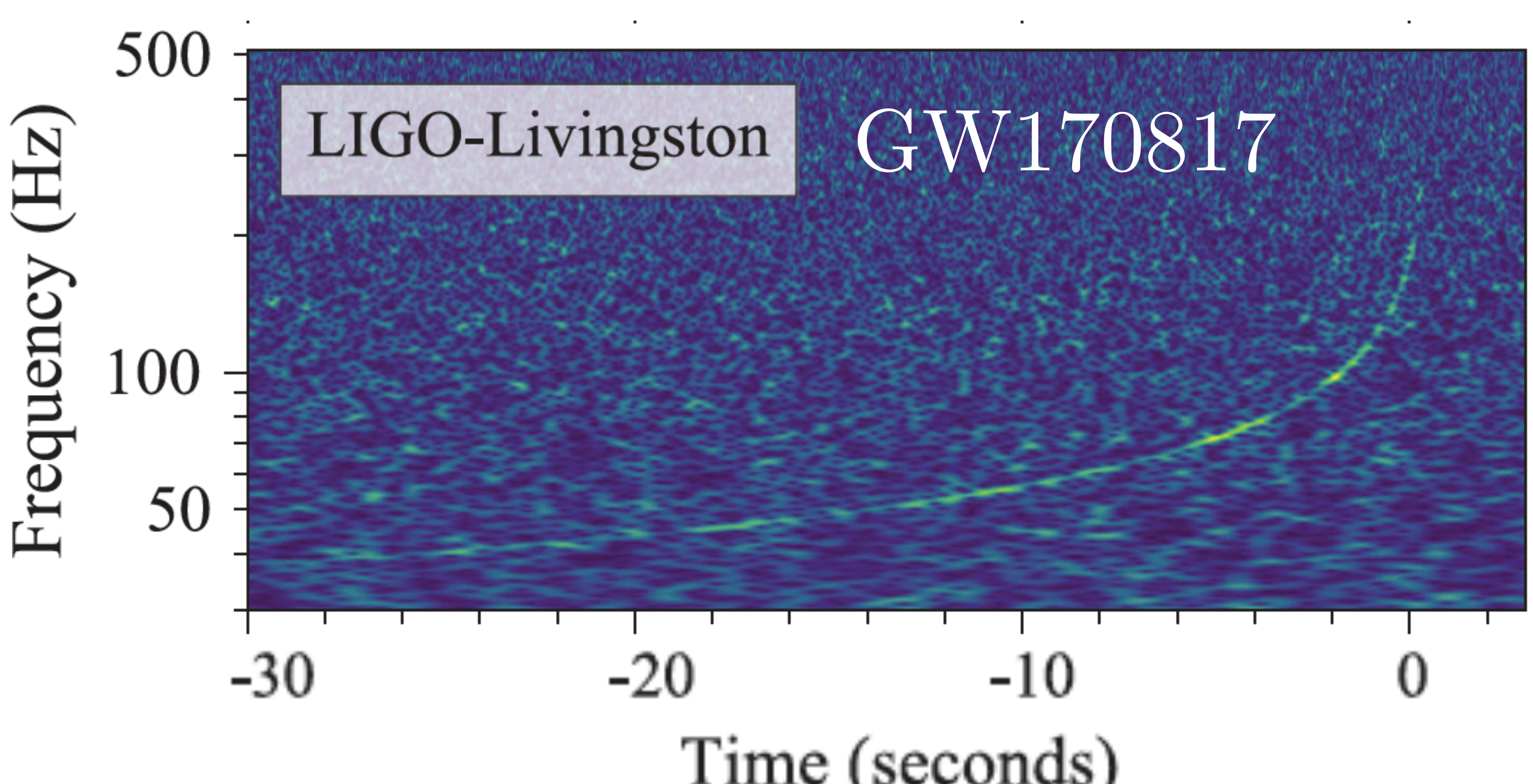


$$m_1 = 35.6^{+4.8}_{-3.0}\, M_\odot$$

$$m_2 = 30.6^{+3.0}_{-4.4}\, M_\odot$$

$$M_c = 28.6^{+1.6}_{-1.5}\, M_\odot$$

$$d_L = 430^{+150}_{-170}\, \mathrm{Mpc}$$



$$m_1 = 1.36 - 1.60\, M_\odot$$

$$m_2 = 1.17 - 1.36\, M_\odot$$

$$M_c = 1.188^{+0.004}_{-0.002}\, M_\odot$$

$$d_L = 40^{+8}_{-14}\, \mathrm{Mpc}$$

LIGO/Virgo arXiv:1811.12907

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

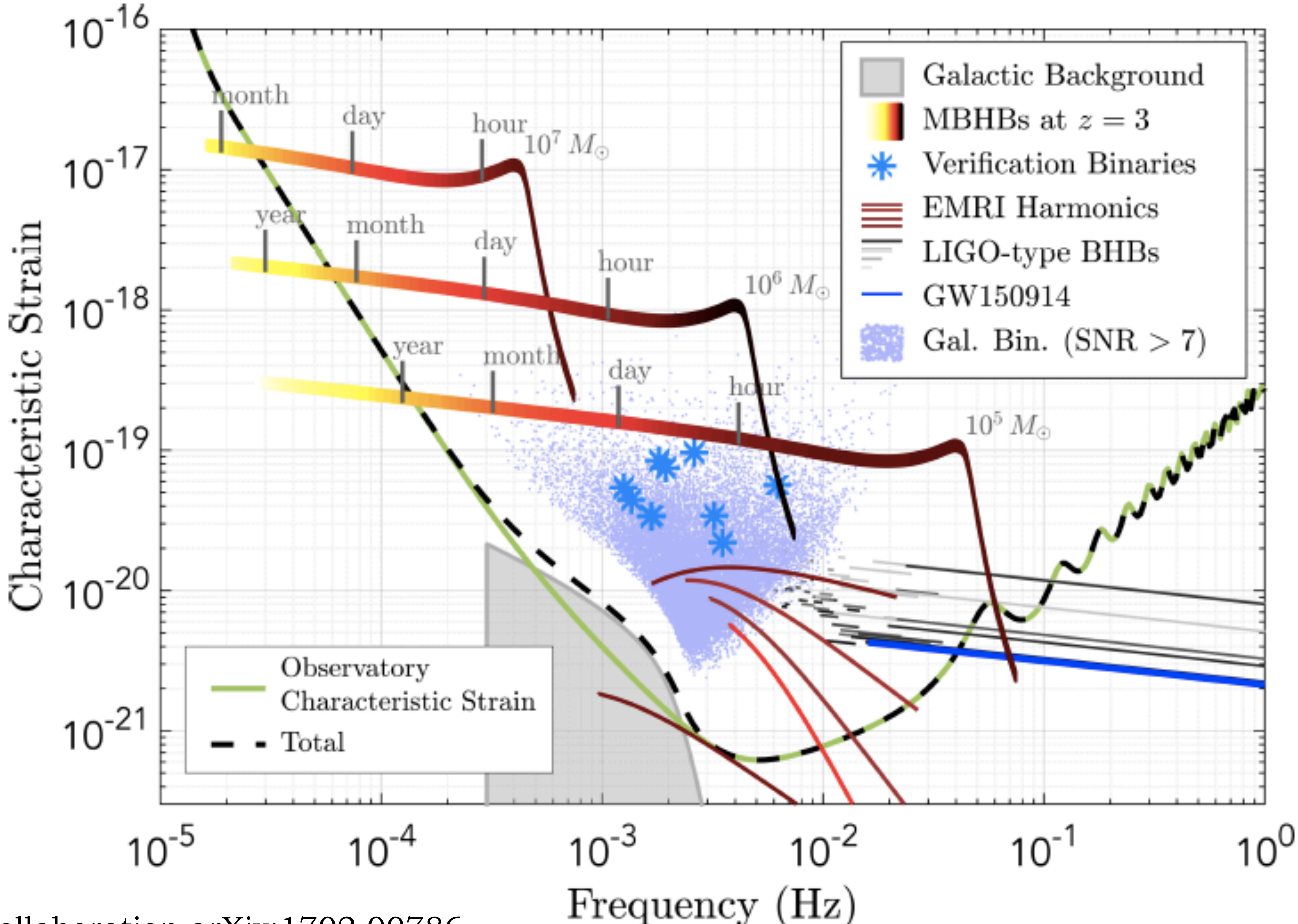
$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 0.2\,\mathrm{sec} \qquad \longrightarrow \qquad f = 37\,\mathrm{Hz}$$

$$M_c = 1.2\,\mathrm{M}_\odot \qquad \tau = 30\,\mathrm{sec} \qquad \longrightarrow \qquad f = 38\,\mathrm{Hz}$$

$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 10\,\mathrm{year} \qquad \longrightarrow \qquad f = 0.01\,\mathrm{Hz}$$

$$M_c = 10^6\,\mathrm{M}_\odot \qquad \tau = 1\,\mathrm{hour} \qquad \longrightarrow \qquad f = 1\,\mathrm{mHz}$$

Space-based interferometers detection targets (LISA)



