Ten dimensional symmetry of $\mathcal{N} = 4$ SYM correlators

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A triality in planar $\mathcal{N} = 4$ SYM

Correlation function of protected Dimension-2 operators

$(x_i - x_{i+1})^2 \rightarrow 0$

4D null limit

$(Square of) massless amplitude$

$(Square of) null Wilson loop$

$\nu_i \equiv x_i - x_{i+1}$

(T-duality in AdS)

[Eden, Korchemsky, Sokatchev; 2010]

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[Almsey, Eden, Korchemsky, Maldacena, Sokatchev; 2010]

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Generalization for massive amplitude

• Massive amplitude in the Coulomb branch (turned on VEVs for scalar fields).

• Amplitude (integrand) has a higher dimensional symmetry that acts on the vector $(p_i, m_i)$.
  [Alday, Henn, Plefka, Schuster; Caron-Huot, O’Connell; Bern, Carrasco, Dennen, Huang, Ita]
Generalization for massive amplitude

? 

- Need object with higher-dimensional structure
- Candidates: BPS operators dual to KK modes in $AdS_5 \times S_5$

$$\mathcal{O}_k(x, y) = \frac{1}{k} \text{Tr} \left( y \cdot \Phi(x) \right)^k$$

- Massive amplitude in the Coulomb branch (turned on VEVs for scalar fields).
- Amplitude (integrand) has a higher dimensional symmetry that acts on the vector $(p_1, m_1)$.

In SUGRA a 10D symmetry emerges when summing all (four-point) correlators of $\mathcal{O}_k(x, y)$

This talk: similar 10D structure in a different coupling regime.

[Caron-Huot, Trinh, 2018; Aprile, Drummond, Heslop, Paul]
Generalization of correlator/massive amplitude

- The four-point function of the “master operator” $O(x, y) \equiv \sum_{k} O_k(x, y)$ has an emergent 10-dimensional structure that combines spacetime and R-charge distances:

$$X^2_{i,i+1} \equiv (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \quad \text{duality} \quad \equiv \quad p_i^2 + m_i^2$$

- The 10D null limit of the “master” correlator is equal to a massive amplitude in the Coulomb branch.

Generating function of all four-point correlators

(Square of) four-point massive amplitude

- Checked at various loop orders.
Outline

• Ten dimensional structure of free correlators.

• 10D symmetry of loop integrands.

• 10D null limit:
  massive amplitude = large R-charge correlator (octagon).

• Amplitude/octagon from integrability and massless limit.
Free correlators

- Computed by Wick contractions:
  \[
  \langle \mathcal{O}_k(x_i, y_i) \mathcal{O}_k(x_j, y_j) \rangle = \frac{1}{k} \left( \frac{-y_{ij}^2}{x_{ij}^2} \right)^k + O(1/N_c^2)
  \]
  \(x_{ij}^2 \equiv (x_i - x_j)^2, y_{ij}^2 \equiv (y_i - y_j)^2\)

- The free four-point correlator of the “master operator” \( O(x, y) = \sum_k \mathcal{O}_k(x, y) \)

\[ G^{\text{free}} \equiv \langle \mathcal{O}(x_1, y_1) \mathcal{O}(x_2, y_2) \mathcal{O}(x_3, y_3) \mathcal{O}(x_4, y_4) \rangle^{(0)} = \sum_{l_{ij}} C_{(l_{ij})} \prod_{1 \leq i < j \leq 4} \left( \frac{-y_{ij}^2}{x_{ij}^2} \right)^{l_{ij}} \]

\[ G^{\text{free}} = D_{12}D_{23}D_{34}D_{41} + D_{12}D_{23}D_{34}D_{41}(2D_{13} + D_{23}^2) + 2D_{12}D_{13}D_{14}D_{23}D_{24}D_{34} + \text{perm.} \]

- Emergente 10D structure:
  \[ D_{ij} \equiv \frac{-y_{ij}^2}{x_{ij}^2 + y_{ij}^2} = \sum_{k=1}^{\infty} \left( \frac{-y_{ij}^2}{x_{ij}^2} \right)^k \]
Loop integrands

- Perturbative series in the 't Hooft coupling, \( g^2 = \frac{g_{YM}^2 N_c}{16\pi^2} \)

\[
\langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \mathcal{O}_{k_3} \mathcal{O}_{k_4} \rangle_c = G_{k_1 k_2 k_3 k_4}^{\text{free}} + \sum_{\ell=1}^{\infty} G_{k_1 k_2 k_3 k_4}^{(\ell)} + O(1/N_c^2)
\]

