Progress in String Compactification

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(arXiv:2103.10454)

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(arXiv:2012.04656)

STRINGS 2021

ICTP-SAIFR, São Paulo (Online)

June 28th, 2021

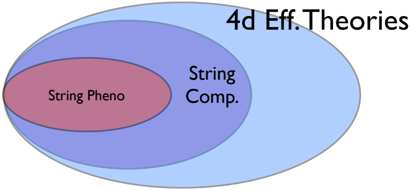
Motivation

String theory is a powerful extension of quantum field theory, but extracting low-energy physics from string geometry is mathematically challenging...

Higher dimensional theory → String Comp. → *4d* physics

- Need a good toolkit in any corner of string theory to extract the full low energy physics: (e.g. *the mass of the electron?*)
- Rules for "top down" model building?

  Patterns/Constraints/Predictions?



What possible EFTs?

⇕

Which geometries?

Today: Progress in $\mathcal{N}=1$, 4D Heterotic String Compactifications...

Heterotic Trilinear Yukawa Couplings

For a heterotic compactification, need: a manifold, X, and bundle, $\pi : V \to X$.

- Perturbative, tri-linear couplings defined around a background:

$$\mathcal{A}_b = A_b^0 + \phi^I v_{Ib}^x T_x + \dots$$

where v_I is the bundle valued harmonic 1-form on X which counts the multiplicity of the 4D fields ϕ^I.

- For matter-fields, ϕ^I, then the superpotential tri-linear coupling $\lambda_{IJK}\phi^I\phi^J\phi^K$ is given by the integral

$$\lambda_{IJK} \sim \int_X v_I \wedge v_J \wedge v_K \wedge \Omega$$

where Ω is the holomorphic $(3,0)$ form on X.

- Interested in particular textures/hierarchies. E.g. Standard model (Heavy top quark), SUSY "mu problem" (want to forbid a mass term allowed by gauge symmetry, i.e. $\mu H_d H_u$), Forbid rapid proton decay operators, etc...

Topological Vanishings

- In general: to understand heterotic effective theories, we need

  a) Good tools to effectively compute couplings/masses

  b) Ways to study/predict general structure (which could lead to hierarchies like those above).

- Observation: Heterotic theories can exhibit "Topological Vanishings" (i.e. vanishing couplings that must be zero due to the structure of the geometric background (X, V), rather than gauge invariance)

- Goals:
  - Understand what geometric effects can lead to topological vanishings
  - How can these geometric effects differ/be related?
  - How generic are such vanishings?
  - Hidden symmetries?

Algebraic vs. Differential Geometry Approaches

- How to actually compute?
- First E.g.s, $V = TX$ here some Yukawa couplings are simply triple intersection numbers ($d_{rst} = D_r \cap D_s \cap D_t$) and can use mirror symmetry (Strominger, Candelas,...)
- Algebro-geometric approach: Can show that integral form of λ_{IJK} is 1-1 with a triple product in cohomology (i.e. a Cup/Yoneda product). Eg. for $\mathbf{27}^3$ coupling in an E_6-theory:

$$H^1(X,V) \times H^1(X,V) \times H^1(X,V) \rightarrow H^3(X, \wedge^3 V) \simeq \mathbb{C}$$

- Given algebraic representation of $H^1(X,V)$, can turn this into a problem in polynomial multiplication (i.e. Groebner basis calculation). (Distler, et al),(Ovrut, Donagi, Pantev, et al), (LA, Gray, Lukas-et al...)

Algebraic vs. Differential Geometry Approaches

- Differential Geometry Approach (Blesneag, et al):
- E.g. Suppose $X \subset \mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$ is defined via $p(x) = 0$ and the bundles/forms descend from objects on $\mathcal{A}$:

$$\lambda(v_1, v_2, v_3) \sim -\frac{1}{2i} \int_{\mathcal{A}} \hat{v}_1 \wedge \hat{v}_2 \wedge \hat{v}_3 \wedge \delta^2(p) dp \wedge d\bar{p}$$

with $\delta^2(p)$ is a delta-function current satisfying

$$\delta^2(p) d\bar{p} = \frac{1}{\pi} \bar{\partial}(\frac{1}{p})$$

- Defining $\hat{\Omega} \wedge dp = \mu$ one has an explicit form

$$\lambda(v_1, v_2, v_3) = -\frac{1}{2\pi i} \int_{\mathcal{A}} \frac{\mu}{p} \wedge \left[\bar{\partial}\hat{v}_1 \wedge \hat{v}_2 \wedge \hat{v}_3 - \hat{v}_1 \wedge \bar{\partial}\hat{v}_2 \wedge \hat{v}_3 + \hat{v}_1 \wedge \hat{v}_2 \wedge \bar{\partial}\hat{v}_3\right]$$

note: $\hat{v}_I$ is not necessarily closed. Can easily see that some couplings must vanish.