LISA collaboration arXiv:1702.00786

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 0.2\,\mathrm{sec} \qquad \longrightarrow \qquad f = 37\,\mathrm{Hz}$$

$$M_c = 1.2\,\mathrm{M}_\odot \qquad \tau = 30\,\mathrm{sec} \qquad \longrightarrow \qquad f = 38\,\mathrm{Hz}$$

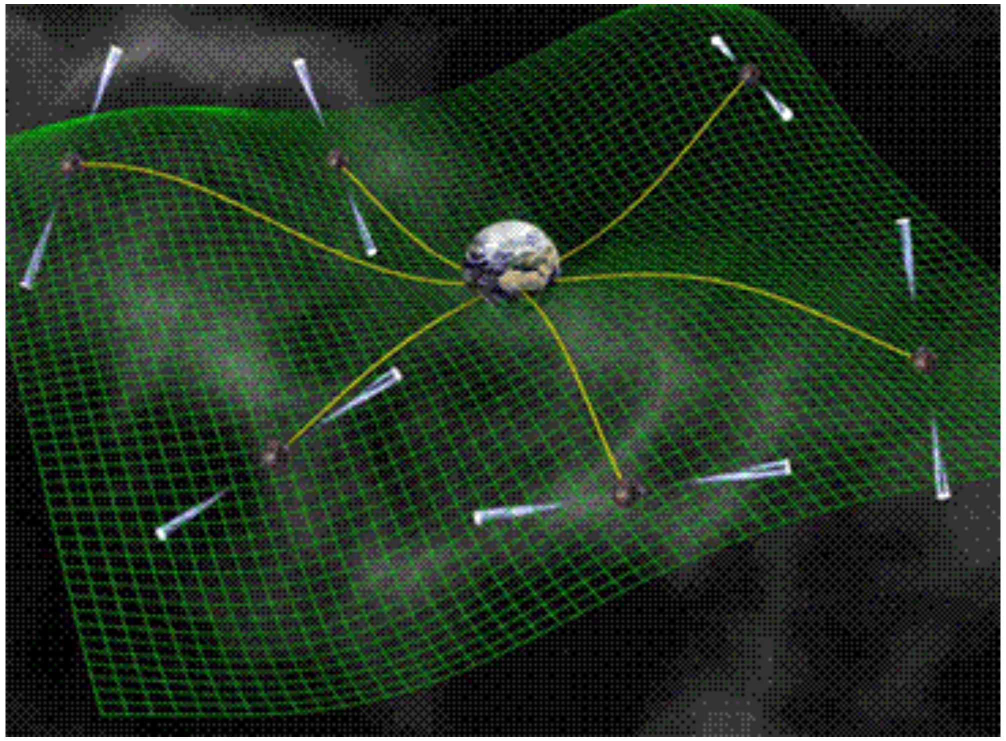
$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 10\,\mathrm{year} \qquad \longrightarrow \qquad f = 0.01\,\mathrm{Hz}$$

$$M_c = 10^6\,\mathrm{M}_\odot \qquad \tau = 1\,\mathrm{hour} \qquad \longrightarrow \qquad f = 1\,\mathrm{mHz}$$

$$M_c = 10^9\,\mathrm{M}_\odot \qquad \tau = 10^5\,\mathrm{year} \qquad \longrightarrow \qquad f = 7 \cdot 10^{-9}\,\mathrm{Hz}$$