- We can define an integrand by the Lagrangian insertion method:

\[
G_{k_1 k_2 k_3 k_4}^{(\ell)} = \frac{(-g^2)^\ell}{\ell!} \int \frac{d^4 x_5}{\pi^2} \cdots \frac{d^4 x_{4+\ell}}{\pi^2} G_{k_1 k_2 k_3 k_4}^{(\ell)} ,
\]

- The \( \ell \)-loop integrand is a \((4 + \ell)\)-point correlator evaluated at leading order:

\[
G_{k_1 k_2 k_3 k_4}^{(\ell)} = \langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \mathcal{O}_{k_3} \mathcal{O}_{k_4} \mathcal{L}(x_5) \cdots \mathcal{L}(x_{4+\ell}) \rangle^{(0)}
\]

\[
\text{SUSY} \quad R_{1234} = \frac{(y_{12}^2 y_{24}^2)}{x_{13}^2 x_{24}^2} + \frac{y_{12}^2 y_{23}^2 y_{24}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} (x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2)
\]

\[
+ (1 \leftrightarrow 2) + (1 \leftrightarrow 4) .
\]

- Advantage: integrand is a rational function with simple poles. It treats external and integration points almost in the same footing (e.g. \( \mathcal{H}_{2222} \) has a full permutation symmetry).

[Eden, Petkou, Schubert, Sokatchev]
• Decomposition in R-charge:

\[ \mathcal{H}_{k_1 k_2 k_3 k_4}^{(l)} = \sum_{k_i - 2 = \sum_j b_{ij}} \mathcal{F}_{\{b_{ij}\}}^{(l)}(x_{ij}^2) \times \prod_{1 \leq i < j \leq 4} \left( \frac{-y_{ij}^2}{x_{ij}^2} \right)^{b_{ij}} \]

The number of inequivalent structures \( \mathcal{F}_{\{b_{ij}\}}^{(l)} \) is finite and depends on the loop order.

• Saturation: thanks to planarity, a bridge becomes uncrossable when the number of propagators is larger than the loop order.

After saturation, we have an infinite tail forming a geometric series.

\[ \mathcal{F}_{\{b_{12}, \ldots\}}^{(l)} \equiv \mathcal{F}_{\{l-1, \ldots\}}^{(l)} \]

if \( b_{12} \geq l - 1 \).

[Chicherin, Drummond, Heslop, Sokatchev, 2015]
One-loop integrands

• At one loop, saturation implies that all R-charge structures are identical:

\[ \mathcal{F}_{\{b_{ij}\}}^{(1)} = \mathcal{F}_{\{0,0,0,0,0\}}^{(1)} \]

• The reduced integrands:

\[
\mathcal{H}_{2222}^{(1)} = \mathcal{F}_{\{0,0,0,0,0\}}^{(1)} = \frac{1}{\prod_{1 \leq i < j \leq 5} x_{ij}^2} 
\]

\[
\mathcal{H}_{k_1k_2k_3k_4}^{(1)} = \sum_{\{b_{ij}\}} \frac{1}{\prod_{1 \leq i < j \leq 5} x_{ij}^2} \times \prod_{1 \leq i < j \leq 4} \left( \frac{-y_{ij}^2}{x_{ij}^2} \right)^{b_{ij}} 
\]

• Resumming the geometric series:

\[
\mathcal{H}^{(1)} = \sum_{k_i \geq 2} \mathcal{H}_{k_1k_2k_3k_4}^{(1)} = \frac{1}{\prod_{1 \leq i < j \leq 5} (x_{ij}^2 + y_{ij}^2)} \quad \text{with } y_{5i}^2 = 0 
\]

• Higher-loop data shows similar pattern.
10D symmetry of loop integrands

- At each loop order, all (reduced) integrands form a geometric series that resums into a function which depends only on $X_{ij}^2 \equiv x_{ij}^2 + y_{ij}^2$.

\[
\mathcal{H}_{k_1k_2k_3k_4}^{(\ell)}(x_{ij}^2, y_{ij}^2) = \text{coefficient of } \left( \prod_{i=1}^{4} \beta_i^{k_i-2} \right) \text{ in } \mathcal{H}^{(\ell)}(X_{ij}^2) \bigg|_{y_{ij}^2 \rightarrow \beta_i \beta_j y_{ij}^2},
\]

- This generating function can be uplifted from the known case $\mathcal{H}_{2222}$ by replacing all four-dimensional distances $x_{ij}^2$ by ten-dimensional ones $X_{ij}^2$.