A homological algebra framework

- We proved a number of theorems, only give a flavor here:
- *Theorem : Given any resolution*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to V_X \to 0$$

can define a notion of 'type' for elements of $H^1(V_X)$*. Namely, a cohomology element has type* $\tau = i$ *if it descends from* $H^i(\mathcal{F}_{i-1})$*. Then if* $H^3(\wedge^3 \mathcal{F}_0) = 0$*, all trilinear couplings between* $\tau = 1$ *fields will vanish. If, in addition,* $H^4(\mathcal{F}_1 \otimes \wedge^2 \mathcal{F}_0) = 0$*, then all Yukawa couplings between one type* $\tau = 2$ *and two type* $\tau = 1$ *fields will also vanish.*

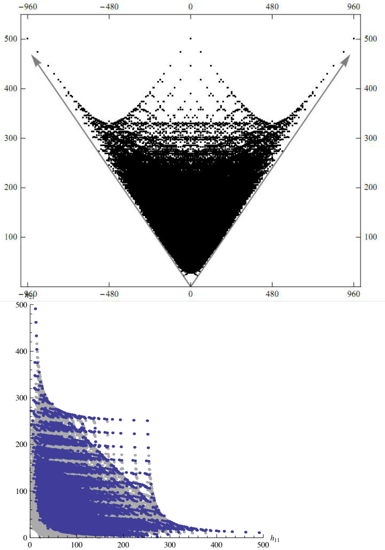
- Special case: (Generalization of work of Blesneag, et al): Koszul resolution for $X \subset \mathcal{A}$ (a complete intersection in co-dimension k):

$$0 \to \wedge^k \mathcal{N}^\vee \otimes \mathcal{V} \to \wedge^{k-1} \mathcal{N}^\vee \otimes \mathcal{V} \to \ldots \to \mathcal{V} \to V_X \to 0$$

Geometric origins for topological vanishings

- There are three main classes of geometric structures in the literature that lead to vanishing criteria (not necessarily exhaustive)
- Broadly, constraints arise from either – the manifold – and/or – the bundle. These include constraints on couplings due to
  1. The description of the CY manifold inside an ambient space (i.e. as a toric complete intersection) (Candelas, Lukas, et al)
  2. The description of the CY manifold as a fibration (Braun, Pantev, Ovrut, Donagi, et al),(Bouchard, Cvetic, Donagi), etc
  3. The stability of the bundle and Kähler cone substructure (i.e. "stability walls") (Watari et al), (LA, Gray, Ovrut)
- Homological algebra tools can also describe fibrations/stability walls.
- Simple Question: Given diverse geometric origins for topological vanishings of Yukawa couplings, must different descriptions agree? $\Rightarrow$ No.

Genericity of vanishing criteria



(from Taylor 1205.0952)

- CY3 Fibrations are actively studied (LA,Gao,Gray,Lee), (Taylor-Huang)
- Observation 1: Almost all known CY 3-folds are fibered.
- Observation 2: generic manifolds do not admit just one elliptic fibration, they admit many ($\sim$ 10s or 100s).
- Each of these distinct fibrations can induce topological vanishings! Overall Conclusion: Heterotic couplings *generically* highly constrained (beyond any one basis).

Determining the 4D theory in string compactification

- Physical quantities in low energy string theory depend on the metric and gauge connections in the extra dimensions.
- For example:
  - Yukawa couplings in Heterotic string theory descend from a term in the 10-dimensional action of the form $\sim \int d^{10}\sqrt{-g}\bar{\psi}A\psi$. Normalization of fields and coefficients of the superpotential depend on g.
  - Matter field Kahler potential unknown except for special cases.
  - Modes of V-twisted Dirac Operator: $\nabla\!\!\!/_X \Psi = 0$ depend on the metric and a connection on a vector bundle, V on X (gauge field vevs on X).
  - Problems in moduli stabilization
- **How to determine the metric and the connection?**
- Only current *general* approach via $\Rightarrow$ numeric approximation.