Pulsar timing array

EPTA, NANOGrav, PPTA, IPTA

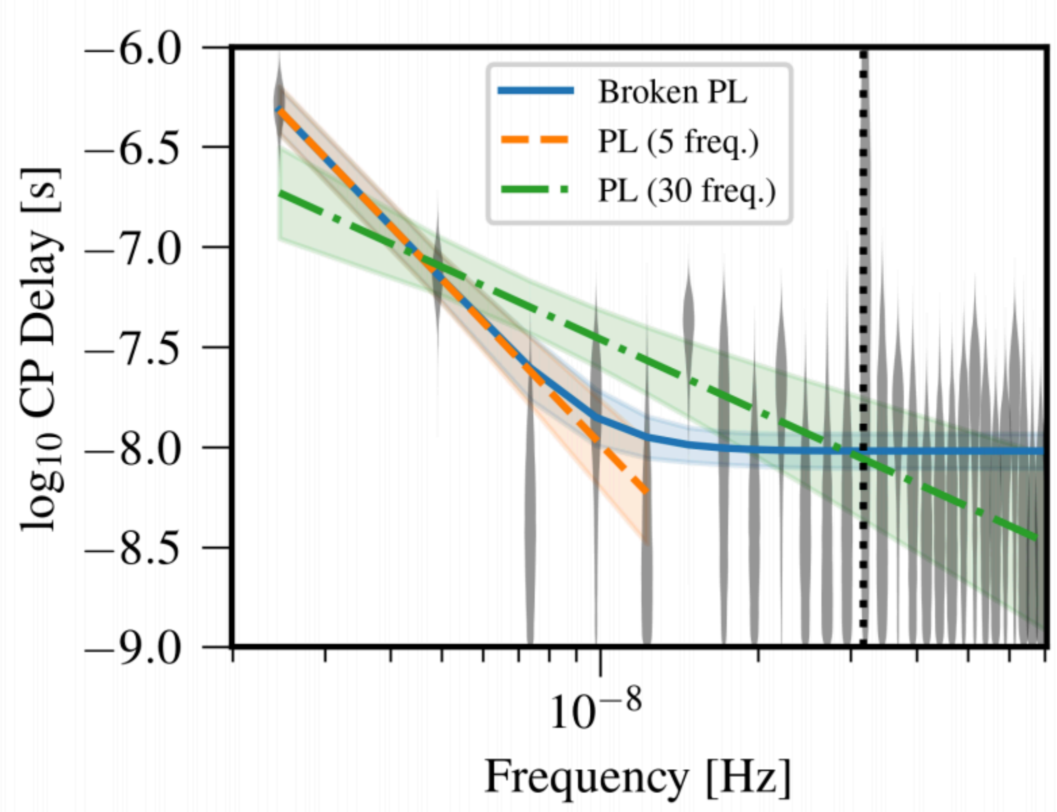


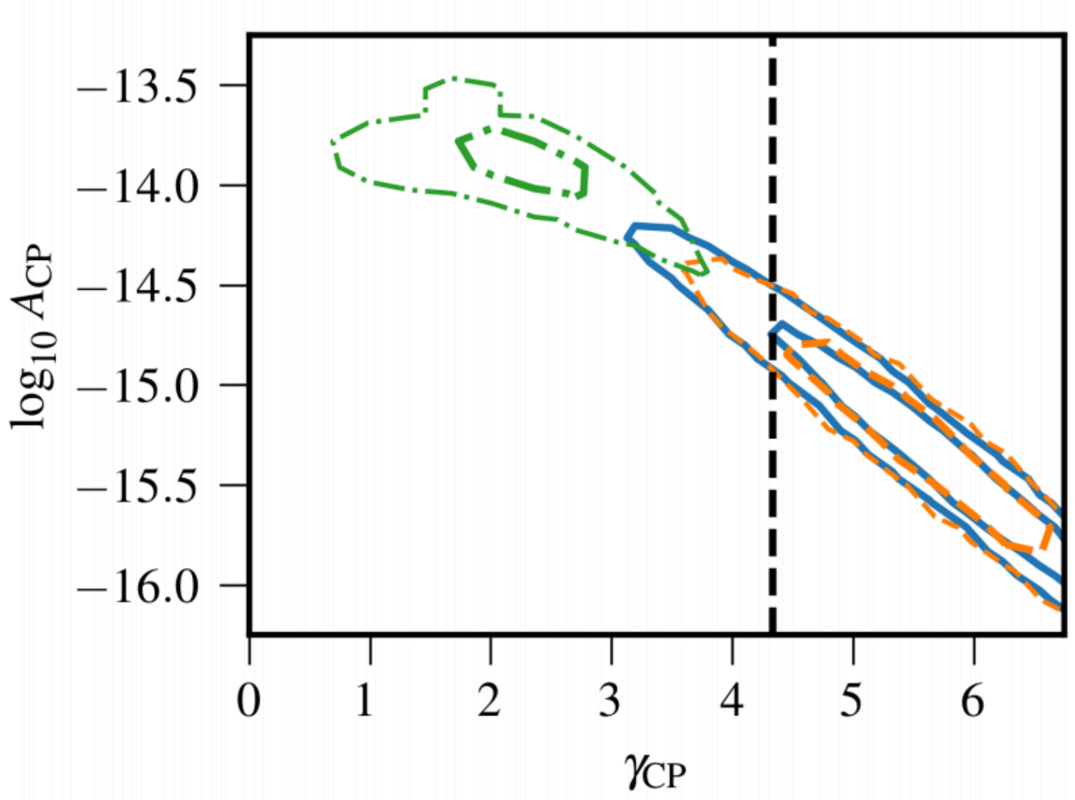
frequency range of detection: 10^{-9} Hz < f < 10^{-7} HZ

DETECTION TARGETS:

stochastic background from inspiralling SMBH binaries
(masses of order 10^9 solar masses)

Recent NANOGrav result! First SGWB detection?





NANOGrav collaboration: arXiv:2009.04496