Theory-Changing Interfaces
and Quantum Algebras

Mykola Dedushenko

Simons Center for Geometry and Physics

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Photographic evidence:
Introduction

The topic of this talk lies at the intersection of several general ideas.

- **QFTs come in families.**
  - Interfaces between different members of the family\(^1\), their OPE:
    \[
    \text{QFT}_1 \parallel \text{QFT}_2 \parallel \text{QFT}_3 \quad \longrightarrow \quad \text{QFT}_1 \parallel \text{QFT}_3
    \]
  - “Symmetries” acting on families. They can act between different QFTs, with the generators realized via interfaces.
    - Extend usual symmetries acting within a theory (via codim-1 defects).
    - SUSY interfaces \(\Rightarrow\) maps in the Q-cohomology.
  - More generally, derived structures, higher operations, etc.

\(^1\)Interface for us is any codimension-1 defect.
Introduction

A few ideas that play role:

- **Wall-crossing.**
  - How spaces of SUSY vacua transform as we vary mass/FI parameters across walls.

- **Mirror symmetry and symplectic duality in 3d $\mathcal{N} = 4$.**
  - Interfaces (walls) between Higgs, Coulomb, mixed, and CFT phases.
Main motivation: Bethe-Gauge correspondence [Nekrasov-Shatashvili]

- Hilbert space on the left $\cong \bigoplus_i$ (SUSY vacua in QFT$_i$) on the right.
- $\cong$ rep of $\mathcal{A}$ (spectrum-generating algebra, $Y_\hbar(g)$, $U_\hbar(Lg)$, $U_\hbar(Eg)$)
- Can we realize the action of $\mathcal{A}$ on the family of QFTs via interfaces?
- The main example of a family of QFTs:

$\rightarrow$ XXX, XXY, or XYZ spin chain (depending on $d$ and $Q$)
Another big motivation comes from geometric constructions:

- Stable envelopes [Maulik-Okounkov’12, Aganagic-Okounkov’16]
  - Geometric building blocks, from which R-matrices can be constructed.
  - Once R-matrix interfaces are known, you know the $\mathcal{A}$-action.

- For $X$ a Nakajima quiver variety, such that the quiver diagram determines an algebra $\mathfrak{g}$:
  - Nakajima constructed the action of $U_h(\mathfrak{g})$ on $\bigoplus_i K_T(X_i)$
  - Varagnolo constructed the action of $Y_h(\mathfrak{g})$ on $\bigoplus_i H_T(X_i)$
  - Both constructions are via Lagrangian correspondences $L \subset X_i \times X_j$, which look like $(B, A, A)$ branes.
Introduction

Some literature:

- [Nakajima’00] and [Varagnolo’00] constructions as precursors.
- Bethe-Gauge correspondence [Nekrasov-Shatashvili’09]. The idea to realize $\mathcal{A}$ via branes, already proposed in [Nekrasov, Strings-2009 talk].
- A big advance: geometric construction of [Maulik-Okounkov’12]; elliptic case in [Aganagic-Okounkov’16]; K-theoretic case e.g. in [Okounkov-Smirnov’16]. Physics construction remained open.
- Theory-changing interfaces played role in [Gaiotto-Moore-Witten’15], who explored structures relevant to $Q_A$ in 2d.
- See [Bullimore-Kim-Lukowski’17] for a discussion on R-matrices in the context of Bethe/Gauge correspondence.
- Connection to half-indices in 3d and factorization [Beem-Dimofte-Pasquetti’12, Gadde-Gukov-Putrov’13, Dimofte-Gaiotto-Paquette’17, Bullimore-Crew-Zhang-Dorey’20, Okazaki’20]
- Some ideas from [MD’18,”Gluing II”] initially played role.
- Use some tools from [Bullimore-Dimofte-Gaiotto-Hilburn’16], as well as earlier [Hori-Iqbal-Vafa’00]. Also relation to [Cecotti-Vafa’10].
Basic setup

Gauge theory with eight supercharges. The flavor group is $G_H$. Fix $A \subset G_H$ – a maximal torus; $U(1)_\hbar$ – R-symmetry that is flavor symmetry for theory with four supercharges. $T = A \times U(1)_\hbar$.