- It inherits the full permutation symmetry of $\mathcal{H}_{2222}$.

The dimension-2 operator and the chiral Lagrangian belong to the stress-tensor super-multiplet.

\[\mathcal{H}^{(1)} = \frac{1}{\prod_{1 \leq i < j \leq 5} X_{ij}^2},\]

\[\mathcal{H}^{(2)} = \frac{1}{48} \frac{X_{12}^2 X_{34}^2 X_{56}^2 + S_6 \text{ permutations}}{\prod_{1 \leq i < j \leq 6} X_{ij}^2},\]

\[\mathcal{H}^{(3)} = \frac{1}{20} \frac{(X_{12}^2)^2 (X_{34}^2 X_{45}^2 X_{56}^2 X_{67}^2 X_{73}^2) + S_7 \text{ permutations}}{\prod_{1 \leq i < j \leq 7} X_{ij}^2} .\]
10D structure of four-point correlators

- We set the 6D null condition for the external points and turn off the R-charge of the internal points.

\[ y_i \cdot y_i = 0 \quad \text{when } i = 1, 2, 3, 4 \quad \text{and} \quad y_i = 0 \quad \text{when } i = 5, \ldots, 4 + \ell \]

and integrate:

\[ G^{(\ell)} = \sum_{k_i \geq 2} G_{k_1 k_2 k_3 k_4}^{(\ell)} \left( \frac{-g^2}{\ell!} R_{1234} \left( 2 x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \right) \int \frac{dx_5^4}{\pi^2} \cdots \frac{dx_{4+\ell}^4}{\pi^2} \mathcal{H}^{(\ell)} \right) \]

- One and two-loop examples:

\[ G^{(1)} = -2g^2 R_{1234} g_{1234} \prod_{1 \leq i < j \leq 4} \frac{1}{1 - d_{ij}} \]

\[ G^{(2)} = 2g^4 R_{1234} \left( c_h^1 h_{12;34} + c_h^2 h_{13;24} + c_h^3 h_{14;23} + \frac{1}{2} (c_{gg}^1 x_{12}^2 x_{34}^2 + c_{gg}^2 x_{13}^2 x_{24}^2 + c_{gg}^3 x_{14}^2 x_{23}^2) \right) \]

\[ c_h^i = \left( 1 - d_{12} \right) \prod_{1 \leq i < j \leq 4} \left( 1 - d_{ij} \right) \quad \text{and} \quad c_{gg}^i = \left( 1 - d_{12} \right) \frac{1}{2} \prod_{1 \leq i < j \leq 4} \left( 1 - d_{ij} \right) \quad \text{with} \quad d_{ij} = \frac{-y_{ij}^2}{x_{ij}^2}. \]

\[ g_{1234} = \frac{1}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \quad \text{and} \quad h_{13;24} = \frac{x_{24}^2}{\pi^4} \int \frac{d^4 x_5 d^4 x_6}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2)(x_{56}^2 x_{26}^2 x_{36}^2 x_{46}^2)}. \]

- Similar checks up to 5 loops. [Chicherin, Georgoudis, Goncalves, Pereira, 2018] [Chicherin, Drummond, Heslop, Sokatchev, 2015]

- Predictions at higher loops (up to ten loops from knowledge of the seed \( \mathcal{H}_{2222} \)). [Bourjaily, Heslop, Tran, 2016]

- Higher loop integrals are hard to evaluate.

10D null limit: octagon = amplitude

- The simplest correlators factorized into squares (octagons).

- Their integrands receive contributions only from \( \mathcal{F}\{a, \infty, \infty, \infty, \infty, b\} \)

- Only the terms of “G” with the four poles contribute to the simplest correlators. We can select them by taking the 10D null limit:

\[
\lim_{x_{i,i+1}^2 \to 0} G = \sum_{a,b,\ell} \mathcal{L}_{i+\ell} = \sum_{\ell} \mathcal{L}_{\ell} = 0 \times 0 = M \times M
\]

- Octagon and amplitude are identical at the integrand and integrated level. They are IR finite.