The Donaldson Algorithm

- Idea: Use projective embeddings to generate simple metrics that can be parametrically tuned to the Ricci-flat solution.
- Kodaira embedding: Given an ample line bundle $\mathcal{L}$ on X then an embedding

$$i_k : \ X \to \mathbb{P}^{n_k - 1}, \quad (x_0, \ldots, x_2) \mapsto \left[s_0(x) : \ldots : s_{n_k - 1}(x)\right]$$

exists for all $\mathcal{L}^k = \mathcal{L}^{\otimes k}$ with $k \geq k_0$ for some k_0, where $s_\alpha \in H^0(X, \mathcal{L}^k)$.

- What do we know about metrics on $\mathbb{P}^n$? Fubini-Study:

$$(g_{FS})_{i\bar{j}} = \frac{i}{2} \partial_i \bar{\partial}_{\bar{j}} K_{FS} \quad \text{where} \quad K_{FS} = \frac{1}{\pi} \ln \sum_{i\bar{j}} h^{i\bar{j}} z_i \bar{z}_{\bar{j}}$$

and $h^{i\bar{j}}$ is a non-singular, hermitian matrix.

- FS metric restricted to X is not Ricci-flat. But...

- Generalize: $K_{h,k} = \frac{1}{k\pi} \ln \sum_{\alpha,\bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}} = \ln ||s||^2_{h,k}$
- $h^{\alpha\bar{\beta}}$ is a hermitian fiber metric on $\mathcal{L}^{\otimes k}$.
- Such Kähler potentials are dense in the moduli space (Tian)
- Fixed point of Donaldson's "T-operator" $\leftrightarrow$ "balanced metric".

$T(h)_{\alpha\bar{\beta}} = \frac{n_k}{Vol_{CY}(X)} \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum_{\gamma\bar{\delta}} h^{\gamma\bar{\delta}} s_\gamma \bar{s}_{\bar{\delta}}} \mathrm{dVol}_{CY}$

Theorem (Donaldson)

For each $k \geq 1$, the balanced metric, h, on $\mathcal{L}^{\otimes k}$ exists and is unique. As $k \to \infty$, the sequence of metrics

$$g^{(k)}_{i\bar{j}} = \frac{1}{k\pi} \partial_i \bar{\partial}_{\bar{j}} \ln \sum_{\alpha,\bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}$$

on X converges to the unique Ricci-flat metric for the given Kähler class and complex structure.

A new approach

- Existing numeric implementations of Donaldson's algorithm (Douglas et al, Ovrut et al). Computationally intensive. (Accurate enough? Don't know...)
- Moduli dependence difficult to obtain.
- New Approach ⇒ Machine Learning. What we did:

1. Supervised learning of moduli dependence of Calabi-Yau metrics using the Donaldson algorithm to generate training data.
2. Direct learning of moduli dependent Calabi-Yau metrics both using the metric ansatz and without it.
3. Direct learning of metrics associated to $SU(3)$ structures with torsion.

- I'll give a brief flavor of these results...(see also, (Douglas et al), (Jejjala, et al))

Preliminaries

- One definition of a Calabi-Yau three-fold: A complex 3-fold admitting a nowhere vanishing real two-form, J, and a complex three-form, Ω, such that:

$$J \wedge \Omega = 0 \qquad J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \overline{\Omega}$$

$$dJ = 0 \qquad d\Omega = 0$$

- Metric is related to the two form as $ig_{a\bar{b}} = J_{a\bar{b}}$
- Example CY manifold: "Quintic" hypersurface: $X = \mathbb{P}^4[5]$
- e.g. $p(\vec{z}) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \psi z_0 z_1 z_2 z_3 z_4 = 0$
- The holomorphic $(3,0)$ form can explicitly constructed for such manifolds (Candelas, et al).

$$\Omega = \frac{1}{\partial p_\psi(\vec{z})/\partial z_b} \bigwedge_{\substack{c=0,\ldots 3, \\ c \neq a,b}} dz_c$$

Direct Learning of the Kahler Potential

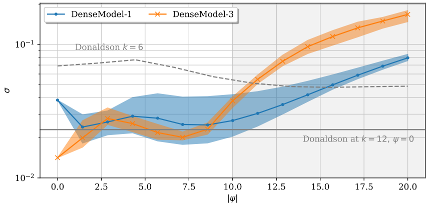
- The balanced metric output by Donaldson's algorithm at given finite k is not necessarily the most accurate approximation to the Ricci-flat metric - maybe we can do better?
- Can generate networks to find the parameters that are trained directly using a loss such as:

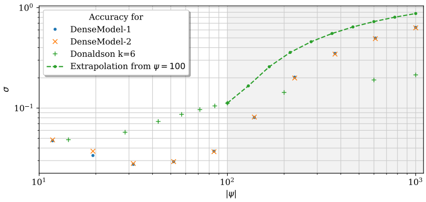
$$\mathcal{L}_{MA} = \left| 1 + \frac{4i}{3} \frac{J^3}{\Omega \wedge \overline{\Omega}} \right|$$

- C.f.: Headrick and Nassar (although note that we are obtaining moduli dependent results and using ML). Network Architecture:

Layer	Number of Nodes	Activation	Number of Parameters
input	17	–	–
hidden 1	100	leaky ReLU	1800
hidden 2	100	leaky ReLU	10 100
hidden 3	100	leaky ReLU	10 100
output	d^2	identity	101 d^2

- These networks were optimized for $0 \leq |\psi| \leq 10$





(shaded region denotes extrapolation of the networks).

- Note Donaldson algorithm with $k = 12$ takes order days to run even for the single case of $\psi = 0$. This network at $k = 6$ takes only minutes and gives comparable accuracy for a whole range of ψ.
- We do better than Donaldson Alg. at $k = 6$ and that this improvement extends up to $|\psi| \simeq 175$, nearly a factor of 2 beyond the regime used during training.

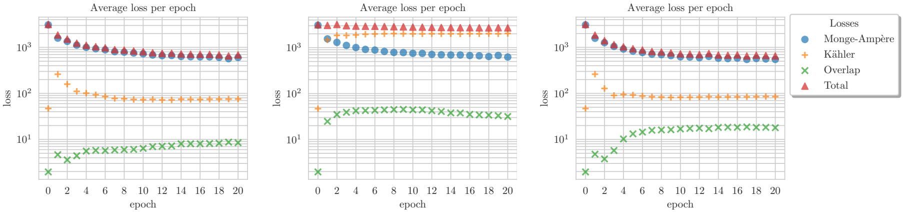
Direct learning of the metric

- Instead of learning parameters in an ansatz for the Kahler potential we can try to learn the CY metric directly.
- Why try?
  - Perhaps we can improve performance by not being tied to an ansatz at fixed k.
  - We will be able to generalize this approach to more complicated geometries.
- One disadvantage:
  - We now need loss functions to check that the metric is globally well defined and Kahler! We use $\mathcal{L} = \lambda_1 \mathcal{L}_{\mathrm{MA}} + \lambda_2 \mathcal{L}_{\mathrm{dJ}} + \lambda_3 \mathcal{L}_{\mathrm{overlap}}$
  - Here $\mathcal{L}_{\mathrm{MA}}$ is the loss described before and we add to this

$$\mathcal{L}_{dJ} = \frac{1}{2} ||dJ||_1$$

$$\mathcal{L}_{\mathrm{overlap}} = \frac{1}{d} \sum_{k,j} ||g_{\mathrm{NN}}^{(k)}(\vec{z}) - T_{jk}(\vec{z}) \cdot g_{\mathrm{NN}}^{(j)}(\vec{z}) \cdot T_{jk}^{\dagger}(\vec{\bar{z}})||_n$$

- Input: $Re(z_i)$, $Im(z_i)$ (homogeneous coords describing pt in CY) $Re(\psi)$, $Im(\psi)$. Output: d^2 real and imaginary parts of a metric at point.
- To give a concrete example: optimized at $\psi = 10$ on a data set of $10,000$ points. We split the points according to train:test$=90:10$ and we train for 20 epochs.
- Accuracy reaches same level as Donaldson Alg. at $k = 5$ (we expect more points and better architecture will easily improve this).



Left: Optimizing the NN with all three losses. *Middle:* Optimizing the NN without Kähler loss (i.e. $\lambda_2 = 0$). *Right:* Optimizing the NN without overlap loss (i.e. $\lambda_3 = 0$).