$3d \mathcal{N} = 4$ on $E_\tau \times \mathbb{R}$. Interface is wrapped on the elliptic curve $E_\tau$, acts on the space of ground states $\mathcal{V}[E_\tau] \subset \mathcal{H}[E_\tau]$.

$2d \mathcal{N} = (4, 4)$ on $S^1 \times \mathbb{R}$. Interface on $S^1$ acts on $\mathcal{V}[S^1] \subset \mathcal{H}[S^1]$.

$1d \mathcal{N} = 8$ on $\mathbb{R}$. Interface is a local operator acting on $\mathcal{V}[pt] \subset \mathcal{H}[pt]$.

In 3d: also “Coulomb” $G_C$; the maximal torus $A' \subset G_C$ is given by topological symmetries, whose currents, for every $U(1)$ gauge group, are $J = *F$. 
Basic setup

The structures we study exist in the $Q$-cohomology. Which $Q$?

- 2d $(2, 2)$ supercharges:
  - $\bar{Q}_+ \text{ and } Q_B \rightarrow \text{“holomorphic-topological” } Q \text{ in 3d}^2$
  - $Q_A \rightarrow \text{lifts to “3d A-twist” in 3d}^3$
  - $Q = Q_A + Q_A^\dagger = Q_B + Q_B^\dagger \rightarrow \text{the Omega-deformation } Q$

$$Q^2 = 2D\bar{z} \quad \text{along } E_\tau.$$

Operators in the $Q$-cohomology are interfaces on $E_\tau$.

One more description: $Q$ is a 3d $\mathcal{N} = 1$ supercharge.

Today: realize stable envelopes as such interfaces.
Basic setup

Equivariant parameters

\[ x \in \text{Hom}(\pi_1(E_\tau), \mathbb{H})/\mathbb{H}; \quad \hbar \in \text{Hom}(\pi_1(E_\tau), U(1)\hbar); \]
\[ z \in \text{Hom}(\pi_1(E_\tau), G_C)/G_C \]

Kähler parameters

Turn on flat connections on $E_\tau$:

In 2d: $z$ gets replaced by the $\theta$-angles
In 1d: $z$ completely disappears.
Focus on the Higgs phase, $X = \text{Higgs branch}$. It is well-known that:

- $\mathcal{V}[\text{pt}] \cong H_T(X)$. [Witten’82]
- Similarly, one can identify $\mathcal{V}[S^1]$ with $K_T(X)$.
- $\mathcal{V}[E_\tau]$ is related to the equivariant elliptic cohomology $\text{Ell}_T(X)$.

$\text{Ell}_T(X)$ is a scheme; an elliptic generalization of $\text{Spec } H_T(X)$ and $\text{Spec } K_T(X)$. However, it is not affine, i.e., not a Spec of anything ⇒ should study bundles on $\text{Ell}_T(X)$, and $\mathcal{V}[E_\tau]$ are sections of a “bundle of vacua”. No time for this story.

We will focus on the cohomological case in the following. Elliptic generalization (3d lift) comes with two new phenomena: Kähler parameters and boundary anomalies.
Stable envelopes

Let $\mathbf{A}$ be a torus of flavor group, and $X^A \subset X$ a set of $A$-fixed points.

To $p \in X^A$, attach its full $A$-attractor $Attr_p$ (with “broken” trajectories):

There exists a natural map:

$$\text{Stab} : H_T(X^A) \to H_T(X),$$

which extends cohomology classes from fixed locus along the full attractor. [Maulik-Okounkov’12]

(Similarly in the K-theoretic case, while in the elliptic case, one extends sections of line bundles on $\text{Ell}_T$)
Janus interface

$X^A$ can be identified with the Higgs branch of the theory with large generic real masses for the flavor symmetry $A$ turned on.