\[
O_1 = \text{Tr}(\bar{X}^{2K+a})
\]
\[
O_2 = \text{Tr}(X^K \bar{Y}^b) + \text{cyclic permutations}
\]
\[
O_3 = \text{Tr}(Z^{2K} X^a) + \text{cyclic permutations}
\]
\[
O_4 = \text{Tr}(Z^K X^K Y^b) + \text{cyclic permutations}
\]

\[
\begin{bmatrix}
 p_i^\mu & \equiv & x_{i,i+1}^\mu & \text{and} & m_i^2 & \equiv & y_{i,i+1}^2
\end{bmatrix}
\]
Octagon from Integrability

$$\mathcal{O} = \mathcal{O}_0 + \sum_{l=1}^{\infty} (d_{13})^l \mathcal{O}_l + (d_{24})^l \mathcal{O}_l$$

Gluing two hexagons by summing over mirror particles:

$$\mathcal{O}_l(z, \bar{z}, d_{13}, d_{24}) = \lim_{X_{i,i+1} \to 0} \sum_{\psi}$$

Octagon is given by an infinite determinant

$$\mathcal{O}_l = \det(1 - K_l)$$

$$\chi(\tau) = \frac{(1 - d_{13}d_{24})}{\sqrt{z\bar{z}(1-z)(1-\bar{z})}} \frac{1}{\cosh(\sqrt{\zeta^2 + \tau^2}) - \cos \phi}, \text{ with } e^{-2\zeta} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, e^{2i\phi} = \frac{z(1-\bar{z})}{\bar{z}(1-z)}.$$
Steinmann condition for amplitudes:

Double discontinuities vanish in overlapping channels

Octagon has a vanishing double discontinuity

\[ \text{disc}_s \text{disc}_t \mathcal{O} = 0 \]

with \( s = x_{13}^2, t = x_{24}^2 \)

Weak coupling:

\[ \mathcal{O} = \text{sum of det (ladder)} \]

Provides analytic results for (unknown) conformal integrals such as fishnets and deformations

Strong coupling:

\[ \mathcal{O} = e^{-g \text{Area}} \]

[Alhay, Maldacena, 2007; Kruczenski]

Double discontinuities vanish in overlapping channels

Perimeter is null in \( AdS_5 \times S_5 \)

\[ x_{i,i+1}^2 \rightarrow 0 \]

4D null limit of octagon or massless limit of the amplitude

Four-cusped null Wilson loop
**Massless limit**

**Coulomb-branch amplitudes with double logarithmic scaling:**

\[
\left( m_i^{\text{ext}} \right)^2 = y_{i,i+1}^2 = -x_{i,i+1}^2
\]

\[
\left( m_i^{\text{int}} \right)^2 = y_i^2 \neq 0
\]

\[
m^{\text{ext}} \to 0 \quad \Rightarrow \quad \Gamma_{\text{cusp}} \quad \Rightarrow \quad \Gamma_{\text{oct}} = \frac{2}{\pi^2} \log \cosh(2\pi g)
\]

\[
\begin{align*}
 m^{\text{int}} & \to 0 \\
 m^{\text{ext}} & \to 0 \\
 m^{\text{ext}} & \to 0
\end{align*}
\]

\[
\begin{align*}
 e^{-\frac{1}{2} \Gamma_{\text{cusp}} \log \frac{\left( m_i^{\text{int}} \right)^2}{x_i^{13}} \log \frac{\left( m_i^{\text{int}} \right)^2}{x_i^{24}}} = 1 \\
 e^{-\frac{1}{16} \Gamma_{\text{oct}} \log^2 \left( \frac{\sum x_i^{12} x_i^{23} x_i^{34} x_i^{41}}{x_i^{13} x_i^{24}} \right)^2} = 1
\end{align*}
\]

[Alady, Henn, Plefka, Schuster, 2007; Bern, Dixon, Smirnov, 2005]

**Controlled by functions of the coupling satisfying a deformed BES equation:**

[Beisert, Eden, Staudacher, 2006]

[Basso, Dixon, Papathanasious, 2020]

[Belitsky, Korchemsky, 2019]
Summary

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<tr>
<th>10D</th>
<th>( G(X) )</th>
<th>Amplitude</th>
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<tbody>
<tr>
<td>( X_{i,i+1} \to 0 )</td>
<td>( [M(x,y)]^2 = \Omega^2 )</td>
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- Can we relax the null condition?
- Higher-points?
- Relation to 10D sym. in SUGRA?
- Integrability in the Coulomb branch?