Learning $SU(3)$ structures from an anzatz

- Important reason to directly learn the metric: Can be generalized to non-Kähler geometry!
- One important class of geometries for $\mathcal{N} = 1$ compactifications: $SU(3)$ structure manifolds
- These are six-manifolds with a nowhere vanishing two form J and three form Ω obeying the same algebraic properties as the Calabi-Yau threefold case:

$$J \wedge \Omega = 0 \qquad\qquad J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \overline{\Omega}$$

But with different differential properties...

- An $SU(3)$ structure can be classified by its torsion classes:

$$\begin{aligned} dJ &= -\frac{3}{2}\mathrm{Im}(W_1\overline{\Omega}) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + W_5 \wedge \Omega \,, \end{aligned}$$

- Where torsion classes are given the defining forms:

$$W_1 = -\frac{1}{6} i\Omega \lrcorner dJ = \frac{1}{12} J^2 \lrcorner d\Omega \ , \ W_4 = \frac{1}{2} J \lrcorner dJ \ , \ W_5 = -\frac{1}{2} \Omega_+ \lrcorner d\Omega_+$$

- Given string theories place different constraints on the torsion classes for there to be an associated solution to the theory of the type we want.
- E.g. heterotic string theory: $W_1 = W_2 = 0$, $W_4 = \frac{1}{2} W_5 = d\phi$, W_3 free.
- Note that a CY structure is a special case: $W_i = 0 \ \forall i = 1, \ldots 5$.
- Need to start with some well-controlled/simple example.
- Observation: Some CY manifolds admit not only Ricci-flat metrics, but other $SU(3)$ structures as well.
- E.g. (generalization of work by Larfors, Lukas and Ruehle)

$$J = \sum_{i=1}^{h^{1,1}(x)} a_i J_i \qquad\qquad \Omega = A_1 \Omega_0 + A_2 \bar{\Omega}_0$$

- The a_i are real functions and A_1 and A_2 are complex functions. CY taken to be a complete intersection in a product of projective spaces.

- Quintic E.g. (from Larfors, et al)

$$a_1 = \frac{1}{\pi^3} \frac{|\nabla p|^2}{\sigma^4} \ , \ A_1 = a_1^2 \ , \ A_2 = 0 \ , \text{(where} \ \ \sigma = \sum_{a=0}^{4} |z_a|^2 \ )$$

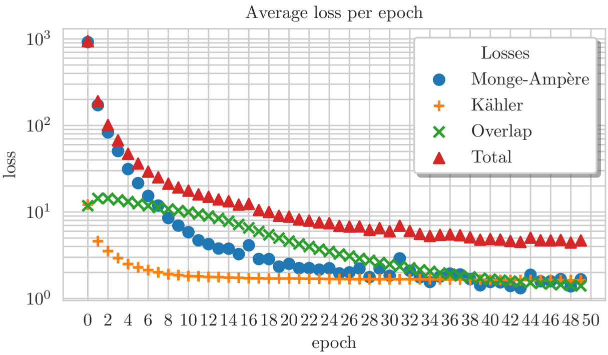
with p the defining equation of the hypersurface.

- This has torsion classes $W_1 = W_2 = W_3 = 0$, $W_5 = 2W_4 = 2d(ln(a_1))$ and thus provides a solution to heterotic string theory.
- We aimed to reproduce this known analytic solution using direct learning of the metric.
- Such a check is particularly important in learning such metrics as we have no analogue of Yau's theorem to use to argue we are converging towards an exact/unique solution.

- Can use the same losses as in the Calabi-Yau case, then, with the exception of replacing the Kahler loss by the following.

$$\mathcal{L}_{W_4} = \| dJ - dln(a_1 \wedge J) \|_n$$

- We ran this for the $\psi = 10$ quintic, using multiplicative boosting from g_{FS}.



Results and Future Work

- Progress is being made in characterizing generic features of heterotic CY vacua – E.g. when/why topological vanishings arise in heterotic couplings.
- Can phrase (all) such constraints (Koszul/fibrations/etc) in a common language (using homological algebra). Constraints ubiquitous.
- Open questions:
  - Higher order and non-perturbative contributions?
  - Dual theories? Hidden Symmetries? Link to Swampland conjectures?
- Control of the metric is necessary to specify the 4D theory in compactifciation $\Rightarrow$ ML techniques can provide a powerful new tool.
- In particular, we have provided the first numeric approx. to $SU(3)$ structure metrics.
- Open questions: What next? $\Rightarrow$ Aspects of 4D theory? Moduli stabilization problems? Stay tuned for the discussion session....