It is possible to vary real masses in the $y \in \mathbb{R}$ direction while preserving half of SUSY.\(^4\) In particular, we can have:

\[ \text{Proposal: such an interface (call it mass Janus) gives a physics realization of the map Stab.} \]

\(^4\)This requires an extra term $-\phi m'(y) \phi$ in the Euclidean action.
Janus interface

The reason: BPS equations for $Q$ include $A_C$ flows parametrized by $y$.

$$(D_y + \sigma + m(y))\phi = 0, \quad D_y\sigma = \mu_R.$$

This is a gradient flow for the function

$$\bar{\phi}(m(y) + \sigma)\phi = m(y) \cdot \mu^f_R + \sigma \cdot \mu^g_R$$

On the Higgs branch, it restricts to the Morse function

$$f = \bar{\phi}m(y)\phi.$$

For theories with eight supercharges, all critical points of this function (if isolated) have indices equal to half the target dimension.

Remark: Such gradient trajectories do not contribute to the differential of the MSW complex, i.e., critical points give the exact vacua.
So, we need to consider SQM as in [Witten’82] (NLSM into the Higgs branch + the Morse function $f$ representing the effect of masses.\footnote{In practice, all our computations are done in gauge theory.})

But with time-dependent Morse function.

With time-independent $f$, the action is $Q$-exact, up to a “topological term”, $S = \{Q, \ldots\} - df$.

We still want to use this action when $f$ is time-dependent $\Rightarrow$ need to include $-\frac{\partial f}{\partial y}$ in the action.\footnote{This is the term $-\bar{\phi} m'(y) \phi$ we added earlier.}

Variations $\delta f$ that vanish at $y \to \pm \infty$ correspond to $Q$-exact deformations.
Tension between SUSY and unitarity

This modifies the standard formulas:

\[ Q = d + df, \quad Q^\dagger = d^* + i \nabla f, \quad H = \frac{1}{2} \{ Q, Q^\dagger \} - i \frac{\partial f}{\partial t}, \quad i \{ H, Q \} + \frac{\partial Q}{\partial t} = 0. \]

Unitary, no SUSY

Without \( \frac{\partial f}{\partial t} \) in \( H \), the evolution is unitary, but SUSY is broken if \( \frac{\partial f}{\partial t} \neq 0 \).

Non-unitary, SUSY preserved

With \( \frac{\partial f}{\partial t} \), \( Q \) is conserved, but the evolution is non-unitary if \( \frac{\partial f}{\partial t} \neq 0 \).

We want a \( Q \)-closed interface between theories with different \( f \)'s. There is no reason for the corresponding operator to be unitary.

Hence, choose option 2. Have to be very careful: make sure the operator is well-defined!
Conjugate by $e^f$: $Q = e^f Q e^{-f} = d$, $G = e^f Q^\dagger e^{-f} = d^* + 2i \nabla f$,

$H = e^f H e^{-f} + i \frac{\partial f}{\partial t}$, so that $H = \frac{1}{2} \{Q, G\}$.

(This corresponds to dropping the topological term, and using the Q-exact action $S = \delta(...)$ $= \frac{1}{2}(\dot{x} + \nabla f)^2 + ...$)

$H$ is $d$-exact, $\Rightarrow$ naively, evolution is trivial in the de Rham cohomology. This logic is correct if the target manifold is compact.

The story gets much richer if the target is non-compact.

Non-unitary evolution can make an $L^2$ function unnormalizable. Yet, matrix elements between $L^2$ functions are well-defined.
Toy example

Let the target be $\mathbb{C}$, with $f = \frac{m}{2}|z|^2$, where $m \in \mathbb{R}$ can have either sign. Two states are in the kernel of $H$:

$L^2$ only if $m > 0$
\[ \psi_0 = e^{-f}, \quad \psi_2 = e^{f} dz \wedge d\bar{z}. \text{(}\psi_0\text{ is a solution even for time-dependent } f!\text{)} \]

$L^2$ only if $m < 0$

Consider $m(t)$, s.t. $m(-\infty) > 0$ and $m(+\infty) = 0$. Start with $\psi_0 = e^{-f}$ in the past. It evolves into $e^{-f}\big|_{t\to+\infty} = 1$, which is not $L^2$. Are we in trouble? Equivariance saves the day. Replace $d$ by $D = d + \iota_\epsilon \partial_\phi$.

\[ \mathcal{Q} = d + \iota_\epsilon V_\epsilon, \quad \mathcal{G} = d^* + V_\epsilon^b \wedge + 2\iota_\nabla f, \quad V_\epsilon = \epsilon \partial_\phi. \]
Define $\omega^2 = m^2 + |\varepsilon|^2$. For constant $m$ (including $m = 0$), the normalizable ground state is

$$\Psi^{(m)} = e^{-f} \Omega^{(m)}, \quad \Omega^{(m)} = \frac{\varepsilon}{\sqrt{2\pi(\omega - m)}} e^{-\frac{1}{2}(\omega - m)(x^2 + y^2) - \omega - m} dx \wedge dy,$$

where $z = x + iy$. To compute the transition amplitude, use: (1) shape independence of $m(y)$ to assume

$$m(t) = m \Theta(-t);$$

(2) continuity of $\Omega = e^f \Psi$ across the jump. This results in

$$\langle \Omega^{(0)} | \Omega^{(m)} \rangle = \int \ast \bar{\Omega}^{(0)} \wedge \Omega^{(m)} = \sqrt{\frac{|\varepsilon|}{\omega - m}}$$
We can compute all possible transitions:

- **[0| + m]** \(\rightarrow \langle \Omega^{(0)}|\Omega^{(m)} \rangle = \sqrt{\frac{|\epsilon|}{\omega - m}} = S_m\)
- **[+m|0]** \(\rightarrow \langle \Omega^{(m)}|e^{-m|z|^2}|\Omega^{(0)} \rangle = \sqrt{\frac{\omega - m}{|\epsilon|}} = S_m^{-1}\)
- **[0| − m]** \(\rightarrow \langle \Omega^{(0)}|\Omega^{(-m)} \rangle = \sqrt{\frac{\omega - m}{|\epsilon|}} = S_{-m}\)
- **[−m|0]** \(\rightarrow \langle \Omega^{(-m)}|e^{m|z|^2}|\Omega^{(0)} \rangle = \sqrt{\frac{|\epsilon|}{\omega - m}} = S_{-m}^{-1}\)
- **[−m|m]** \(\rightarrow \langle \Omega^{(-m)}|e^{m|z|^2}|\Omega^{(m)} \rangle = \frac{|\epsilon|}{\omega - m} = R_m\)

Here \(S_{\pm m}\) is the analog of \(\text{Stab}_{\pm m}\), and \(R_m\) is the analog of R-matrix. The relation:

\[ R_m = S_{-m}^{-1} \circ S_m \]

is exactly how the R-matrix is built from stable envelopes.

This confirms our expectations, e.g., independence on the shape of \(m(t)\).
In general, the region with masses prepares an equivariant form, which (in the limit of large masses and metric) looks like a delta-form supported on the attractor of a fixed point.

We then compute its overlap with a “probe” equivariant form representing some vacuum in the massless region.
Less simplified view

- What we actually do:
  - Like in the toy example, but the target has many fixed points, $S_m$ are non-trivial upper-triangular matrices.
  - In practice, all computations are done in gauge theory.
  - ... Via SUSY localization.
  - Which is also why the (Euclidean) time $\mathbb{R}$ is replaced by an interval.
  - And the choice of vacua at $t \to \pm \infty$ is represented by special boundary conditions. (Thimble boundary conditions [Hori-Iqbal-Vafa’10, Gaiotto-Moore-Witten’15, Bullimore-Dimofte-Gaiotto-Hilburn’16])

(It is problematic to localize on a non-compact spacetime. Interval is more straightforward, but still can be a challenge, depending on the boundary conditions.)
Key ideas:

- Theory on the left: our gauge theory $T$.
- Theory on the right: turn on large real masses $m \in \mathcal{C}$, integrate out fields that are massive in vacuum $\beta$, the remaining gauge theory is $T^m$.
- Boundary conditions $B_m$ on fields that “terminate” at the middle are naturally induced by the SUSY jump of masses.
- On the left: boundary condition corresponding to vacuum $\alpha$ realized via Dirichlet b.c. for gauge fields (exceptional Dirichlet).
- On the right: vacuum $\beta$ realized via Neumann boundary conditions for the gauge fields.
- The $\mathcal{N}_\beta$ boundary contains extra matter to cancel anomalies in 3d case.
Interval index

In \( d \geq 2 \) spacetime dimensions, we can regard the interval direction as space, and one of the circles as time.

Then the answer can be more conventionally interpreted as the index in a certain soliton sector.

But in 1d, the interval direction can only be time.
R-matrices

R-matrices are realized as $\text{Stab}^{-1}_{m_2} \circ \text{Stab}_{m_1}$, i.e., instead of changing mass from $m$ to 0, we change it from $m_1 \in \mathcal{C}_1$ to $m_2 \in \mathcal{C}_2$ (different chambers).

If there is only one mass, and the chambers are $m > 0$, $m < 0$, then the R-matrix is realized by

Let me explain how to construct an interface realizing a raising operator of the $\mathfrak{sl}_2$ Yangian. We want an interface between theories:

$$ \begin{align*}
\begin{array}{c}
\begin{array}{c}
L \quad U(N) \quad L
\end{array}
\end{array}
\end{align*} $$
R-matrix and Higgsing

Inspired by [Maulik-Okounkov’12], consider a larger theory $T$:

$$L + 1 \quad \overset{}{\longrightarrow} \quad U(N + 1).$$

It has an extra $U(1)$ flavor symmetry that rotates the added hyper. Let $m$ be the real mass for it. Then at $m \to \infty$, the theory decomposes into

$$\begin{bmatrix} L \quad U(N + 1) \end{bmatrix}_A \bigoplus \begin{bmatrix} \left( \begin{array}{c} L \quad U(N) \end{array} \right) \otimes \left( \begin{array}{c} 1 \quad U(1) \end{array} \right) \end{bmatrix}_B$$

In sector $B$, $U(N + 1)$ is broken to $U(N) \times U(1)$ via Higgsing.
Consider an R-matrix $R_m$ constructed as above via changing mass from $m \gg 0$ to $m \ll 0$. It has a block form:

$$R_m(u) = \begin{bmatrix} A & B \\ A^* & B^* \end{bmatrix}$$

The AB and BA blocks represent interfaces that change the gauge group (we can “forget” the $1 - U(1)$ factor). They provide realization of the basic raising and lowering operators in the Yangian $Y_{\hbar}(\mathfrak{sl}_2)$ (in the 1d case).

**Generalizations are clear...**
Further comments

- Didn’t have time to talk about:
  - The elliptic case in details.
  - Details of interval computations.
  - Construction of general R-matrices for general quiver varieties.
  - Janus for FI parameters.
  - Half-index. Can stretch the 3d index and half-index, proving that the squashing parameter $b$ is a trivial deformation of the THF using techniques of [Closset-Dumitrescu-Festuccia-Komargodski’13].
  - It connects holomorphic blocks [B-D-P’12] to half-indices [G-G-P’13,D-G-P’17].
  - Our interfaces describe wall-crossing of the half-index. Acting with an interface, one can transport half-index between chambers, or from the Higgs to the Coulomb phase.
  - Brane constructions of our systems via Type IIA on the ALE spaces. Dualities: relation between supercharges and also to 4d CS approach.
What else?

For the future:

- We construct interfaces “up to quasi-isomorphism” ⇒ can we study derived structure, higher operations?
- Generalization to fewer supercharges?
- $Q$ in d-dim can be lifted to $Q_A$ in $d + 1$. Analogs of our constructions in quantum cohomology theories?
- Lifting $Q$ in 1d to $Q_A$ in 2d, explore connections to [Gaiotto-Moore-Witten’15]?
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Thank you!

Questions?